A NON-LINEAR HIERARCHY FOR DISCRETE EVENT DYNAMICAL SYSTEMS

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Abstract

Dynamical systems of monotone homogeneous functions appear in Markov decision theory, in discrete event systems and in Perron-Frobenius theory. We consider the case when these functions are given by finite algebraic expressions involving the operations max, min, convex hull, translations, and an infinite family of binary operations, of which max and min are limit cases. We set up a hierarchy of monotone homogeneous functions that reflects the complexity of their defining algebraic expressions. For two classes of this hierarchy, we show that the trajectories of the corresponding dynamical systems admit a linear growth rate (cycle time). The first class generalizes the min-max functions considered previously in the literature. The second class generalizes both max-plus linear maps and ordinary non-negative linear maps.

Keywords
Nonexpansive maps, Fixed points, Cycle Time.

1 Introduction

For all integers $n \geq 1$, we denote by $\leq$ the usual partial order on $\mathbb{R}^n$ ($x \leq y \iff x_i \leq y_i$, for all $i = 1, \dots, n$). For all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we set $\lambda + x \stackrel{\text{def}}{=} (\lambda + x_1, \dots, \lambda + x_n)$.

A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is *monotone* if

(M) $\quad \forall x, y \in \mathbb{R}^n, \qquad x \leq y \implies f(x) \leq f(y) \;;$

it is (additively) *homogeneous* if

(H) $\quad \forall \lambda \in \mathbb{R}, x \in \mathbb{R}^n, \qquad f(\lambda + x) = \lambda + f(x) \;.$

Monotone homogeneous functions $\mathbb{R}^n \to \mathbb{R}^p$ are called *topical functions* (see [14]).

We consider dynamics of the form:

$$(1) \qquad \begin{aligned} x(k) &= f(x(k-1)), \quad \forall k \geq 1, \\ x(0) &= \xi \in \mathbb{R}^n \;, \end{aligned}$$

where f is a topical function $\mathbb{R}^n \to \mathbb{R}^n$. The following questions are natural:

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Question 1. Does the limit

$$(2) \qquad \chi(f, \xi) \stackrel{\text{def}}{=} \lim_{k \to \infty} x(k)/k$$

exist ?

Question 2. Does f admit an (additive, non-linear) *eigenvector* $x \in \mathbb{R}^n$:

$$(3) \qquad f(x) = \lambda + x, \qquad \text{with} \quad \lambda \in \mathbb{R} \;?$$

If such a λ exists, we call it the *eigenvalue* of f associated with x. We say that $(\eta, v) \in (\mathbb{R}^n)^2$ is an *ultimately affine regime* of f if there exists an integer K such that

$$(4) \qquad \forall k \geq K, \quad f(v + k\eta) = v + (k+1)\eta \;.$$

Clearly, f has an eigenvector x with eigenvalue λ iff it has an ultimately affine regime of the form (η, x), with $\eta = (\lambda, \dots, \lambda)$. This leads us to ask, as a generalization of Question 2:

Question 3. Does f admit an ultimately affine regime?

In the sequel, we will write $\chi(f)$ instead of $\chi(f, \xi)$, since, for all topical functions f and for all $\xi, \xi' \in \mathbb{R}^n$,

$$(5) \qquad \chi(f, \xi) \text{ exists} \implies \chi(f, \xi') \text{ exists and } \chi(f, \xi') = \chi(f, \xi) \;.$$

This follows from a remarkable observation due to Crandall and Tartar [5]: if the homogeneity property (H) holds, the monotonicity property (M) becomes equivalent to f being *non-expansive*:

(N) $\quad \forall x, y \in \mathbb{R}^n, \qquad \|f(x) - f(y)\| \leq \|x - y\| \;,$

where $\| \; \|$ denotes the sup-norm ($\|x\| = \max_{1 \leq i \leq n} |x_i|$). Then, by (N), for all $\xi, \xi' \in \mathbb{R}^n$, $\|f^k(\xi)/k - f^k(\xi')/k\| \leq \|(\xi - \xi')/k\|$ which goes to 0 when k goes to ∞. This shows (5).

If f has an ultimately affine regime, $\chi(f)$ exists. Indeed,

$$(6) \qquad f \text{ has an ultimately affine regime } (\eta, v) \in \mathbb{R}^n \implies \chi(f) = \eta \;,$$

for (4) implies that $\chi(f) = \lim_{k \to \infty} f^k(v + K\eta)/k = \lim_{k \to \infty}(v + (K+k)\eta)/k = \eta$. A fortiori:

$$(7) \qquad f \text{ has an eigenvalue } \lambda \implies \chi(f) = (\lambda, \dots, \lambda) \;.$$

Property (6) implies that the vector η, which satisfies (4), if it exists, is unique. A fortiori, the eigenvalue λ, if it exists, is unique.

In Discrete Event Systems applications (see e.g. [1, 13]), $x_i(k)$ represents the time of k-th occurrence of a repetitive event of type i, say the time of the k-th production of a part of type i. Then, $\chi_i(f)$ represents the asymptotic mean time between two consecutive events of type i. For this reason, $\chi(f)$ is called *cycle time* vector. A remarkable well understood case is that of *timed event graphs* [1], whose dynamics involve linear maps over the max-plus semiring, which can be written as:

$$f_i(x) = \max_{1\leq j\leq n} (A_{ij} + x_j) \enspace , \tag{8}$$

where A is a $n\times n$-matrix with entries in $\mathbb{R}\cup\{-\infty\}$, with at least one finite entry per row (this condition is required for f to send $\mathbb{R}^n$ to $\mathbb{R}^n$). The max-plus spectral theory (see e.g. [1, 6, 9]) gives a complete answer to Questions 1 and 2. Dynamics of the form (1) also appear in dynamic programming and, up to a change of variables, in Perron-Frobenius theory and in some of its extensions (see [19]).

The paper is organized as follows. In section 2, we present a hierarchy of topical functions, which provides a common setting for the above applications. Then, we answer Question 1 and to some extent Questions 2 and 3, for two classes of this hierarchy. 1) The first class encompasses dynamic programming operators of sequential zero-sum two players stochastic games with finite state and action spaces, and in particular, the "min-max functions" studied in [20, 12, 3] which appear for instance in the time evaluation of digital circuits [2]. The effective answer that we give in section 3 solves in more generality the duality conjecture stated in [11]: the cycle time of a min-max function does exist and we can compute it efficiently. 2) The second class extends both the non-negative linear operators which appear in Perron-Frobenius theory, and the max-plus linear operators familiar in the discrete event system literature. The answer to Questions 1,2 for this class, which is given in section 4, can be regarded as a common generalization of the classical Perron-Frobenius theorem and of the max-plus spectral theorem.

Let us conclude this introduction with some bibliographical indications. Monotone homogeneous maps have appeared in the work of several authors, in particular in [5, 19, 14, 17]. A systematic study can be found in the monograph [19], where, in particular, Question 2 is addressed under different assumptions. Min-max functions were introduced in [12] following earlier work on special cases in [20]. A counter example showing that the cycle time of a topical function need not exist is given in [14]. The theorem of existence of ultimately affine regimes for maps in the $\mathcal{D}^*$ class (Theorem 15 below) generalizes the results given in [3, 7] for min-max functions. The proof of Theorem 15 can be seen as an extension to the case of sequential zero-sum two players stochastic games of the vanishing discount/policy iteration approach to average cost programming in stochastic control —see e.g. [22, Chap. 31,§7]. Some classes of the hierarchy introduced here where applied in [4] to the modelling and analysis of fluid timed Petri nets. The definition of the hierarchy and the notation were inspired by numerous discussions with Jean-Pierre Quadrat, on the special (max, $+$, $\mathbb{E}$) class.

2 A Hierarchy of Topical Functions

2.1 Definitions and first examples

For all $\epsilon\in\mathbb{R}\setminus\{0\}$, for all $a,b\in\mathbb{R}$, we set:

$$a\oplus_\epsilon b = \epsilon\log(e^{a/\epsilon}+e^{b/\epsilon}) \enspace .$$

This law is associative and commutative. The laws max and min are limits of $\oplus_\epsilon$:

$$\lim_{\epsilon\to 0^+} a\oplus_\epsilon b = \max(a,b),\qquad \lim_{\epsilon\to 0^-} a\oplus_\epsilon b = \min(a,b) \enspace .$$

In the sequel, we will use the notations max, min, $\oplus_\epsilon$ for functions $\mathbb{R}^n\to\mathbb{R}^n$, with a pointwise meaning, e.g. $(f\oplus_\epsilon g)_i(x)\stackrel{\text{def}}{=} f_i(x)\oplus_\epsilon g_i(x)$. The (easy) verification of the following proposition is left to the reader.

PROPOSITION 4. *Let $f,g:\mathbb{R}^n\to\mathbb{R}^n$ denote topical functions. Let $\epsilon\in\mathbb{R}\setminus\{0\}$, $\alpha\in\mathbb{R}$, with $0\leq\alpha\leq 1$, $c\in\mathbb{R}^n$. The following functions all are topical:*

$$\begin{aligned}&\max(f,g),\quad \min(f,g),\quad f\oplus_\epsilon g,\\ &\alpha f+(1-\alpha)g,\quad c+f,\quad f\circ g \enspace .\end{aligned}$$

These closure properties allow us to build inductively complex topical functions from simple ones. If $\mathcal{F}$ is a set of topical functions $\mathbb{R}^n\to\mathbb{R}^n$, we define (the probably surprising notation will be justified soon):

$$\begin{aligned}
(\max,\mathcal{F}) &= \{\max_{f\in F} f \mid F\subset\mathcal{F},\quad F \text{ finite}\} \enspace ,\\
(\min,\mathcal{F}) &= \{\min_{f\in F} f \mid F\subset\mathcal{F},\quad F \text{ finite}\} \enspace ,\\
(\oplus_\epsilon,\mathcal{F}) &= \{\bigoplus_{f\in F}{}_\epsilon f \mid F\subset\mathcal{F},\quad F \text{ finite}\} \enspace ,\\
(\mathbb{E},\mathcal{F}) &= \{\sum_{f\in F}\alpha_f f \mid \begin{array}{l} F\subset\mathcal{F},\quad F\text{ finite}\\ 0\leq\alpha_f,\ \sum_{f\in F}\alpha_f=1\end{array}\} \enspace ,\\
(+,\mathcal{F}) &= \{c+f \mid f\in\mathcal{F}, c\in\mathbb{R}^n\} \enspace .
\end{aligned}$$

We say that a topical function $\mathbb{R}^n\to\mathbb{R}^n$ is *simple* if there exists a map $\pi:\{1,\dots,n\}\to\{1,\dots,n\}$, possibly non-bijective, such that $f_i(x)=x_{\pi(i)}$. We denote by () (parentheses with empty content) the set of simple functions $\mathbb{R}^n\to\mathbb{R}^n$.

Starting from simple functions, and using the above inductive constructions, we define a hierarchy of topical functions, as follows. We first introduce the following sets of symbols:

$$\begin{aligned}
\mathcal{D} &= \{\max,\min,+,\mathbb{E}\} \enspace ,\\
\mathcal{H} &= \mathcal{D}\cup\{\oplus_\epsilon \mid \epsilon\in\mathbb{R}\setminus\{0\}\} \enspace ,\\
\mathcal{P} &= \mathcal{H}\setminus\{\mathbb{E}\} \enspace ,\\
\mathcal{M} &= \mathcal{H}\setminus\{\max,\min\} \enspace .
\end{aligned}$$

Then, we define inductively, for all $\mathcal{X} \subset \mathcal{H}$:

$$\begin{aligned}
\mathcal{X}^0 &= () \ , \\
\mathcal{X}^{k+1} &= \bigcup_{\top \in \mathcal{X}} (\top, \mathcal{X}^k), \qquad \forall k \geq 0 \ , \\
\mathcal{X}^* &= \bigcup_{k \geq 0} \mathcal{X}^k \ .
\end{aligned}$$

We define a *sign function* sgn : $\mathcal{H} \to \{1, -1, 0\}$, whose values are given by the following table:

max	$\oplus_\epsilon$ with $\epsilon > 0$	min	$\oplus_\epsilon$ with $\epsilon < 0$	$+$	$\mathbb{E}$
1	1	-1	-1	0	0

For all $\mathcal{X} \subset \mathcal{H}$, we set $\mathcal{X}_+ = \{\top \in \mathcal{X} \mid \operatorname{sgn}(\top) \geq 0\}$, $\mathcal{X}_- = \{\top \in \mathcal{X} \mid \operatorname{sgn}(\top) \leq 0\}$.

For instance, (max, ()) belongs to $\mathcal{D}^1_+ \cap \mathcal{P}^1_+$. If $A, B, C, \ldots, X$ are any operators in $\mathcal{H}$, we will write $(A, B, C, \ldots, X)$ instead of $(A, (B, (C, \ldots, ()))))$, to simplify the notation. E.g. (max, $+$) stands for (max, ($+$, ())). We next tabulate some elementary classes of the hierarchy.

Example 5. A function in ($+$) can be written as

$$f_i(x) = c_i + x_{\pi(i)} \ ,$$

where c is a vector of $\mathbb{R}^n$ and π a map $\{1, \ldots, n\} \to \{1, \ldots, n\}$. E.g., for $n = 2$, $f(x) = (x_2 + 4, x_2 + 6)$ belongs to ($+$).

Example 6. The max-plus linear map given by (8) above belongs to (max, $+$) $\subset \mathcal{D}^2_+$. Indeed, let $\Pi = \{\pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid A_{i\pi(i)} \neq -\infty\}$. For all $\pi \in \Pi$, we define $f^\pi : \mathbb{R}^n \to \mathbb{R}^n$, $f^\pi_i(x) = A_{i\pi(i)} + x_{\pi(i)}$. By construction, $f = \max_{\pi \in \Pi} f^\pi$, and by Example 5, $f^\pi \in (+)$, for all $\pi \in \Pi$. This shows that $f \in$ (max, $+$).

The last example justifies a posteriori the notation for the classes of the hierarchy: an element in the class "(max, $+$)" is exactly what is traditionally called a (max, $+$) (or max-plus) linear map.

Example 7. Let P denote a $n \times n$ row-stochastic matrix. The function $f(x) = Px$ belongs[1] to $(\mathbb{E}) \subset \mathcal{D}^1_+ \cap \mathcal{D}^1_-$.

Example 8. Let $f(x) = \max_{u \in U}(c^u + P^u x)$, where $\{c^u\}_{u \in U}$ is a finite family of vectors of $\mathbb{R}^n$, and $\{P^u\}_{u \in U}$ is a family of $n \times n$ row-stochastic matrices. By Example 7, for all $u \in U$, the function $x \mapsto c^u + P^u x$ belongs to $(+, \mathbb{E})$. Hence, f belongs to (max, $+$, $\mathbb{E}$). Such maps classically arise as the Bellman operators of stochastic control problems with average reward (see e.g. [22]).

Example 9. The function

$$\begin{aligned}
f_1(x_1, x_2, x_3) &= \min(\max(x_2 + 2, x_3 + 5), x_1) \\
f_2(x_1, x_2, x_3) &= \min(x_2 + 1, x_3 + 2) \\
f_3(x_1, x_2, x_3) &= \max(x_1 - 1, x_2 + 3)
\end{aligned}$$

belongs to (min, max, $+$) $\subset \mathcal{D}^3$. More generally, the class (min, max, $+$) corresponds exactly to the *min-max functions* studied in [12, 11, 3, 21].

[1] This result is a consequence of the following elementary row-stochastic analogue of Birkhoff's theorem for bi-stochastic matrices: a matrix is row-stochastic iff it is a finite convex combination of substitution matrices. The (easy) proof is detailed in [8].

2.2 First properties and other examples

We will say that a set $\mathcal{F}$ of functions $\mathbb{R}^n \to \mathbb{R}^n$ is *rectangular* if $\mathcal{F} = \pi_1(\mathcal{F}) \times \cdots \times \pi_n(\mathcal{F})$, where π_i denotes the canonical projection on the i-th coordinate. The following technical lemma will allow us to establish "canonical forms" for some classes of the hierarchy.

LEMMA 10. *For all sets $\mathcal{F}$ of functions $\mathbb{R}^n \to \mathbb{R}^n$, for all $\top \in \mathcal{H}$, there holds*

1. $(\top, \top, \mathcal{F}) = (\top, \mathcal{F})$*;*
2. $(+, \mathbb{E}, \mathcal{F}) = (\mathbb{E}, +, \mathcal{F})$*;*
3. $(+, \top, \mathcal{F}) \subset (\top, +, \mathcal{F})$.
4. *If $\mathcal{F}$ is rectangular,* $(\top, \min, \mathcal{F}) \subset (\min, \top, \mathcal{F})$.
5. *If $\mathcal{F}$ is rectangular and $\top \in \mathcal{D}$, then $(\top, \mathcal{F})$ is rectangular.*
6. *If $\mathcal{F}$ is rectangular and satisfies $\mathcal{F} = (+, \mathcal{F})$, then $(\oplus_\epsilon, \mathcal{F})$ is rectangular, for all $\epsilon \in \mathbb{R} \setminus \{0\}$.*

This lemma is proved in [8]. As an illustration of the use of Lemma 10, we show that, up to a change of variables, non-negative linear maps belong to $\mathcal{H}^*$.

Example 11. Let M denote a $n \times n$ matrix with nonnegative entries, and at least one non-zero entry per row. Consider the non-negative linear map: $\overline{f} : (\mathbb{R}^*_+)^n \to (\mathbb{R}^*_+)^n$, $y \mapsto My$, where $\mathbb{R}^*_+$ denotes the set of strictly positive reals. Let log denote the function: $(\mathbb{R}^*_+)^n \to \mathbb{R}^n$, $\log(y_1, \ldots, y_n) = (\log(y_1), \ldots, \log(y_n))$, and let exp denote the inverse function. Then, $f = \log \circ \overline{f} \circ \exp$ belongs to $(\oplus_1, +) \subset \mathcal{P}^2_+$. Indeed, we can write, for all $1 \leq i \leq n$:

$$f_i(x) = \bigoplus_{1 \leq j \leq n}{}_1 (\log M_{ij} + x_j) \ .$$

Hence, for all $1 \leq i \leq n$, the map $f^i = (f_i, \ldots, f_i)$, $\mathbb{R}^n \to \mathbb{R}^n$ clearly belongs to $(\oplus_1, +)$. Since by Lemma 10,6, $(\oplus_1, +)$ is rectangular, it follows that $f \in (\oplus_1, +)$.

Related examples of maps are given in [8]. The following result can be easily derived from Lemma 10.

PROPOSITION 12. *We have:*

$$\mathcal{D}^* = \mathcal{D}^4 = (\min, \max, +, \mathbb{E}) \ .$$

The following proposition allows us to write a function in $\mathcal{H}^*$ in a simple "canonical" form, in which all the min and max operators are gathered.

PROPOSITION 13. *We have:*

$$\mathcal{H}^* = (\min, \max, \mathcal{M}^*) \ .$$

Proof. This is a consequence of Lemma 10,4 and of its dual (with max instead of min). □

Since the functions in $\mathcal{H}^*$ are given by *finite* expressions involving operations in $\mathcal{H}$, it is not difficult to see that there are topical functions that do not belong to $\mathcal{H}^*$, even for $n = 2$.

2.3 Main results

We next answer Questions 1,2 and 3 for some subclasses of the hierarchy $\mathcal{H}^*$.

THEOREM 14. *Any function in* $\mathcal{P}_+^*$ *has a cycle time.*

This theorem can be seen as an extension of the Perron-Frobenius theorem and of the max-plus spectral theorem, which deal with functions in $(\oplus_1, +)$ and $(\max, +)$, respectively. The proof of Theorem 14 is sketched in section 4 below, where it is shown how we can compute $\chi(f)$ from a structural decomposition of f (Theorem 29).

THEOREM 15. *Any function in* $\mathcal{D}^*$ *has an ultimately affine regime.*

The proof of Theorem 15 is sketched in Section 3 below. We will just mention here some useful corollaries.

COROLLARY 16. *Any function* f *in* $\mathcal{D}^*$ *has a cycle time. Moreover,* $\chi(f) = \eta$*, for all ultimately affine regimes* $(\eta, v) \in (\mathbb{R}^n)^2$ *of* f.

Proof. This follows from (6). □

COROLLARY 17. *A function* f *in* $\mathcal{D}^*$ *has an eigenvector in* $\mathbb{R}^n$ *iff* $\chi(f) = (\lambda, \dots, \lambda)$*, for some* $\lambda \in \mathbb{R}$.

Proof. The implication $\Rightarrow$ follows from (7). Conversely, assume that $\chi(f) = (\lambda, \dots, \lambda)$. We know by Theorem 15 that (4) holds for some $(\eta, v) \in (\mathbb{R}^n)^2$. By Corollary 16, we have $\eta = \chi(f) = (\lambda, \dots, \lambda)$. Then, (4) means precisely that v is an eigenvector of f with associated eigenvalue λ. □

The importance of rectangular subsets stems from the following general observation (rectangularity is defined as in §2.2, with E instead of $\mathbb{R}$).

LEMMA 18. *If* $(E, \leq)$ *is a linearly ordered set, if* $\mathcal{F}$ *is a finite rectangular set of functions* $E^n \to E^n$*, for all* $x \in E^n$*, there exists* $g \in \mathcal{F}$ *such that*

$$\max_{f \in \mathcal{F}} f(x) = g(x) \ . \tag{9}$$

Proof. Since E is linearly ordered, and since $\mathcal{F}$ is finite, for all $i = 1, \dots, n$, there exists an $f^i \in \mathcal{F}$ such that $f_i^i(x) = \max_{f \in \mathcal{F}} f_i(x)$. Let g denote the function whose i-th coordinate is f^i, for all $i = 1, \dots, n$. Since $\mathcal{F}$ is rectangular, g belongs to $\mathcal{F}$. We have $\max_{f \in \mathcal{F}} f(x) = g(x)$. □

Let $\mathcal{F}$ denote a set of topical functions $\mathbb{R}^n \to \mathbb{R}^n$ that admit a cycle time. By an immediate monotonicity argument, we have:

$$\chi(\max_{f \in \mathcal{F}} f) \geq \max_{f \in \mathcal{F}} \chi(f) \ , \tag{10}$$

provided that the left hand side is well defined (we use the notation max for the least upper bound of a possibly infinite set). Due to (9), it is very natural to ask whether the equality holds in (10) when $\mathcal{F}$ is finite and rectangular. The answer is positive in $\mathcal{D}^*$, as shown by the following theorem which will be derived from an asymptotic expansion result in Section 3.

THEOREM 19. *For all finite rectangular subsets* $\mathcal{F}$ *of* $\mathcal{D}^*$*, the equality holds in* (10).

By symmetry, under the same assumptions, the dual equality $\chi(\min_{f \in \mathcal{F}} f) = \min_{f \in \mathcal{F}} \chi(f)$ also holds.

3 Computing Cycle Times using Germs of Laurent Series and Policy Iteration

In this section, we sketch the main ideas of the proofs of Theorem 15 and 19. The proofs are detailed in [8].

Let $\mathbb{L}$ be the set of Laurent series of the form

$$x(\alpha) = \frac{a_{-1}}{1-\alpha} + a_0 + a_1(1-\alpha) + \cdots \ , \qquad \text{with } a_{-1}, a_0, a_1 \ldots \in \mathbb{R} \ , \tag{11}$$

that converge in some interval $(\alpha_0, 1)$. The set $\mathbb{L}$ can be identified with the set of sequences of the form $(a_{-1}, a_0, ...)$ subject to the condition that (11) converges. A *germ* of function at $\alpha = 1^-$ is an equivalence class of a real function by the equivalence relation: $x \sim y \iff \exists \alpha_0 < 1, \forall \alpha \in (\alpha_0, 1),\ x(\alpha) = y(\alpha)$. Let M be the set of functions which have a convergent Laurent series expansion of the form (11) in some interval $(\alpha_0, 1)$. The set $\mathbb{L}$ can be identified with the set of germs of M (any two functions having the same germ have the same Laurent series). The usual order $\leq$ on functions clearly passes to germs, and hence to $\mathbb{L}$, where it can easily be seen to correspond to the lexicographic ordering on sequences. Hence, $(\mathbb{L}, \leq)$ is linearly ordered. We will extend the notation $\sim$ to vectors (entrywise), and we will also speak of germs for vector functions.

For all $0 \leq \alpha < 1$, we associate with a topical function $f : \mathbb{R}^n \to \mathbb{R}^n$ the unique $\xi_\alpha(f) \in \mathbb{R}^n$ such that

$$f(\alpha \xi_\alpha(f)) = \xi_\alpha(f) \ . \tag{12}$$

Due to (N), the map $x \mapsto f(\alpha x)$ is α-*contracting* for the sup-norm. Hence, the Banach Contraction Theorem [10, Th. 2.1] shows that the fixed-point $\xi_\alpha(f)$ is well defined. The key of the proof of Theorem 15 and Theorem 19 is the following asymptotic expansion result.

THEOREM 20. *For all* $f \in \mathcal{D}^*$*, the germ of* $\xi_\alpha(f)$ *belongs to* $\mathbb{L}^n$.

I.e., the entries of $\xi_\alpha(f)$ have expansions of the form (11), in some interval $(\alpha_0, 1)$. Theorem 20 is connected to Theorems 15 and 19 by the following proposition, which is proved in [8].

PROPOSITION 21. *Let* $f \in \mathcal{D}^*$*, with*

$$\xi_\alpha(f) = \frac{a_{-1}}{1-\alpha} + a_0 + o(1-\alpha) \ , \tag{13}$$

and $a_{-1}, a_0 \in \mathbb{R}^n$. Then, for $\alpha < 1$, α sufficiently close to 1,

$$f\left(\frac{a_{-1}}{1-\alpha} + a_0 - a_{-1}\right) = \frac{a_{-1}}{1-\alpha} + a_0 \quad . \tag{14}$$

This remarkable property is due to the *locally affine* character of maps in $\mathcal{D}^*$: this allows us to derive the algebraic identity (14) from the approximate result (13).

Using Theorem 20, we get that if $f \in \mathcal{D}^*$, then $\xi_\alpha(f)$ has an expansion of the form (13). Then, by specializing (14) to a large integer $k = (1-\alpha)^{-1}$, we get that (a_{-1}, a_0) is an ultimately affine regime of f, which shows Theorem 15. □

Using Corollary 16, we obtain immediately the following useful fact.

COROLLARY 22. *For all $f \in \mathcal{D}^*$, $\chi(f)$ coincides with the term a_{-1} in the asymptotic expansion of $\xi_\alpha(f)$ (see (13)).* □

We next derive Theorem 19 from Theorem 20. Let $\mathcal{F}$ be a finite rectangular subset of $\mathcal{D}^*$, and let $h = \max_{f\in\mathcal{F}} f$. We will prove that there exists $g \in \mathcal{F}$ such that $\chi(h) = \chi(g)$. By Theorem 20, $\xi_\alpha(h) = a_{-1}(1-\alpha)^{-1} + a_0 + \cdots \in \mathbb{L}^n$ (we denote by the same symbol the element $\xi_\alpha(f)$ and its germ). By[2] Lemma 18 (with $E = \mathbb{L}$), $h(\alpha\xi_\alpha(h)) = g(\alpha\xi_\alpha(h))$, for some $g \in \mathcal{F}$. Then, $\xi_\alpha(g) = \xi_\alpha(h)$. By Corollary 22, $\chi(h) = \chi(g) = a_{-1}$, which concludes the proof of Theorem 19. □

It remains to prove Theorem 20.

LEMMA 23. *If $f = c + Px \in (+, \mathbb{E})$, where $c \in \mathbb{R}^n$ and P is a $n \times n$ row-stochastic matrix, then $\xi_\alpha(f) = (I - \alpha P)^{-1}c$ has a Laurent series expansion in $\mathbb{L}^n$.*

This is a standard result on resolvents [16, Chap. 1,§3, pp-38,39]: due to Perron-Frobenius theorem, 1 is a semisimple eigenvalue of P, hence $(1-\alpha P)^{-1}$ has a pole of order one at $\alpha = 1$. One key of the proof of Theorem 20 is the following elementary version of the maximum principle.

LEMMA 24. *Let $f \in (+, \mathbb{E})$. If $x \in \mathbb{L}^n$ is such that $f(\alpha x) < x$, then $\xi_\alpha(f) < x$.*

Proof. Let $g : y \mapsto f(\alpha y)$. By Banach contraction theorem, $\lim_k g^k(x) = \xi_\alpha(f)$. Since g is monotone, we get $x > g(x) \geq g^2(x) \geq \lim_k g^k(x) = \xi_\alpha(f)$. □

LEMMA 25. *Let $f = \min_{u\in U} g^u \in (\min, +, \mathbb{E})$, with U finite, $(+, \mathbb{E}) \supset \{g^u\}_{u\in U}$ rectangular. Then, there exists $\overline{u} \in U$ such that $\xi_\alpha(f) = \xi_\alpha(g^{\overline{u}}) \in \mathbb{L}^n$.*

[2]We can apply Lemma 18 because $\mathcal{F} \subset \mathcal{D}^*$ can be identified to a set of maps $\mathbb{L}^n \to \mathbb{L}^n$. Indeed, usual laws on real functions pass to germs, and $\mathbb{L}$ is stable by: translation, product by a scalar, max, min. Since only these operations are involved in the syntactic definition of a map in $\mathcal{D}^*$, a map $f \in \mathcal{D}^*$ defines a map $\mathbb{L}^n \to \mathbb{L}^n$, that to the germ of a function $x \in M^n$, associates the germ of the function $f(x)$. We still denote by f this map.

Proof. We select $\overline{u}$ such that $\xi_\alpha(g^{\overline{u}})$ is minimal in $\{\xi_\alpha(g^u)\}_{u\in U}$. We set $\overline{\xi}_\alpha = \xi_\alpha(g^{\overline{u}})$, to simplify the notation. We have

$$f(\alpha\overline{\xi}_\alpha) \leq g^{\overline{u}}(\alpha\overline{\xi}_\alpha) = \overline{\xi}_\alpha \quad . \tag{15}$$

By Lemma 18 (applied to $E = \mathbb{L}$), there exists $u \in U$ such that $f(\alpha\overline{\xi}_\alpha) = g^u(\alpha\overline{\xi}_\alpha)$. If the inequality in (15) is strict, we have $g^u(\alpha\overline{\xi}_\alpha) < \overline{\xi}_\alpha$. By the Maximum Principle (Lemma 24), $\xi_\alpha(g^u) < \overline{\xi}_\alpha$, which contradicts the minimality of $\overline{\xi}_\alpha$. Thus $f(\alpha\overline{\xi}_\alpha) = \overline{\xi}_\alpha$, that is $\xi_\alpha(f) = \overline{\xi}_\alpha$. □

Theorem 20 is an immediate consequence of the following more precise lemma, whose proof is similar in essence to that of Lemma 25.

LEMMA 26. *Let $f = \min_{u\in U} g^u \in (\min, \max, +, \mathbb{E})$, with U finite and $(\max, +, \mathbb{E}) \supset \{g^u\}_{u\in U}$ rectangular. Then, there exists $\overline{u} \in U$ such that $\xi_\alpha(f) = \xi_\alpha(g^{\overline{u}}) \in \mathbb{L}^n$.*

Example 27. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{aligned} f_1(x_1, x_2) &= \min(3 + x_1, 5 + \tfrac{1}{2}(x_1 + x_2)) \\ f_2(x_1, x_2) &= \max(\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2, x_2) \quad . \end{aligned}$$

We can write $f = \min(g^1, g^2)$, $g^1 = \max(h^{11}, h^{12})$, $g^2 = \max(h^{21}, h^{22})$, where

$$h^{11} = \begin{pmatrix} 3 + x_1 \\ \frac{1}{3}x_1 + \frac{2}{3}x_2 \end{pmatrix}, \; h^{12} = \begin{pmatrix} 3 + x_1 \\ x_2 \end{pmatrix}$$

$$h^{21} = \begin{pmatrix} 5 + \frac{1}{2}(x_1 + x_2) \\ \frac{1}{3}x_1 + \frac{2}{3}x_2 \end{pmatrix}, \; h^{22} = \begin{pmatrix} 5 + \frac{1}{2}(x_1 + x_2) \\ x_2 \end{pmatrix} \quad .$$

Some elementary linear algebra yields

$$\xi_\alpha(h^{21}) = \begin{pmatrix} \frac{2}{1-\alpha} + \frac{18}{6-\alpha} \\ \frac{2}{1-\alpha} - \frac{12}{6-\alpha} \end{pmatrix} \quad .$$

We can check easily that $\xi_\alpha(h^{21}) = g^2(\alpha\xi_\alpha(h^{21})) = f(\alpha\xi_\alpha(h^{21}))$. Thus, $\xi_\alpha(h^{21}) = \xi_\alpha(g^2) = \xi_\alpha(f)$, and by Corollary 22, $\chi(h^{21}) = \chi(g^2) = \chi(f) = \binom{2}{2}$.

In the above example, the main difficulty is to "guess" that the maps g^2 and h^{21} are the ones which determine the cycle time of f and of g^2, respectively. As detailed in [3, 7] in the special case of maps in (min, max, +), this can be done systematically by using a policy improvement scheme. A similar algorithm allows us to compute efficiently the cycle time of maps in $\mathcal{D}^*$.

4 Structural Analysis of Topical Functions

We equip $\underline{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$ with the metric $d(x, y) = |e^x - e^y|$. Clearly, any map in $\mathcal{H}^*$ admits a continuous extension $\underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^n$. We will consider more generally continuous functions $\underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^n$, that satisfy property (H) and (M), with $x, y \in \underline{\mathbb{R}}^n$, $\lambda \in \underline{\mathbb{R}}$. We will still call topical the functions $\underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^n$ that satisfy these two properties.

For a continuous topical function $f : \underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^n$, we can easily prove via Brouwer's fixed point theorem that

$f(x) = \lambda + x$, for some $x \in \underline{\mathbb{R}}^n \setminus \{-\infty\}$, $\lambda \in \underline{\mathbb{R}}$ (we denote by $-\infty$ the vector whose components are all equal $-\infty$)[3]. But to show that $\chi(f) = \lambda$ using (7), we need a *finite* eigenvector $x \in \mathbb{R}^n$. We next introduce the natural class of *irreducible* topical functions, for which eigenvectors are automatically finite.

Let $e_i \in \underline{\mathbb{R}}^n$ denote the vector such that $(e_i)_i = 0$, and $(e_i)_j = -\infty$, if $i \neq j$. We say that $i \in \{1, \dots, n\}$ *has access* to $j \in \{1, \dots, n\}$ (and we write $i \to j$) if $f_i^k(e_j) > -\infty$, for some $k \geq 0$. By convention, f^0 is the identity map, hence $i \to i$, for all $i \in \{1, \dots, n\}$. The relation $\to$ is clearly transitive, hence the relation $\mathcal{R} : i\mathcal{R}j \iff (i \to j)$ and$(j \to i)$ is an equivalence relation. We will simply call *classes of* f the equivalence classes for $\mathcal{R}$. A map f is *irreducible* if it has a single class.

THEOREM 28. *An irreducible continuous topical function $\underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^n$ admits a finite eigenvector.*

Proof. We have just observed that $f(x) = \lambda + x$, for some $x \in \underline{\mathbb{R}}^n \setminus \{-\infty\}$, and $\lambda \in \underline{\mathbb{R}}$. Since x has at least one finite coordinate, x_j, we can write, for all $i \in \{1, \dots, n\}$, and for all k such that $f_i^k(e_j) > -\infty$, $k \times \lambda + x_i = f_i^k(x) \geq f_i^k(x_j + e_j) = x_j + f_i^k(e_j) > -\infty$. Hence, $x_i > -\infty$. □

If C is a class of f, we denote by $\iota^C : \underline{\mathbb{R}}^C \to \underline{\mathbb{R}}^n$ the canonical injection, such that $\iota_i^C(x) = x_i$ if $i \in C$, and $\iota_i^C(x) = -\infty$ otherwise. We denote by $\pi^C : \underline{\mathbb{R}}^n \to \underline{\mathbb{R}}^C$ the canonical projection, such that $\pi_i^C(x) = x_i$, for all $i \in C$. We set $f^C = \pi^C \circ f \circ \iota^C$. We say that the class C is *non-degenerate* if f^C is not an identically $-\infty$ scalar map. By Theorem 28, for all classes C, f^C has a finite eigenvector. If C is non-degenerate, the associated eigenvalue λ must be finite. It is unique, since $\lambda = \chi(f^C)$. We will set $\rho(f^C) = \lambda$. We say that $i \in \{1, \dots, n\}$ has access to the class C, and we write $i \to C$, if $i \to j$, for some $j \in C$.

THEOREM 29 (CYCLE TIME FORMULA). *The cycle time of a map $f \in \mathcal{P}_+^*$ can be computed by the following formula:*

$$\chi_i(f) = \max_{i \to C} \rho(f^C) \ , \tag{16}$$

where the max *is taken over the non-degenerate classes C to which i has access.*

The proof, which is essentially similar to the classical Perron-Frobenius case (when $f \in (\oplus_1, +)$), is detailed in [8].

Example 30. Let $\{A^u\}_{u \in U}$ denote a finite family of nonnegative matrices. Let $\overline{f}(y) = \max_{u \in U} A^u y$. The map $f = \log \circ \overline{f} \circ \exp$, which belongs to $(\max, \oplus_1, +) \subset \mathcal{P}_+^*$, is irreducible iff the matrix $\sum_{u \in U} A^u$ is irreducible in the sense of Perron-Frobenius theory.

[3] The proof consists in noting that, if $f(x) \neq -\infty$ for all $x \neq -\infty$, the map $\tilde{f}(x) = f(x) - \log(\sum_{1 \leq i \leq n} e^{f_i(x)})$ sends continuously $K = \{x \in \underline{\mathbb{R}}^n \mid \sum_{1 \leq i \leq n} e^{x_i} = 1\}$ to itself, and that K is compact and homeomorphic to a convex.

Example 31. Consider the map f such that $f_i(x) = n^{-1}(\sum_{1 \leq j \leq n} x_j)$, for all $1 \leq i \leq n$. We have $f(e_i) = -\infty$, thus, there are n degenerate classes $\{1\}, \dots, \{n\}$, and we cannot get any useful information from the partition of $\{1, \dots, n\}$ into classes. This raises the natural question of defining a structural accessibility notion that encompasses both maps in $\mathcal{P}_+^*$ and maps in $\mathcal{D}_+^*$. See [8].

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