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A convergent hierarchy of non-linear eigenproblems to compute the joint spectral radius of nonnegative matrices

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Abstract— **We show that the joint spectral radius of a finite collection of nonnegative matrices can be bounded by the eigenvalue of a non-linear operator. This eigenvalue coincides with the ergodic constant of a risk-sensitive control problem, or of an entropy game, in which the state space consists of all switching sequences of a given length. We show that, by increasing this length, we arrive at a convergent approximation scheme to compute the joint spectral radius. The complexity of this method is exponential in the length of the switching sequences, but it is quite insensitive to the size of the matrices, allowing us to solve very large scale instances (several matrices in dimensions of order 1000 within a minute). An idea of this method is to replace a hierarchy of optimization problems, introduced by Ahmadi, Jungers, Parrilo and Roozbehani, by a hierarchy of nonlinear eigenproblems. We solve the latter problems by a power type iteration, avoiding the recourse to linear or semidefinite programming techniques, which allows for scalability. This is also related to maxplus-type curse of dimensionality attenuation schemes in dynamic programming.**

Index Terms— **Joint spectral radius, Nonlinear eigenproblem, nonnegative matrices, iterative method.**

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I. Introduction

A. Motivation

A fundamental issue, in optimal control, is to develop efficient numerical schemes that provide globally optimal solutions. Dynamic programming does provide a guaranteed global optimum but it is subject to the well known curse of dimensionality. Indeed, the main numerical methods, including monotone finite difference or semi-Lagrangean schemes [CL84], [CD83], [FF94], [CFF04], and the anti-diffusive schemes [BZ07], are grid-based. It follows that the time needed to obtain an approximate solution with a given accuracy is exponential in the dimension of the state space.

Recently, some innovative methods have been introduced in optimal control, which somehow attenuate the curse of dimensionality, for structured classes of problems.

McEneaney considered in [McE07] hybrid optimal control problems in which a discrete control allows one to switch between different linear quadratic models. The max-plus type method that he introduced approximates the value function by a supremum of quadratic forms. Its complexity, which is exponential in some parameters, has the remarkable feature of being polynomial in the dimension [MK10], [Qu14]. To produce approximations of the value function as concise as possible, the method makes an intensive use of semidefinite programming [GMQ11].

A different problem consists in computing the joint spectral radius of a finite set of matrices [Jun09]. This boils down to computing an ergodic value function, known as the Barabanov norm. Specific numerical methods have been developed, which approximate the Barabanov ball by a polytope [GZ14], or are of semi-Lagrangean type [Koz10]. Ahmadi, Jungers, Parrilo and Roozbehani [AJPR14a] developed a new method, based on a path complete automaton. It approximates the Barabanov norm by a supremum of quadratic norms. Whereas the worst case complexity estimates in [AJPR14a] are still subject to a curse of dimensionality, in practice, the efficiency of the method is determined by the complexity of the optimal switching law rather than by the dimension itself. This allows one to solve instances of dimension inaccessible by grid-based method.

In the max-plus method of McEneaney, and in the method of Ahmadi et al., solving large scale semidefinite programs appears to be the bottleneck, limiting the applicability range.

In our recent work [GS17], [Sto17], we introduced a new method to approximate the joint spectral radius. We replaced the solution of large scale SDP problems by the solution of eigenproblems involving non-linear operators, the "tropical Kraus maps". The latter are the analogues of completely positive maps, or of "quantum channels" acting on the space of positive semidefinite matrices, the operation of addition being now replaced by a multivalued supremum operation in the Löwner order. To solve these eigenproblems, we used iterative power type schemes, allowing us to deal with large scale instances (the algorithm of [GS17], [Sto17] could handle several matrices of order 500 in a few minutes). The convergence of these iterative schemes, however, is only guaranteed so far under restrictive assumptions, since the "tropical Kraus maps" are typically nonmonotone and expansive in some natural metrics.

B. Contribution

In this paper, we develop a non-linear fixed point approach to approximate the joint spectral radius in the special case of *nonnegative matrices*. We exploit a result of Guglielmi and Protasov [GP13], showing that for nonnegative matrices, it suffices to look for a *monotone norm*. We show here that such a monotone norm can be approximated by a finite

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supremum of linear forms, which are found as the solution of a non-linear eigenproblem. This is in contrast to earlier polyhedral approximation schemes, relying for instance on linear programming.

More precisely, we introduce a hierarchy of linear eigenproblems, parametrized by a certain "depth", inspired by [AJPR14a], [GS17], and we show that, as the depth tends to infinity, the non-linear eigenvalue does converge to the joint spectral radius. We remark that the initial ("depth 0") bound in our hierarchy coincides with the bound of the joint spectral radius introduced by Blondel and Nesterov [BN09].

The non-linear operator arising in our construction actually belongs to a known class: it can be identified to the dynamic programming operator of an ergodic risk sensitive control problems [AB15], or of a (one player) "entropy game" [ACD$^+$16], [AGGCG17]. This operator enjoys remarkable properties, like log-convexity, monotonicity, non-expansiveness with respect to Thompson's part metric or Hilbert's projective metric. As a result, computing the non-linear eigenvalue is a tractable problem. It is shown to be polynomial time in [AGGCG17]. It can also be solved in a scalable way by power-type schemes, like an adapted version of Krasnoselskii-Mann iteration, as we do here.

We report numerical results on large scale instances, up to dimension 5000, obtained by an OCaml implementation of the present algorithm.

In summary, the present contribution may be thought of as a "dequantization" of the non-linear fixed point approach of [GS17]. By "dequantization", we mean that we use here operators acting on the standard orthant, whereas the operator in [GS17] acts on the cone of positive semidefinite matrices. Whereas the approach of [GS17] is more general, leading to a convergent approximation scheme for any family of matrices, the present algorithm only applies to families of *nonnegative matrices*. However, it is experimentally faster, and it has stronger theoretical convergence guarantees. This suggests that the joint spectral radius problem is easier for nonnegative matrices.

C. Organization of the paper

In Section II, we recall some basic results on Barabanov norms of nonnegative matrices. In Section III, we introduce the family of non-linear eigenproblems to approximate the joint spectral radius. We show that these eigenproblems are solvable, under an appropriate irreducibility condition. In Section IV, we show that the non-linear eigenvalues in this hierarchy do converge to the joint spectral radius. The adapted version of Krasnoselskii-Mann iterative scheme is presented in Section V. Benchmarks are presented in Section VI.

The proofs will appear in an extended version of the present work.

II. The joint spectral radius of nonnegative matrices

The joint spectral radius $\rho(\mathcal{A})$ of a finite collection of $n\times n$ real matrices $\mathcal{A} = \{A_1, \dots, A_p\}$ is defined by

$$\rho(\mathcal{A}) \coloneqq \lim_{k\to\infty} \max_{1\leqslant i_1,\dots,i_k\leqslant p} \|A_{i_1}\cdots A_{i_k}\|^{1/k} .$$

When the set of matrices $\mathcal{A}$ is irreducible (meaning that there is no nontrivial subspace of $\mathbb{R}^n$ that is left invariant by all matrices), a fundamental result by Barabanov [Bar88] shows that there is a norm ν on $\mathbb{R}^n$ such that

$$\max_{1\leqslant i\leqslant p} \nu(A_i x) = \lambda\nu(x) \; , \; \forall x \in \mathbb{R}^n \; , \tag{1}$$

for some positive real number λ. The scalar λ is unique and coincides with the joint spectral radius $\rho(\mathcal{A})$.

A norm that satisfies Equation (1) is called an *invariant norm*. A norm that only satisfies the inequality $\max_{1\leqslant i\leqslant p} \nu(A_i x) \leqslant \lambda\nu(x)$ for all vector $x \in \mathbb{R}^n$ is called a λ-*extremal norm*. In that case, it is readily seen that $\lambda \geqslant \rho(\mathcal{A})$, so that λ-extremal norms provide safe upper bounds of the joint spectral radius.

We now assume that the matrices in $\mathcal{A}$ are nonnegative, i.e. their entries take nonnegative values. It is then readily seen that all matrices in $\mathcal{A}$ leave the (closed) cone of nonnegative vectors invariant. The latter cone, denoted by $\mathbb{R}^n_+$, induces an ordering on $\mathbb{R}^n$: we have $x \leqslant y$ if and only if $y - x$ is nonnegative. We note that a vector belongs to the interior of $\mathbb{R}^n_+$ if its entries are positive. Recall that the cone $\mathbb{R}^n_+$ is self-dual, so that $x \leqslant y$ if and only if $\langle u, y - x\rangle \geqslant 0$ for all $u \in \mathbb{R}^n_+$. This cone also induces a lattice structure on $\mathbb{R}^n$, meaning that the supremum of two vectors x, y always exists and is given coordinate-wise by $\big[\sup(x,y)\big]_i = \sup(x_i, y_i)$. A norm defined on $\mathbb{R}^n$ is called *monotone* if $0 \leqslant x \leqslant y$ implies $\nu(x) \leqslant \nu(y)$.

When the matrices in $\mathcal{A}$ are nonnegative, the irreducibility assumption on $\mathcal{A}$ can be weakened to *positive-irreducibility*, meaning that there is no non-trivial face of the cone of nonnegative vectors that is left invariant by all matrices in $\mathcal{A}$. A theorem by Guglielmi and Protasov [GP13] shows that, in this setting, the norm in Equation (1) can be chosen to be monotone.

Theorem 1 (Corollary 1 in [GP13]). *A positively-irreducible family of nonnegative matrices has a monotone invariant norm.*

We shall say that a map ν from $\mathbb{R}^n_+$ to $\mathbb{R}$ is a *monotone hemi-norm* if it is convex and positively homogegenous of degree 1, if $0 \leqslant x \leqslant y$ implies $\nu(x) \leqslant \nu(y)$, and if $\nu(x) = 0$ with $x \geqslant 0$ implies $x = 0$. The term hemi-norm is borrowed to [GV12], functions of this kind are also known as weak Minkowski norms in metric geometry [PT09]

Note that a monotone hemi-norm ν defined on $\mathbb{R}^n_+$ can be extended to a monotone norm on $\mathbb{R}^n$ by setting

$$\widehat{\nu}(x) \coloneqq \inf\{\nu(y) \vee \nu(z) \colon x = y - z \, , \text{with}\, y, z \geqslant 0\} . \tag{2}$$

The norm $\widehat{\nu}$ is a λ-extremal norm whenever ν is a monotone λ-extremal hemi-norm on $\mathbb{R}^n_+$, meaning that:

$$\max_{1\leqslant i\leqslant p} \nu(A_i x) \leqslant \lambda\nu(x)\,,\ \forall x \in \mathbb{R}^n_+\,.$$

In this way, it suffices to study monotone λ-extremal hemi-norms defined on $\mathbb{R}^n_+$.

III. A HIERARCHY OF NON-LINEAR EIGENPROBLEMS

A. Definition of the operators

In the sequel, we consider a finite set of $n \times n$ nonnegative matrices $\mathcal{A} = \{A_1, \dots, A_p\}$. We denote by $[\![p]\!] = \{1, \dots, p\}$.

The operator considered at the 0-level of the hierarchy is given by

$$T^0(x) \coloneqq \sup_{1\leqslant a\leqslant p} A_a^T x\,.$$

Higher levels of the hierarchy are built by introducing a memory process that keeps track of the past matrix products, up to a given depth. More precisely, given an integer d, the operator considered in the d-level is a self-map of the product cone $\prod_{s\in[\![p]\!]^d} \mathbb{R}^n_+$. It maps the vector $x = (x_s)_{s\in[\![p]\!]^d}$, where each $x_s \in \mathbb{R}^n_+$, to the vector $T^d(x)$, whose s-component is the vector of $\mathbb{R}^n_+$ given by

$$T^d_s(x) \coloneqq \sup_{r,a\colon\, \tau^d(r,a)=s} A_a^T x_r\,.$$

Here, the map $\tau^d\colon [\![p]\!]^d \times [\![p]\!] \to [\![p]\!]^d$ is the transition map of the *De Bruijn automaton* of length d on p symbols: given a word $i_1 \cdots i_d \in [\![p]\!]^d$, we have

$$\tau^d(i_1 \cdots i_d, a) = i_2 \cdots i_d a\,.$$

In other words, the transition forgets the initial symbol of a sequence, and concatenates the letter a representing the most recent switch, to this sequence.

The map T^d is monotone with respect to the cone $\prod_{s\in[\![p]\!]^d} \mathbb{R}^n_+$, i.e., $x \leqslant y \implies T^d(x) \leqslant T^d(y)$, and it is (positively) homogeneous, meaning that $T^d(\lambda x) = \lambda T^d(x)$ holds for all positive λ.

B. Some results of non-linear Perron-Frobenius theory

Monotone and homogeneous maps are studied in non-linear Perron-Frobenius theory. We recall some basic results here, referring the reader to [Nus88], [LN12] for background.

The *spectral radius* of a monotone and homogeneous map f defined on a cone $\mathcal{C}$, denoted by $r(f)$ is defined by:

$$r(f) \coloneqq \lim_{k\to+\infty} \|f^k(x)\|^{1/k}$$

for $x \in \operatorname{int}\mathcal{C}$. This value is independent of the choice of x and the norm $\|\cdot\|$, see [MPN02], [AGN11], [LN12].

We say that a monotone and homogeneous map $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is *positively irreducible* if it does not leave invariant a non-trivial face of $\mathbb{R}^n_+$. A basic result of non-linear Perron-Frobenius theory, which follows as a consequence of Brouwer theorem, shows that a positively irreducible map has an eigenvector in the interior of the cone. Then, the associated eigenvalue λ coincides with the spectral radius $r(f)$. The same conclusion holds, in fact, under less demanding assumptions [GG04], however, for the present class of operators, positive irreducibility will suffice.

Nussbaum proved that the classical variational characterization of the Perron root of a nonnegative matrix carries over to the non-linear setting. More precisely, the (non-linear) spectral radius coincides with the infimum of all "super-eigenvalues" associated with a vector in the interior of the cone, regardless of the positive-irreducibility of the map f:

Theorem 2 (Non-linear Collatz-Wielandt formula [Nus86])**.** *Given a continuous, monotone and homogeneous map f on the cone $\mathbb{R}^n_+$, we have*

$$r(f) = \inf\left\{\rho > 0\colon \exists u \in \operatorname{int}\mathbb{R}^n_+, f(u) \leqslant \rho u\right\}.$$

In particular, if the map f is not positively-irreducible, it may be the case that $f(u) = \lambda u$ holds for some nonzero vector u in the boundary of the cone $\mathbb{R}^n_+$. Then, we can only conclude that $\lambda \leqslant r(f)$. However, we do have $r(f) = \lambda$ if u belongs to the interior of $\mathbb{R}^n_+$.

C. Construction of the hierarchy

For every integer $d \geqslant 0$, the d-level of the hierarchy consists in solving the non-linear eigenproblem:

$$\begin{cases} T^d(u) = \lambda_d u \\ u \in \prod_{s\in[\![p]\!]^d} \mathbb{R}^n_+\,, u \neq 0 \end{cases} \tag{E_d}$$

The first main result shows that every problem (E_d) has a solution, and that a solution provides an upper bound on the joint spectral radius $\rho(\mathcal{A})$ and a corresponding monotone λ_d-extremal hemi-norm.

Theorem 3. *Suppose that the set of nonnegative matrices $\mathcal{A}$ is positively-irreducible. Then Problem (E_d) has a solution. Any such solution (λ_d, u) satisfies*

$$\rho(\mathcal{A}) \leqslant \lambda_d \leqslant r(T^d)\,.$$

Moreover, the map $\|x\|_u \coloneqq \max_s \langle u_s, x\rangle$ is a monotone λ_d-extremal hemi-norm:

$$\max_a \|A_a x\|_u \leqslant \lambda_d \|x\|_u\,, \forall x \in \mathbb{R}^n_+\,.$$

By the Perron-Frobenius theorem, the equality $\lambda_d = r(T^d)$ holds in Theorem 3 when the map T^d is positively-irreducible. The latter property can be decided by checking whether a lifted version of the set of matrices $\mathcal{A}$ is positively irreducible. In the following, the set $\{e_r\colon r \in [\![p]\!]^d\}$ denotes the canonical basis of the space $\mathbb{R}^{p^d}$ and $\otimes$ is the Kronecker product.

Proposition 1. *The map T^d is positively-irreducible if and only if the set of matrices $\{(e_r e_s^T) \otimes A_a\colon \tau^d(r,a) = s,\ r,s \in [\![p]\!]^d\,, a \in [\![p]\!]\}$ is positively irreducible.*

The term "hierarchy" for the sequence of problems (E_d) is justified by the following proposition.

Proposition 2. *Suppose that the set of nonnegative matrices $\mathcal{A}$ is positively-irreducible. Then $r(T^{d+1}) \leqslant r(T^d)$ for all d.*

IV. Convergence of the hierarchy of nonlinear eigenproblems

The next theorem shows that the spectral radius of the map T^d approximates the joint spectral radius $\rho(\mathcal{A})$ up to a factor $n^{1/(d+1)}$. The proof of this result is inspired by the ones found in [AJPR14b], [PEDJ16]. However, instead of using the *Löwner-John ellipsoid*, we rely on the fact that a monotone hemi-norm ν can be approximated by a monotone linear map, up to a factor n. More precisely, the vector c defined by $c_i = \nu(e_i)$ is positive and satisfies $\nu(x) \leqslant \langle c, x\rangle \leqslant n\nu(x)\,,\ \forall x \geqslant 0$.

Theorem 4. *Suppose that the set of nonnegative matrices $\mathcal{A}$ is positively-irreducible. Then*

$$r(T^d) \leqslant n^{1/(d+1)}\rho(\mathcal{A})\,.$$

We obtain as an immediate corollary of Theorems 3 and 4 that the hierarchy is convergent, in the sense that any sequence of eigenvalues of the map T^d converges towards the joint spectral radius.

Corollary 1. *Suppose that the set of nonnegative matrices $\mathcal{A}$ is positively-irreducible. If λ_d denotes an eigenvalue of the map T^d for all d, then*

$$\lim_{d\to\infty} \lambda_d = \rho(\mathcal{A})\,.$$

In particular, the sequence of spectral radii $r(T^d)$ is non-increasing and its limit is equal to $\rho(\mathcal{A})$.

V. Solving the non-linear eigenproblem

Several numerical methods allow one to solve the non-linear eigenproblem (E_d). First, the log-convexity property of T^d allows a reduction to convex programing, which entails a polynomial time bound (see for instance the part of [AGGCG17] concerning "Despot free" entropy games). There are also algorithms, more efficient in practice, that do not have polynomial time bounds. Protasov proposed a "spectral simplex" algorithm [Pro16]. A policy iteration scheme was proposed in [AGGCG17]. The spectral simplex, like policy iteration, involve at each step the computation of the spectral radius of a nonnegative matrix. For huge scale instances, it seems more convenient to employ the following iterative scheme, which is essentially a Krasnoselskii-Mann iteration [Man53], [Kra55] in the space of rays of the cone:

$$v^{k+1} = \Big[\frac{T^d(v^k)}{G\big[T^d(v^k)\big]} \circ v^k\Big]^{1/2}\,. \qquad (3)$$

where $\circ$ denotes the entrywise product of two vectors. and $G(x) = (x_1\cdots x_n)^{1/n}$ denotes the geometric means of the components of the vector x. By comparison with the original Krasnoselskii-Mann iteration, the arithmetic mean is replaced by the geometric mean, and a normalization is introduced to deal with the projective setting. We next show that this scheme does converge.

Theorem 5. *Assuming that the set of matrices $\{(e_r e_s^T)\otimes A_a\colon \tau(r,a) = s\}$ is positively-irreducible, the iteration in Equation* (3) *initialized at any positive vector $v^0 \in \prod_{s\in[\![p]\!]^d}\mathbb{R}^n_+$ converges towards an eigenvector of the map T^d.*

Remark 1. The same iteration also converges under the weaker assumption that $\mathcal{A}$ is positively-irreducible, but it must be initialized with a vector belonging to the interior of a minimal invariant face of the cone $\prod_{s\in[\![p]\!]^d}\mathbb{R}^n_+$.

Remark 2. We could also use the simpler power algorithm $x^{k+1} = T(x^k)/\|T(x^k)\|$. As it lacks the "damping" of Krasnoselskii-Mann iteration, the power iteration converges in less general circumstances. However, explicit conditions, with a geometric convergence rate, have been worked out. Indeed, it is shown in [AGN16, Theorem 7.8] that the asymptotic convergence rate of the power algorithm can be bounded by a certain spectral radius $r_H(T'_v)$ of the *semidifferential map* of T at an eigenvector v of T. More precisely, it is shown there that $\lim_{k\to\infty} d_H(x^k, v)^{1/k} \leqslant r_H(T'_v)$ (the spectral radius r_H is defined there with respect to Hilbert's projective metric d_H, it should not be confused with the spectral radius used in the rest of this paper). In particular, if T is differentiable at point v (a property which is expected to hold under some genericity assumptions), the semidifferential of T'_v can be identified to a nonnegative matrix P. If $r(P)$ denotes the Perron root of this matrix, and if λ denotes the second maximal modulus of an eigenvalue of P, then, we have the explicit bound $r_H(T'_v) = |\lambda|/r(P)$ for the convergence rate.

VI. Benchmarks

The present method has been implemented in OCaml and has been run on one core of an 2.2 GHz Intel Core i7 processor with 8 Gb of RAM. We report two numerical experiments, showing respectively the convergence of the scheme and the gain in scalability.

A. Convergence of the hierarchy

We illustrate the convergent nature of the hierarchy on the pair of matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 1 & 2\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

This pair has a spectrum maximizing product of length 6 given by A^2B^4 yielding a joint spectral radius equal to 2.0273. We report in Table I the eigenvalue obtained by solving the hierarchy (E_d) for $1 \leqslant d \leqslant 9$ as well as the computation time. We observe that the hierarchy is stationary at $d = 7$ and that we recover the exact value of the joint spectral radius. The last column indicates the relative error $\big[\lambda_d - \rho(\mathcal{A})\big]/\rho(\mathcal{A})$. Finally, we also observe the exponential cost in computation time at the level d of the hierarchy.

Level d	CPU Time (s)	Eigenvalue λ_d	Relative error
1	0.01	2.165	6.8%
2	0.01	2.102	3.7%
3	0.01	2.086	2.9%
4	0.01	2.059	1.6%
5	0.02	2.041	0.7%
6	0.05	2.030	0.1%
7	0.7	2.027	0.0%
8	0.32	2.027	0.0%
9	1.12	2.027	0.0%

TABLE I

CONVERGENCE OF THE HIERARCHY ON 5×5 MATRICES

Dimension n	Level d	Eigenvalue λ_d	CPU Time
10	2	4.287	0.01 s
	3	4.286	0.03 s
20	2	8.582	0.01 s
	3	8.576	0.03 s
50	2	22.34	0.04 s
	3	22.33	0.16 s
100	2	44.45	0.17 s
	3	44.45	0.53 s
200	2	89.77	0.71 s
	3	89.76	2.46 s
500	2	224.88	5.45 s
	3	224.88	19.7 s
1000	2	449.87	44.0 s
	3	449.87	2.7 min
2000	2	889.96	4.6 min
	3	889.96	19.2 min
5000	2	2249.69	51.9 min
	3	2249.57	2.9 h

TABLE II

COMPUTATION TIME FOR LARGE MATRICES

B. Scalability of the approach

We demonstrate the scalability of our method on quadruplets of matrices of increasing size, with random entries between 0 and 0.9. We show in Table II the computation time associated with each dimension. The iteration process converges in less than 50 iterations in all examples, with a 10^{-6} numerical stopping criterion. A monotone extremal hemi-norm has been computed as the supremum of 16 or 64 linear forms (respectively for $d = 2$ and $d = 3$).

VII. CONCLUSION

We have proposed a new approach for computing a convergent sequence of upper bounds of the joint spectral radius of nonnegative matrices, by solving a hierarchy of non-linear eigenproblems. At any level of this hierarchy, the non-linear eigenvalue λ provides an upper bound for the joint spectral radius, whereas the eigenvector encodes a monotone λ-extremal norm. The non-linear eigenproblem is solved efficiently by a variation of the Krasnoselskii-Mann iteration. We have implemented this approach and numerical results are witnesses of the scalability of this approach, compared to other works based on the solution of optimization problems.

We finally point out one open problem. Guglielmi and Protasov showed in [GP13] that when the joint spectral radius is obtained for a unique periodic product, and when this product has a unique dominant eigenvalue, then, there is a polyhedral invariant norm. Each level of the present hierarchy generates a dictionary of linear forms, whose supremum yields a polyhedral extremal norm. This dictionary becomes richer when the level of the hierarchy is increased. Hence, we may ask whether the hierarchy is exact, i.e., whether there exist a level d such that $r(T^d) = \rho(\mathcal{A})$, under the same assumption.

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