Static / dynamic properties of stochastic volatility models:

a structural connection

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Outline

- Stochastic volatility models produce a smile *AND* generate a dynamics for implied volatilities

- Is there a connection of a structural nature ?
- Enter the *Skew Stickiness Ratio*

  - Long-maturity smiles
    - an expression linking the decay of the ATM and the SSR
  - Historical dynamics of market smiles ?

  - Short-maturity smiles
    - The *realized* skew

- Conclusion

Papers *Smile Dynamics I, II, III, IV* are available on the SSRN web site.

• Start with term-structure of Variance Swap Volatilities $\hat{\sigma}_{tT}$ observed at time $t = 0$ and define forward variances ξ_t^T as :

$$\xi_t^T = \frac{d}{dT}((T-t)\hat{\sigma}_{tT}^2) \qquad \hat{\sigma}_{tT}^2 = \frac{1}{T-t}\int_t^T \xi_t^\tau d\tau$$

• Forward variances can be traded – using Variance Swaps – at no cost : foward variances are driftless processes in the pricing measure.

• The general expression of a diffusive stochastic volatility model can be written as:

$$dS_t = (r-q)S_t\,dt + \sqrt{\xi_t^t}\,S_t\,dZ_t$$
$$d\xi_t^T = \bullet\; dW_t^T$$

• Any diffusive stochastic volatility model can be written this way

• The instantaneous variance of the spot process is ξ_t^t

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- Let us write the dynamics as: $$\begin{cases} dS_t^{\omega} = (r-q)S_t^{\omega}dt + \sqrt{\xi_t^t}\, S_t^{\omega} dZ_t \\ d\xi_t^T = \omega\, \xi_t^T \sum_k \lambda_{kt}^T(\xi_t)\, dW_t^k \end{cases}$$

- At order 1 in ω: $$\xi_t^T = \xi_0^T \left(1 + \omega \int_0^t \sum_k (\lambda_{k\tau}^T)_0\, dW_\tau^k \right)$$

  – In particular, for the instantaneous variance:

$$\xi_t^t = \xi_0^t + \delta\xi_t$$

$$\delta\xi_t = \omega \xi_0^t \int_0^t \sum_k (\lambda_{k\tau}^t)_0\, dW_\tau^k$$

  – ATMF skew as a function of skewness s_T of $x_T = \ln(S_T / F_T)$ is approximately given by:

$$\mathcal{S}_T = \left.\frac{d\hat{\sigma}_{KT}}{d\ln K}\right|_F \approx \frac{s_T}{6\sqrt{T}} \qquad s_T = \frac{M_3^T}{(M_2^T)^{3/2}}\ , \quad M_n^T = (x_T - <x_T>)^n$$

(at order 1 in ω, exactly equivalent to 1st order perturbation of pricing equation in ω)

$$x_T - <x_T> = \int_0^T \sqrt{\xi_0^t}\, dZ_t + \frac{1}{2}\left(\int_0^T \frac{\delta\xi_t}{\sqrt{\xi_0^t}} dZ_t - \int_0^T \delta\xi_t dt \right)$$

The ATMF skew - 3

$$M_2^T = \int_0^T \sqrt{\xi_0^t}\,dt$$

$$M_3^T = \frac{3}{2}E\left[\left(\int_0^T \sqrt{\xi_0^t}\,dZ_t\right)^2\left(-\int_0^T \delta\xi_t dt + \int_0^T \frac{\delta\xi_t}{\sqrt{\xi_0^t}}dZ_t\right)\right]$$

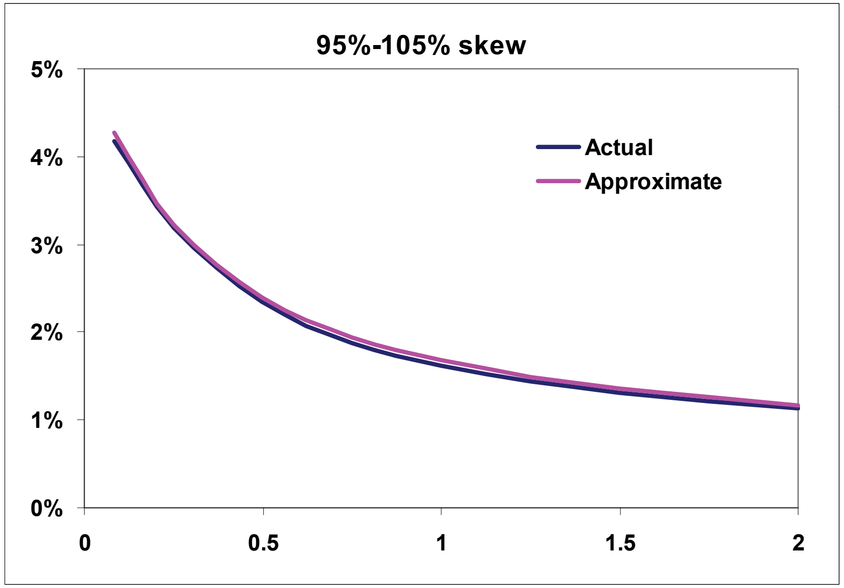
- Computing the expectation: $M_T^3 = 3\omega\int_0^T dt\,\xi_0^t\int_0^t\sqrt{\xi_0^t}\sum_k \rho_{iS}(\lambda_{i\tau}^t)_0 d\tau = 3\int_0^T dt\left(\int_0^t E\left[\frac{dS_\tau^0}{S_\tau^0}d\xi_\tau^t\right]\right)$
  - intuition: $M_T^3 = \left\langle(\Sigma r_i)^3\right\rangle = \Sigma < r_i r_j r_k > = 3\sum_{j>i} < r_i r_j^2 >$

- Introduce spot/volatility covariance function:

$$f(\tau,t) = \frac{1}{d\tau}E\left[\frac{dS_\tau^0}{S_\tau^0}d\xi_\tau^t\right]$$

- At order 1 in volatility-of-volatility, ATMF skew is given by:

$$S_T = \frac{1}{2\sqrt{T}}\frac{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}}$$



- How much does the ATMF volatility move when the spot moves – in units of the skew?

- Market-makers use following ratio: $r_T = \frac{1}{\left.\frac{d\hat{\sigma}_{KT}}{d\ln K}\right|_F} \frac{\Delta\hat{\sigma}_{FT}}{\Delta \ln S}$
  - $r_T = 1$: " sticky-strike " regime
  - $r_T = 0$: " sticky-delta " regime

- Introduce Skew Stickiness Ratio:

$$R_T = \frac{1}{\left.\frac{d\hat{\sigma}_{KT}}{d\ln K}\right|_F} \frac{E\left[d\hat{\sigma}_{FT}\; d\ln S\right]}{E[(d\ln S)^2]}$$

  - Jump / Lévy models : $R_T = 0$
  - Local vol, weak skew : $R_T = 2$
  - Stochastic volatility, short maturity & weak skew: $R_T = 2$

The SSR - 2

- At order 1 in volatility-of-volatility

  – use VS volatility instead of ATMF volatility:

$$E[d\ln S_\tau \, d\hat{\sigma}_T] = \frac{1}{2\hat{\sigma}_T T}\int_0^T E[d\ln S \; d\xi^t]dt = \frac{1}{2\hat{\sigma}_T T}\int_0^T E[\frac{dS_0^0}{S_0^0}\;\delta\xi_t]dt = \frac{1}{2\hat{\sigma}_T T}\int_0^T f(0,t)\,dt$$

  – Final expression for the SSR, at order 1 in volatility-of-volatility:

$$R_T = \frac{\int_0^T \xi_0^t\,dt}{\xi_0^0 T}\;\frac{T\int_0^T f(0,t)\,dt}{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}$$

$$S_T = \frac{1}{2\sqrt{T}}\frac{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}}$$

- Take short-maturity limit:

$$R_0 = \lim_{T\to 0}\frac{T\int_0^T f(0,t)\,dt}{\int_0^T dt\int_0^t f(\tau,t)\,d\tau} = 2$$

$$S_0 = \lim_{T\to 0}\frac{1}{2\sqrt{T}}\frac{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}} = \frac{f(0,0)}{4(\xi_0^0)^{3/2}}$$

  – Skew tends to a finite value which measures the covariance function at origin

  – SSR tends to universal value: 2

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Bounds for the SSR

- Let us make some additional assumptions:
  - VS curve is flat: $\xi_0^T \equiv \xi_0$
  - Stochastic volatility model is time-homogeneous: $f(\tau,t) \equiv f(t-\tau)$
  - ATMF skew and SSR take simpler forms:

$$\mathcal{S}_T = \frac{\int_0^T (T-t) f(t)}{2\xi_0^{3/2} T^2} \qquad \mathcal{R}_T = \frac{\int_0^T f(t)\,dt}{\int_0^T (1-\frac{t}{T}) f(t)\,dt}$$

- Assume that $|f(t)|$ decreases monotonically towards zero. Rewrite $\mathcal{R}_T$ as

$$\mathcal{R}_T = \frac{g(T)}{\frac{1}{T}\int_0^T g(t)\,dt}, \quad g(t) = \int_0^t f(\tau)\,d\tau$$

  - $\frac{g(t)}{g(T)} \leq 1$ implies $\mathcal{R}_T \geq 1$
  - $\frac{g(t)}{g(T)} \geq \frac{t}{T}$ implies $\mathcal{R}_T \leq 2$

⇨ Model-independent range for $\mathcal{R}_T$: $\mathcal{R}_T \in [1, 2]$

Scaling of the SSR

- Assume that for large t $f(t) \propto \frac{1}{t^{\gamma}}$

- Then, for large *T* two types of behaviour, depending on value of γ

  – Type I : If $\gamma > 1$

$$S_T \propto \frac{1}{T} \quad and \quad \lim_{T\to\infty} R_T = 1$$

  – Type II : If $\gamma < 1$

$$S_T \propto \frac{1}{T^{\gamma}} \quad and \quad \lim_{T\to\infty} R_T = 2 - \gamma$$

  – exponential decay: Type I.

- Type I: scaling of S_T analogous to models with independent increments, however model becomes sticky-strike
- Both types of scaling can be summarized by:

$$S_T \propto \frac{1}{T^{2-R_\infty}}$$

where $R_\infty = \lim_{T\to\infty} R_T$

Type II scaling – in a model ?

- Consider a model of the following type (Bergomi, 2005):

$$d\xi_t^T = \omega \xi_t^T \sum w_i e^{-k_i(T-t)} dW_t^i$$

  - expression for f: $f(\tau) = \omega \xi_0^{3/2} \sum w_i \rho_{Si} e^{-k_i \tau}$

  - expressions for skew and SSR:

$$S_T = \frac{\omega}{2} \sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2} \qquad R_T = \frac{\sum w_i \rho_{Si} \frac{1 - e^{-k_i T}}{k_i T}}{\sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2}}$$

  - for large τ $f(\tau) \propto e^{-(\min_i k_i)\tau}$ ⇨ for (really) large T $S_T \propto \frac{1}{T}$, $R_T \to 1$

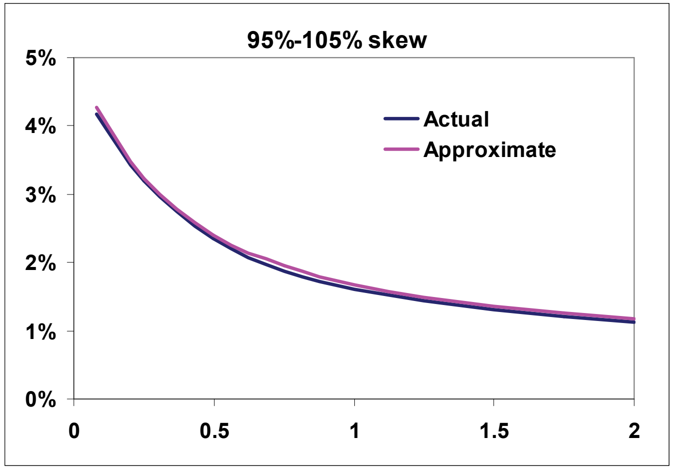
- What about intermediate maturities ?

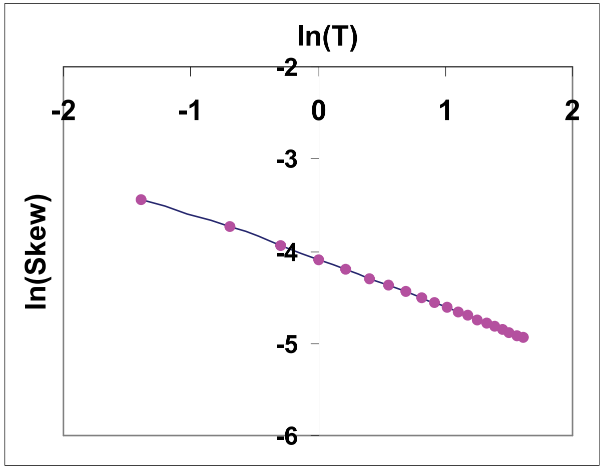
Type II scaling – in a model ?

•By suitably choosing parameter values, get non-trivial scaling over wide range of maturities:

$$k_1 = 8,\ k_2 = 0.35,\quad w_1 = 72\%,\ w_2 = 28\%,\quad \rho_{S1} = -70\%,\ \rho_{S2} = -35.7\%,\quad w = 3.36$$

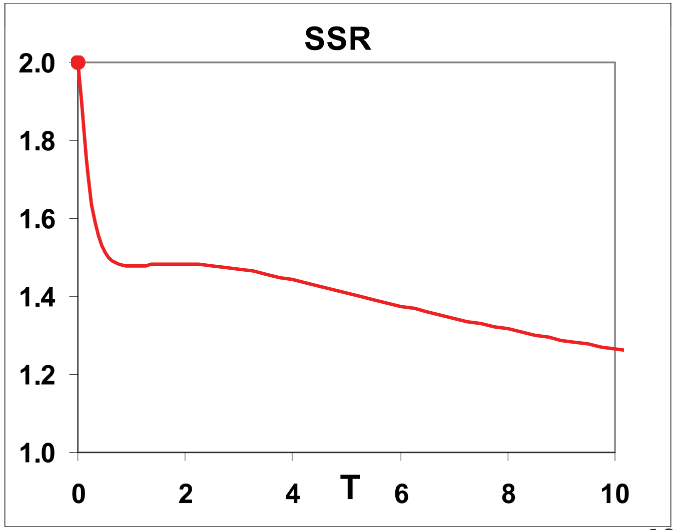
• Look at maturity-dependence of ATMF skew:





⇨ ATMF skew decays like $S_T \propto 1/\sqrt{T}$. What about the SSR ?

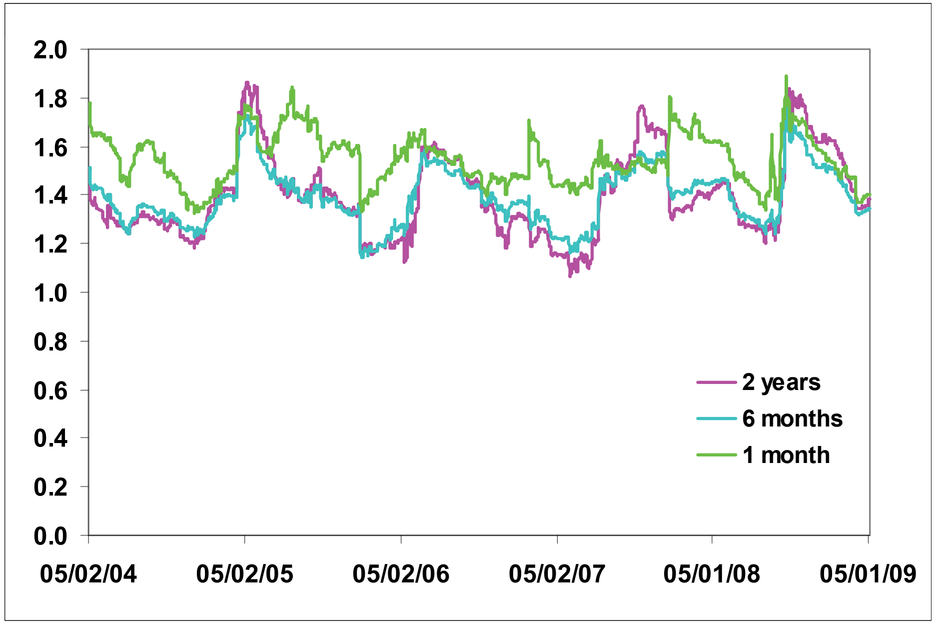
⇨ We get a plateau around 1.5 (*i. e. 2 minus exponent of skew decay*) for an intermediate range of maturities.



⇨ It is possible to get in a model Type II behavior for a range of maturities that is practically relevant.

Type II scaling – in the market ?

- Equity index smiles usually exhibit an ATMF skew that typically decays like $\frac{1}{\sqrt{T}}$
  - looks like Type II, but what about the SSR ?
  - For the Eurostoxx, 3-month running estimates of the SSR for maturities 1 month, 6 months, 2 years:



$$R_T^{Realized} = \frac{\sum (\hat{\sigma}_{i+1} - \hat{\sigma}_i) \ln\left(\frac{S_{i+1}}{S_i}\right)}{\sum \left.\frac{d\hat{\sigma}_{KT}^i}{d\ln S}\right|_F \ln\left(\frac{S_{i+1}}{S_i}\right)^2}$$

⇨ SSR usually in interval [1, 2]

⇨ For longer maturities, average value of the SSR ~ 1.5 – compatible with a decay of the skew ~ $\frac{1}{\sqrt{T}}$

⇨ Equity volatility markets seem to be of Type II

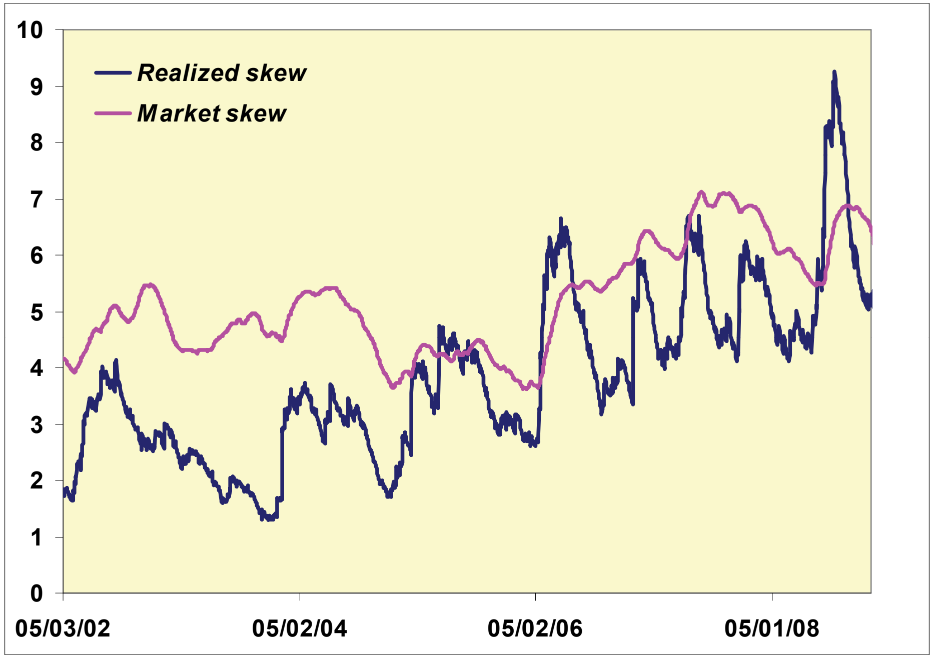
- One puzzle left: short-maturity SSR always lower than 2 – can this be arbitraged ?
  - is it possible materialize $2 - R_0$ as a P&L ? (or are we just plain wrong?)

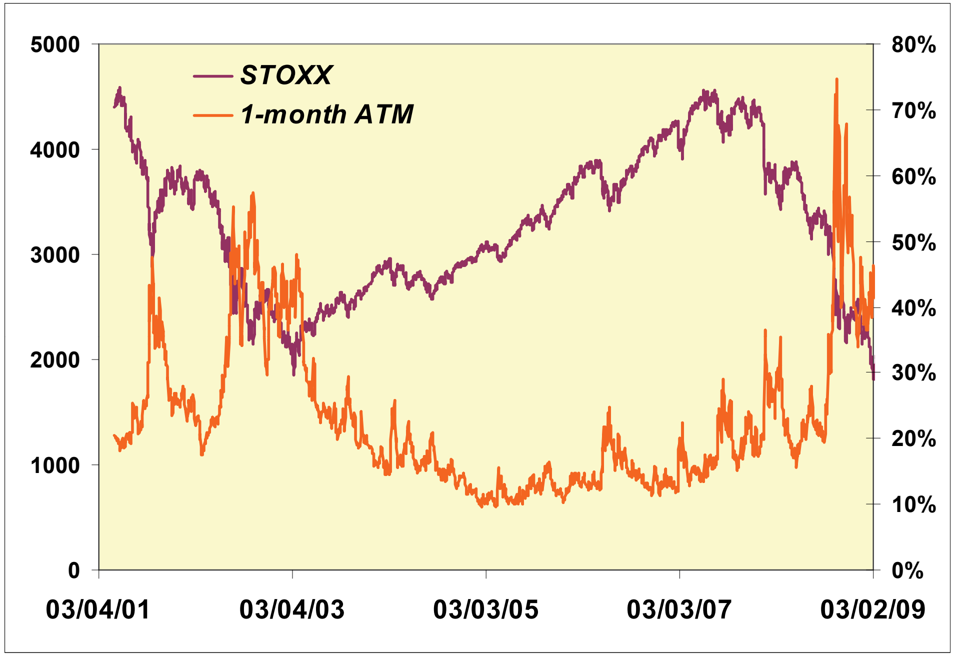
Arbing the realized SSR

- Introduce " realized " skew:

$$\left.\frac{d\hat{\sigma}}{d\ln K}\right|_S^{Realized} = \frac{1}{2\sigma_0\delta t}\left\langle \frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right\rangle$$

– implied versus " realized " 1-month 95 / 105 skew





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Conclusion

- In stochastic volatility models, at order 1 in the vol-of-vol, the Skew Stickiness Ratio and the rate of decay of the ATMF skew are related through the spot/vol covariance function

- For time-homogeneous model & flat VS term-structure:
  - SSR is bounded : $R_T \in [1, 2]$
  - 2 types of models
    - Type I $\quad S_T \propto \frac{1}{T}$ *and* $\lim_{T\to\infty} R_T = 1$
    - Type II $\quad S_T \propto \frac{1}{T^\gamma}$ *and* $\lim_{T\to\infty} R_T = 2-\gamma$

$$S_T \propto \frac{1}{T^{2-R_\infty}}$$

- Index volatility markets' behavior consistent with Type II

- For short maturity smiles, SSR = 2
  - Markets display a *realized* SSR < 2
  - Introduce the notion of " realized skew "

    realized covariance of spot & <u>implied</u> vol
  - $2-R_0$ mismatch can be materialized as a P&L

$$\left.\frac{d\hat{\sigma}}{d\ln K}\right|_S^{Realized} = \frac{1}{2\sigma_0\delta t}\left\langle \frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right\rangle$$

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