Multilevel path simulation for jump-diffusion SDEs

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Outline

Introduction

- General approach to reduce the variance in estimating path-dependent payoff.
- Could work for other simulations involving numerical approximations (high-dimensional problems, sensitivities, SPDE).

Advantages:
Balances the tradeoff between discretisation bias and sample error. in terms of criteria of computational effort to achieve an ε RMS error (equivalent to variance reduction ratio).

- For standard method with discretisation scheme: $O(N_l h_l^{-1})$, after optimising $O(\varepsilon^{-3})$ to achieve ε RMS error.
- For multilevel: once the variance convergence rate $\beta > 1$, $O(\varepsilon^{-2})$.

Multilevel Approach for geometric Brownian SDEs

Given a scalar SDE driven by a Brownian diffusion

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t),$$

to estimate $E[P] := E[f(S(T))]$ where the path-dependent payoff P can be approximated by $\widehat{P}_l$ using 2^l uniform timesteps, we use

$$E[\widehat{P}_L] = E[\widehat{P}_0] + \sum_{l=1}^{L} E[\widehat{P}_l - \widehat{P}_{l-1}].$$

$E[\widehat{P}_l - \widehat{P}_{l-1}]$ is estimated using N_l simulations with same $W(t)$ for both $\widehat{P}_l$ and $\widehat{P}_{l-1}$,

$$\widehat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} \left(\widehat{P}_l^{(i)} - \widehat{P}_{l-1}^{(i)} \right)$$

Using independent samples for each level, the variance of the combined estimator is

$$\mathbb{V}\left[\sum_{l=0}^{L} \widehat{Y}_l\right] = \sum_{l=0}^{L} N_l^{-1} V_l, \qquad V_l \equiv \begin{cases} V[\widehat{P}_l - \widehat{P}_{l-1}], & l > 0 \\ V[\widehat{P}_0], & l = 0 \end{cases}$$

and the computational cost is proportional to $\displaystyle\sum_{l=0}^{L} N_l\, h_l^{-1}$

Hence, the variance is minimised for a fixed computational cost by choosing N_l to be proportional to $\sqrt{V_l\, h_l}$.

MLMC Theorem

Theorem

Let P be a functional of the solution of a stochastic o.d.e., and $\widehat{P}_l$ the discrete approximation using a timestep $h_l = 2^{-l}\, T$.

there exist independent estimators $\widehat{Y}_l$ based on N_l Monte Carlo samples, with computational complexity (cost) C_l, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that

i) $\left| \mathbb{E}[\widehat{P}_l - P] \right| \leq c_1\, h_l^\alpha$

ii) $\mathbb{E}[\widehat{Y}_l] = \begin{cases} \mathbb{E}[\widehat{P}_0], & l = 0 \\ \mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}], & l > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_l] \leq c_2\, N_l^{-1} h_l^\beta$

iv) $C_l \leq c_3\, N_l\, h_l^{-1}$

Theorem

there exists a positive constant c_4 such that for any $\varepsilon < e^{-1}$ there are values L and N_l for which the multilevel estimator

$$\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_l,$$

has Mean Square Error $MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > 1, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4\, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Using Milstein discretization for a European call option where payoff is Lipschitz , first order strong convergence gives

$$V[\widehat{P}_l - P] = O(h_l^2) \Longrightarrow V_l = \propto h_l^2$$

and hence total cost to achieve ε RMS error is $O(\varepsilon^{-2})$ instead of usual $O(\varepsilon^{-3})$.

Jump-diffusion SDEs

To capture the characteristics of fat-tail return distribution and the volatility smile effect, Merton introduced jump-diffusion SDEs to model stock price:

$$\mathrm{d}S(t) = a(S(t-), t)\mathrm{d}t + b(S(t-), t)\mathrm{d}W_t + c(S(t-), t)\mathrm{d}J_t,$$

where the jump item J_t is a compound Poisson process $\sum_{i=1}^{N(t)}(Y_i - 1)$, the jump magnitude Y_i satisfies some probability distribution, and $N(t)$ is a Poisson process with intensity λ, independent of the Brownian motion.

A jump-adapted Milstein discretisation

To do path simulation of jump-diffusion SDEs, the jump times need to be added into the discretisation time grid. We use the so-called jump-adapted approximation proposed by Platen in 1982 [Pla82], which uses superposition of jump times and the fixed-time grid as the time grid.

In the particular case of $c(S(t-),t) = S(t-)$, we propose a jump-adapted Milstein scheme:

$$\widehat{S}_{n+1}^{-} = \widehat{S}_n + a\,h_n + b\,\Delta W_n + \tfrac{1}{2}\frac{\partial b}{\partial S}\,b\,(\Delta W_n^2 - h_n) \quad ;$$

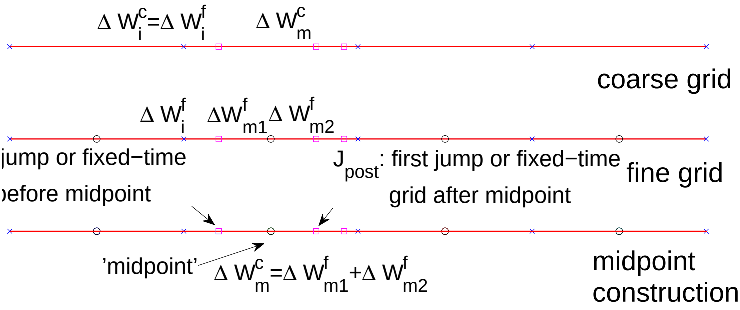
$$\widehat{S}_{n+1} = \begin{cases} \widehat{S}_{n+1}^{-}\ \left(1 + \dfrac{c(S(t-),t)}{S(t-)}(Y_i - 1)\right), & \text{when } t_{n+1} \in \mathbb{J}\ ; \\ \widehat{S}_{n+1}^{-}, & \text{otherwise.} \end{cases}$$

where $\mathbb{J} = \{\tau_1, \tau_2, \ldots, \tau_m\}$ is jump time set.

Multilevel for jump diffusion when rate is constant

- use jump-adapted discretisation, adding jump times to standard uniform timestep discretisation times
- Milstein approximation of pure diffusion model between jumps, with conditional Brownian interpolation within each timestep for barrier and lookback options
- jump intervals are exponential random variables; the same values are used for coarse and fine paths

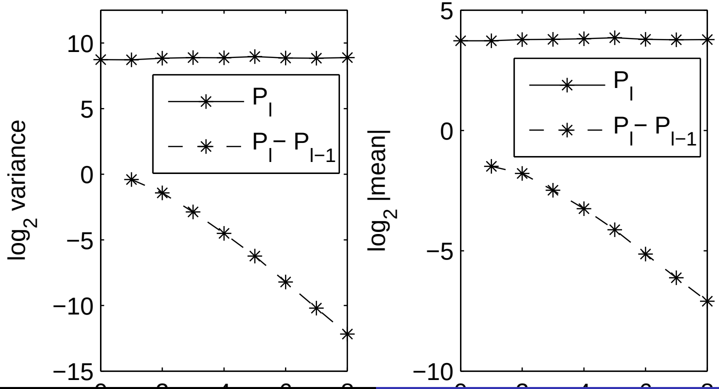
"Midpoint" construction in multilevel approach



Numerical results European vanilla call under

$$\mathrm{d}S(t)/S(t-) = (r - \frac{\sigma^2}{2})\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}J_t,$$

$T = 1, \quad S(0) = 100, \quad r = 0.05, \quad \sigma = 0.2,$
$\log Y \sim N(-0.1, 0.2), \ \lambda = 2; \ P = \exp(-rT)\max(S(T) - K, 0)$



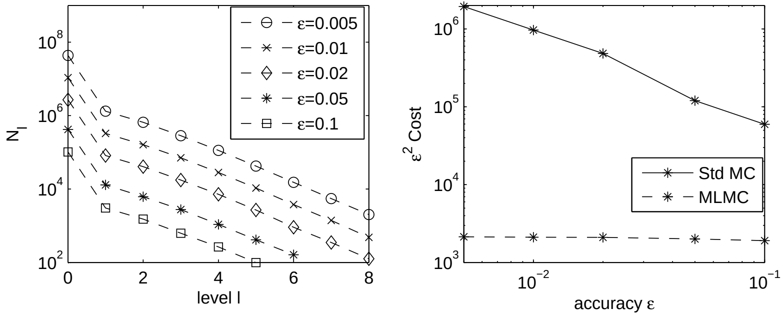


Figure: european vanilla option

Numerical results for variance rate of path-dependent options

Option	numerics for jump-diffusion	proved rate for plain diffusion
Lipschitz	$O(h^2)$	$O(h^2)$
Asian	$O(h^2)$	$O(h^2)$
lookback	$O(h^2)$	$o(h^{2-\delta})$
barrier	$O(h^{3/2})$	$o(h^{3/2-\delta})$
digital	$O(h^{3/2})$	$o(h^{3/2-\delta})$

Table: V_l convergence observed numerically for const rate jump-diffusion SDEs– matches results for standard diffusion (proved by Giles, Debrabant & Rößler), numerical analysis is yet to be done.

General jump-diffusion SDEs

There are several directions to generalise the Merton's model.

- Introduce dependency between parameters. As a particular example, we can let jump rate rely on the stock price, namely $\lambda = \lambda(S(t-), t)$, which is called state-dependent intensity.
- Replace the jump-diffusion SDE with SDE driven by Lévy processes which allows infinite frequency of jump within limited time (done by Dereich &Heidenreich in [DH10]).

The first case is relatively complex for Multilevel implementation because jump times differ in fine and coarse grids due to bias introduced to the approximated intensity by the discretisation scheme.

Two approaches for state-dependent intensity

We can have two methods to cope with state-dependent intensity.

- Thinning method;
  In the literature, the case of bounded state-dependent intensity is discussed by Glasserman & Merener where they use a thinning approach to simulate the jump-diffusion SDEs exactly.
- Cumulative intensity method.
  For the unbounded case, we have the cumulative distribution function of jump times. Thus an inverse transform for the cumulative intensity can be adopted.

Thinning method

The idea of thinning is use a constant rate Poisson process to generate candidate jump time, then accept them with a probability based on current jump rate.

By the terminology of marked point process, the jump part can be written as

$$\begin{aligned}\sum_{i=1}^{N(t)} f(Y_i) &= \int_{z\in E^*} f(S(t-), z)\mu(\mathrm{d}z, \mathrm{d}t) \\ &= \int_0^1 \int_{z\in E^*} f(S(t-), z) 1_A(u)\mu^*(\mathrm{d}z, \mathrm{d}t)\mathrm{d}u,\end{aligned}$$

$$\text{where } A = [0, \frac{\lambda(S(t-), t)}{\lambda_{\sup}}], \;\; 0 < t < T$$

in which $\mu(\cdot,\cdot)$ is a random counting measure with intensity $\lambda(S(t-), t)$, and $\mu^*(\mathrm{d}z, \mathrm{d}t)$ represents the random measure with intensity $\lambda_{\sup}$.

Thinning Algorithm

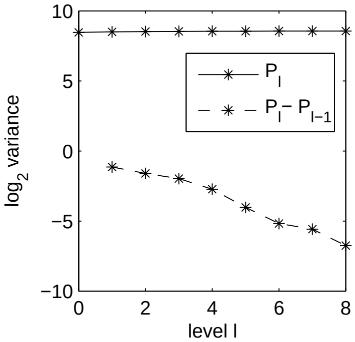
Hence we can have acceptance-rejection procedure:

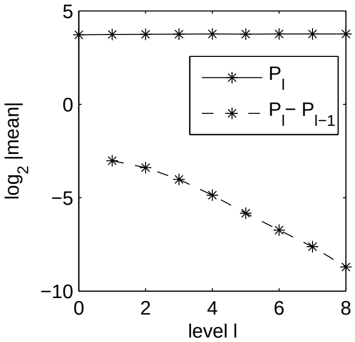
1. Generate the jump-adapted time grid using Poisson process with constant rate λ_{sup};
2. Simulate each interval of time grid using appropriate discretisation scheme;
3. When the endpoint τ is a jump time, generate a uniform random number $U \sim [0, 1]$
    1. If $p = \dfrac{\lambda(S(\tau-), \tau)}{\lambda_{\text{sup}}} > U$, accept τ add jump to the state value;
    2. Otherwise neglect the effect of jump.

Adopting multilevel

Thinning is easy to incorporate into Multilevel framework:
Numerical result for European call

$$\lambda = \frac{\lambda_0}{1+CS(t-)^2}, \ \lambda_0 = 2, \ C = 10^{-4}, \ S_0 = 100$$





Multilevel treatment

The variance convergence rate is approximately 1. What's wrong?

Jump candidates may not be accepted in both fine and coarse grids. Similar to the case of digital option, the proportion of those path where jumps differ is $O(h)$, giving an $O(1)$ value for $\widehat{P}_l - \widehat{P}_{l-1}$, which leads to $V_l = O(h)$.

To improve convergence rate: we change the probability density under which the expectation is taken to force the fine and coarse path to jump simultaneously.

Multilevel treatment

Write the estimator

$$\mathbb{E}_f[P_l] - \mathbb{E}_c[P_{l-1}] = \mathbb{E}_Q[P_l \prod_{\tau} R_f - P_{l-1} \prod_{\tau} R_c],$$

in the measure Q, the acceptance probability for a candidate jump is $\frac{1}{2}$. The Radon-Nikodym derivatives are

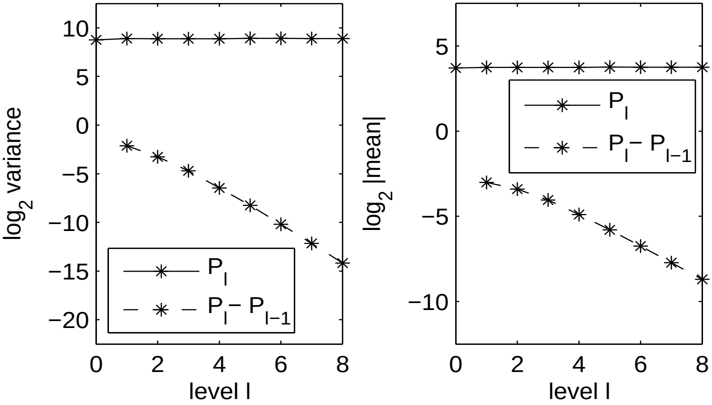
$$R_f = \begin{cases} 2p_f, & \text{if } U < \frac{1}{2}\,; \\ 2(1-p_f), & \text{if } U \geq \frac{1}{2}\,, \end{cases} \quad R_c = \begin{cases} 2p_c, & \text{if } U < \frac{1}{2}\,; \\ 2(1-p_c), & \text{if } U \geq \frac{1}{2}\,, \end{cases}$$

in which the probabilities are

$$p_f = \frac{\lambda^f(S(\tau-),\tau)}{\lambda_{\sup}}, \quad p_c = \frac{\lambda^c(S(\tau-),\tau)}{\lambda_{\sup}}.$$

Numerical result, change of measure

$$\lambda = \frac{\lambda_0}{1 + CS(t-)^2}, \;\; \lambda_0 = 2, \;\; C = 10^{-4} \;, S_0 = 100$$



Numerical result, change of measure

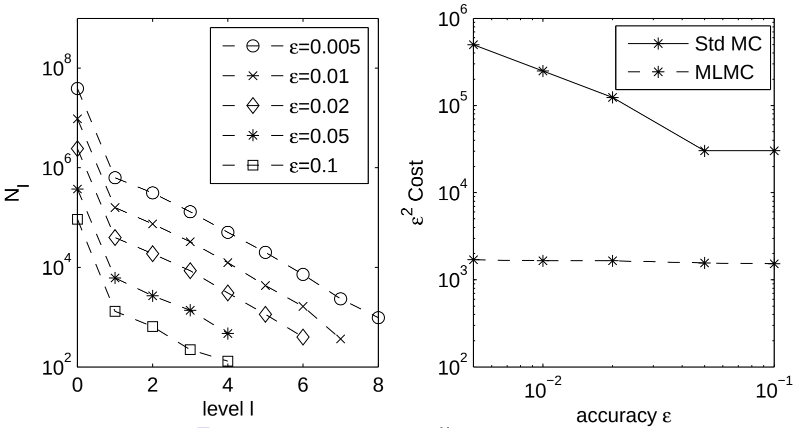


Figure: european vanilla option

Conclusion

- Multilevel approach can be extended to scalar jump-diffusion SDEs by using jump-adapted schemes.
- The variance convergence order is maintained in the constant rate case.
- Extra amendment is needed in the state-dependent rate case by thinning and change of measure.
- Furthermore we have successfully used the cumulative intensity method to cope with the case of unbounded intensity.
- Plan to apply to variance gamma and other processes.

Thanks beneficial discussion with Peter Tankov.

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