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# PERSISTENCE OF BOLTZMANN ENTROPY IN FLUID MODELS

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**Abstract**

Higher order entropies are kinetic entropy estimators suggested by Enskog expansion of Boltzmann entropy. These quantities are quadratic in the density $\rho$, velocity $v$ and temperature $T$ renormalized derivatives. We investigate asymptotic expansions of higher order entropies for compressible flows in terms of the Knudsen $\epsilon_{\rm K}$ and Mach $\epsilon_{\rm M}$ numbers in the natural situation where the volume viscosity, the shear viscosity, and the thermal conductivity depend on temperature, essentially in the form $T^{\varkappa}$. Entropic inequalities are obtained when $\|\log\rho\|_{BMO}$, $\epsilon_{\rm M}\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{BMO}$, $\epsilon_{\rm K}\|h\partial_x\rho/\rho\|_{L^\infty}$, $\epsilon_{\rm K}\epsilon_{\rm M}\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\epsilon_{\rm K}\|h\partial_x T/T\|_{L^\infty}$, and $\epsilon_{\rm K}^2\|h^2\partial_x^2 T/T\|_{L^\infty}$ are small enough, where $h = 1/\rho T^{\frac{1}{2}-\varkappa}$ is a weight associated with the dependence on density and temperature of the mean free path. *AMS subject classifications* : 35K, 76N, 82A40.

## 1 Introduction

The notion of entropy has been shown to be of fundamental importance in fluid modeling from both a physical and mathematical point of view [3, 4, 5, 6, 7, 8, 10, 15, 16, 19, 23, 28]. We have introduced in previous work [11, 12, 13, 14] a notion of kinetic entropy estimators for fluid models, suggested by Enskog expansion of Boltzmann kinetic entropy. Conditional higher order entropic inequalities have been established in the situations of incompressible flows [11, 12, 13] as well as compressible flows [14] spanning the whole space.

In this paper, we investigate asymptotic expansions of higher order entropies for compressible fluids and study the corresponding conditional entropic inequalities when the Mach number $\epsilon_{\rm M}$ and the Knudsen number $\epsilon_{\rm K}$ are small. In contrast, although higher order entropies are suggested by Enskog expansion, only incompressible fluids [11, 12, 13] or compressible fluid equations with coefficients of order unity were considered in previous work [14].

In Section 2 we first summarize the mathematical and physical motivations for higher order entropies which are kinetic entropy estimators for fluid models. The corresponding balance equations may also be seen as a thermodynamic generalization of Bernstein equation to systems of partial differential equations associated with renormalized variables [12].

We introduce in Section 3 the natural rescaled variables and small parameters associated with fluid models, notably the Mach number $\epsilon_{\rm M}$ and the Knudsen number $\epsilon_{\rm K}$. Thanks to the rescaled variables, the governing equations and the higher order entropies are rewriten in terms of $\epsilon_{\rm M}$ and $\epsilon_{\rm K}$. We also introduce the molecular coordinates—associated with the particle collision time and the mean free path—which are such that all small parameters are eliminated from the corresponding system of partial differential equations. The volume viscosity, the shear viscosity and the thermal conductivity are assumed to depend on temperature as given by the kinetic theory, that is, essentially in the form of a power law of temperature $T^{\varkappa}$ with a common exponent $\varkappa$.

In Section 4 we summarize weighted inequalities in Sobolev and Lebesgue spaces [12, 14]. These inequalities are required for renormalized variables with powers of density or temperature as weights as well as for fluid models with temperature dependent thermal conductivity and viscosities. We further specify how the various inequalities are transformed by a change of scale in the coordinate system.

In Section 5 we investigate Boltzmann kinetic entropy estimators taking into account the natural small parameters of fluid models. We derive parameter dependent balance equations for higher order entropic correctors as well as for extra correctors associated with density which is a hyperbolic variable. We then study entropic estimates by combining the correctors balance equations with weighted inequalities. Entropic estimates are obtained when the quantity

$$\chi = \| \log \rho \|_{BMO} + \epsilon_{\mathrm{M}} \| v/\sqrt{T} \|_{L^\infty} + \| \log T \|_{BMO} + \epsilon_{\mathrm{K}} \| h\partial_x \rho/\rho \|_{L^\infty} \\ + \epsilon_{\mathrm{K}}\epsilon_{\mathrm{M}} \| h\partial_x v/\sqrt{T} \|_{L^\infty} + \epsilon_{\mathrm{K}} \| h\partial_x T/T \|_{L^\infty} + \epsilon_{\mathrm{K}}^2 \| h^2 \partial_x^2 T/T \|_{L^\infty}, \tag{1.1}$$

is small enough, where $h = 1/\rho T^{\frac{1}{2}-\varkappa}$ is a weight associated with the dependence on temperature and density of the mean free path.

Note that the quantity $\chi$ is small when the Mach number is small since we formally have $\chi = \mathcal{O}(\epsilon_{\mathrm{M}})$. Assuming that the Mach number is small is equivalent to the underlying assumption of a small Knudsen number since $\epsilon_{\mathrm{M}} = \mathrm{Re}\, \epsilon_{\mathrm{K}}$ where Re is the Reynolds number. In addition, $\chi$ is scaling invariant for the changes of scales naturally associated with the solutions of the compressible Navier-Stokes equations. Finally, kinetic entropy estimators are shown to be closely associated with Sobolev norms of the fluid entropy in molecular coordinates.

# 2 Higher order entropies

In this section we briefly motivate the introduction of higher order entropies by discussing Bernstein equations and inspecting Enskog expansion of Boltzmann kinetic entropy [11, 12, 14]

## 2.1 A thermodynamic interpretation of Bernstein equations

For parabolic or elliptic scalar equations, a priori gradient estimates can be obtained by using Bernstein method [1, 22]. Considering the heat equation

$$\partial_t u - \Delta u = 0,$$

and defining $|\partial^k u|^2 = \sum_{1\leq i_1,\cdots,i_k\leq n} (\partial_{i_1}\cdots\partial_{i_k} u)^2$, Bernstein equation for the $k^{\text{th}}$ derivatives can be written in the form

$$\partial_t |\partial^k u|^2 - \Delta |\partial^k u|^2 + 2|\partial^{k+1} u|^2 = 0. \tag{2.1}$$

The structure of (2.1) appears to be formally similar to that of an entropy balance, where $|\partial^k u|^2$, $k \geq 1$, play the rôle of generalized entropies, even though there also exist zeroth order entropies like $u^2$. In the next section, we introduce a kinetic framework supporting this entropic interpretation.

## 2.2 Enskog expansion of Boltzmann kinetic entropy

In a semi-quantum framework, the state of a polyatomic gas is described by a particle distribution function $f(t, x, c, \mathrm{I})$—governed by Boltzmann equation—where $t$ denotes time, $x$ the $n$-dimensional spatial coordinate, $c$ the particle velocity, $\mathrm{I}$ the index of the particle quantum state, and $\mathcal{I}$ is the corresponding indexing set [4, 6, 8, 10]. Approximate solutions of Boltzmann's equation can be obtained from a first order Enskog expansion $f = f^{(0)}\big(1 + \varepsilon\phi^{(1)} + \mathcal{O}(\varepsilon^2)\big)$ where $f^{(0)}$ is the local Maxwellian distribution, $\phi^{(1)}$ the perturbation associated with the Navier-Stokes regime and $\varepsilon$ the usual Enskog formal expansion parameter. The compressible Navier-Stokes equations can then be obtained upon taking moments of Boltzmann's equation [5, 8, 10].

The kinetic entropy $S^{\text{kin}} = -k_{\mathrm{B}} \sum_{\mathrm{I}\in\mathcal{I}} \int_{\mathbb{R}^n} f\big(\log f - 1\big) dc$, where $k_{\mathrm{B}}$ denotes Boltzmann constant, satisfies the $H$ theorem, that is, the second principle of thermodynamics. Enskog expansion $f/f^{(0)} = 1 + \varepsilon\phi^{(1)} + \cdots + \epsilon^{2k}\phi^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$ then induces expansions for $S^{\text{kin}}$ in the form

$$S^{\text{kin}} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots + \varepsilon^{2k} S^{(2k)} + \mathcal{O}(\varepsilon^{2k+1}), \tag{2.2}$$

where $S^{(0)}$ is the zeroth order fluid entropy evaluated from $f^{(0)}$ and where $S^{(j)}$ is a sum of terms in the form $k_{\mathrm{B}} \sum_{\mathrm{I}\in\mathcal{I}} \int_{\mathbb{R}^n} \prod_{1\leq i\leq j} \big(\phi^{(i)}\big)^{\nu_i} f^{(0)}\, dc$ with nonnegative integers $\nu_i \geq 0$, $1 \leq i \leq j$, such that $j = \sum_{1\leq i\leq j} i\nu_i$. For compressible polyatomic gases, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that

$$-\rho S^{(2)} = \frac{\lambda^2}{2r_{\mathrm{g}} c_p T} \frac{|\partial_x T|^2}{T^2} + \frac{3c_v \kappa^2}{4c_{int} T} \frac{(\partial_x \cdot v)^2}{r_{\mathrm{g}} T} + \frac{\eta^2}{4T} \frac{|d|^2}{r_{\mathrm{g}} T}, \tag{2.3}$$

where $T$ denotes the absolute temperature, $\rho$ the density, $v$ the gas velocity, $d = \partial_x v + \partial_x v^t - \frac{2}{n}(\partial_x \cdot v)\, I$ the deviatoric part of the strain rate tensor, $|d|^2$ the sum $|d|^2 = \sum_{ij} d_{ij}^2$, $c_p$ the constant pressure specific heat per unit mass, $c_v$ the constant volume specific heat per unit mass, $r_g$ the gas constant per unit mass, $c_{int}$ the internal specific heat per unit mass, $\lambda$ the thermal conductivity, $\eta$ the shear viscosity, $\kappa$ the volume viscosity, and the actual values of the numerical factors are evaluated here for $n = 3$.

From the general expression of $\phi^{(i)}$ in the absence of external forces acting on the particles, one can further establish that for any $j \geq 2$

$$S^{(j)} = \rho r_g \Big(\frac{\eta}{\rho\sqrt{r_g T}}\Big)^j \sum_{\nu} c_\nu \prod_{1\leq|\alpha|\leq j} \Big(\frac{\partial_x^\alpha \rho}{\rho}\Big)^{\nu_\alpha} \Big(\frac{\partial_x^\alpha v}{\sqrt{r_g T}}\Big)^{\nu'_\alpha} \Big(\frac{\partial_x^\alpha T}{T}\Big)^{\nu''_\alpha}, \tag{2.4}$$

where $\nu_\alpha, \nu'_\alpha, \nu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leq|\alpha|\leq j}$ must be such that $\sum_{1\leq|\alpha|\leq j} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = j$ and where the coefficients $c_\nu$ are smooth scalar functions of $\log T$ of order unity. In the even case $j = 2k$, after integrations by parts in $\int_{\mathbb{R}^n} S^{(2k)}\, dx$, in order to eliminate spatial derivatives of order strictly greater than $k$, and by using interpolation inequalities, one obtains that $|\int_{\mathbb{R}^n} S^{(2k)}\, dx|$ is essentially controled by the integral of

$$\gamma^{[k]} = \rho r_g \Big(\frac{\eta}{\rho\sqrt{r_g T}}\Big)^{2k} \Big(\big|\frac{\partial_x^k \rho}{\rho}\big|^2 + \big|\frac{\partial_x^k v}{\sqrt{r_g T}}\big|^2 + \frac{c_v}{r_g}\big|\frac{\partial_x^k T}{T}\big|^2\Big), \tag{2.5}$$

or equivalently of

$$\widetilde{\gamma}^{[k]} = \rho r_g \Big(\frac{\eta}{\rho\sqrt{r_g T}}\Big)^{2k} \Big(|\partial_x^k \log\rho|^2 + |\partial_x^k (v/\sqrt{r_g T})|^2 + \frac{c_v}{r_g}|\partial_x^k \log T|^2\Big), \tag{2.6}$$

and in the odd case $j = 2k-1$ $|\int_{\mathbb{R}^n} S^{(2k-1)}\, dx|$ is also controled by $\int_{\mathbb{R}^n} \gamma^{[k]}\, dx$ and $\int_{\mathbb{R}^n} \gamma^{[k-1]}\, dx$. This suggests quantities in the form $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as $(2k)^{\rm th}$ order kinetic entropy correctors—or kinetic entropy deviation estimators [11, 12, 14]. We are thus investigating *majorizing entropic correctors* that we are free to modify for convenience, e.g., by multiplying the temperature derivatives by the factor $c_v/r_g$. These correctors may also be rescaled by mutiplicative constants depending on $k$ and their temperature dependence may be simplified in accordance with that of transport coefficients. Finally, a similar analysis can also be conducted for the Fisher information and suggests the same quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as higher order kinetic information correctors.

### 2.3 Persistence of Boltzmann entropy

Denoting by $\gamma^{[0]}$ a nonnegative quantity associated with the zeroth order entropy $S^{(0)}$, we will investigate entropicity properties of the kinetic entropy estimators $\gamma^{[0]} + \cdots + \gamma^{[k]}$, $0 \leq k \leq l$, for the solutions of a second order system of partial differential equations modeling a compressible fluid. For this fluid system, the zeroth order entropy $S^{(0)}$ is already of fundamental importance as imposed by its hyperbolic-parabolic structure and the corresponding symmetrizing properties [10, 15, 19, 20]. We thus only consider the quantities $\gamma^{[0]} + \cdots + \gamma^{[k]}$, $0 \leq k \leq l$, as a family of mathematical entropy estimators—of kinetic origin—and we will establish that they indeed satisfy conditional entropic principles for solutions of Navier-Stokes type equations, so that, in some sense, there is a persistence of Boltzmann entropy at the fluid level.

This point of view differs from that of thermodynamic theories that have already considered entropies differing from that of zeroth order, that is, entropies depending on transport fluxes or macroscopic variable gradients. These generalized entropies have been associated notably with Burnett type equations or extended thermodynamics. In both situations, new macroscopic equations are correspondingly obtained, that is, 'extended fluid models', which are systems of partial differential equations of higher orders than Navier-Stokes type equations.

## 3 Nondimensionalization

We introduce in this section the rescaled fluid variables, the rescaled fluid equations, and the natural small parameters needed to investigate asymptotic expansions of higher order entropies. We only consider compressible flows spanning the whole space that are 'constant at infinity'.

## 3.1 Rescaled variables

In order to investigate asymptotic expansions of higher order entropies, we need to specify the order of magnitude of the various terms appearing in fluid governing equations. To this purpose, for each quantity $\phi$, we introduce a typical order of magnitude denoted by $<\phi>$. In particular, we introduce a characteristic length $<x>$, velocity $<v>$, density $<\rho>$, viscosity $<\eta>$, and pressure $<p>$. The order of magnitude of the sound velocity $c$ is then $<c>^2 = <p>/<\rho>$ and from the state law we also have $<c>^2 = <r_{\rm g}><T>$ where $r_{\rm g}$ denotes the gas constant per unit mass and $T$ the absolute temperature, and the Reynolds number is given by

$$\mathrm{Re} = \frac{<\rho><v><x>}{<\eta>}. \tag{3.1}$$

An important aerodynamic length upon consideration is the dissipation length $<x>^{\rm dis}$ defined such that the corresponding Reynolds number is unity $<x>^{\rm dis} = <\eta>/<\rho><v>$ and we can then write $\mathrm{Re} = <x>/<x>^{\rm dis}$. We define the characteristic time from the characteristic length $<x>$ and the characteristic velocity $<v>$ by letting $<t> = <x>/<v>$.

We also introduce a typical mean free path $<l>$ and from the kinetic theory of gases we have $<\eta> = <\rho><c><l>$ [5, 8]. Denoting by $\epsilon_{\rm K}$ the Knudsen number $<l>/<x>$ and $\epsilon_{\rm M}$ the Mach number $<v>/<c>$ we then have the Von Karman relation

$$\epsilon_{\rm M} = \frac{<v>}{<c>} = \frac{<l>}{<x>}\mathrm{Re} = \epsilon_{\rm K}\mathrm{Re}, \tag{3.2}$$

which relates $\epsilon_{\rm K}$ and $\epsilon_{\rm M}$. We will assume in the following that the Knudsen number $\epsilon_{\rm K}$ is small and the Mach number $\epsilon_{\rm M}$ will also be small, since we are especially interested in flows where the characteristic length $<x>$ is the dissipative length $<x>^{\rm dis}$ and the Reynolds number is then unity.

Upon defining the reduced quantity $\widehat{\phi} = \phi/<\phi>$ associated with each quantity $\phi$ of the fluid model, we can now estimate the order of magnitude of each term in the governing partial differential equations and in the definition of higher order entropies. We will assume, for the sake of simplicity, that we have $\widehat{r_{\rm g}} = 1$ so that $\widehat{c}^2 = \widehat{T}$ in particular.

## 3.2 Rescaled governing equations

The equations governing compressible flows can be written in the form [10, 23]

$$\partial_t\rho + \partial_x\cdot(\rho v) = 0, \tag{3.3}$$

$$\partial_t(\rho v) + \partial_x\cdot(\rho v\otimes v) + \partial_x p + \partial_x\cdot \Pi = 0, \tag{3.4}$$

$$\partial_t(\rho e) + \partial_x\cdot(\rho e v) + \partial_x\cdot \mathcal{Q} = -\Pi : \partial_x v - p\,\partial_x\cdot v, \tag{3.5}$$

where $t$ is time, $x$ the $n$-dimensional spatial coordinate, $\rho$ the mass density, $v$ the velocity vector, $p$ the pressure, $\Pi$ the viscous tensor, $e$ the internal energy per unit mass, and $\mathcal{Q}$ the heat flux. The viscous tensor and the heat flux are given by

$$\Pi = -\kappa\,\partial_x\cdot v I - \eta\,(\partial_x v + \partial_x v^t - \tfrac{2}{n}\partial_x\cdot v I), \tag{3.6}$$

$$\mathcal{Q} = -\lambda\,\partial_x T, \tag{3.7}$$

where $\kappa$ is the volume viscosity, $I$ the unit tensor, $\eta$ the shear viscosity, $\lambda$ the thermal conductivity, and we denote by $d = \partial_x v + \partial_x v^t - \frac{2}{n}\partial_x\cdot v I$ the deviatoric part of the strain rate tensor. For the sake of simplicity the internal energy per unit mass $e$ is taken in the form $e = c_v T$ where $c_v$ is a constant.

Upon using the general notation of Section 3.1 the reduced equations can then be written

$$\partial_{\widehat{t}}\widehat{\rho} + \partial_{\widehat{x}}\cdot(\widehat{\rho}\,\widehat{v}) = 0, \tag{3.8}$$

$$\partial_{\widehat{t}}(\widehat{\rho}\,\widehat{v}) + \partial_{\widehat{x}}\cdot(\widehat{\rho}\,\widehat{v}\otimes\widehat{v}) + \tfrac{1}{\epsilon_{\rm M}^2}\partial_{\widehat{x}}\,\widehat{p} + \tfrac{1}{\mathrm{Re}}\partial_{\widehat{x}}\cdot\widehat{\Pi} = 0, \tag{3.9}$$

$$\partial_{\widehat{t}}(\widehat{\rho}\,\widehat{e}) + \partial_{\widehat{x}}\cdot(\widehat{\rho}\,\widehat{v}\,\widehat{e}) + \tfrac{1}{\mathrm{Re}}\partial_{\widehat{x}}\cdot\widehat{\mathcal{Q}} = -\tfrac{\epsilon_{\rm M}^2}{\mathrm{Re}}\widehat{\Pi} : \partial_{\widehat{x}}\widehat{v} - \widehat{p}\,\partial_{\widehat{x}}\cdot\widehat{v}. \tag{3.10}$$

Note that, in contrast with the rescaled system (3.8)–(3.10), the compressible flow model previously considered in [14] did not contain any small parameter. The asymptotic analysis of higher order entropies for small $\epsilon_{\rm K}$ and $\epsilon_{\rm M}$ numbers has only been investigated in the simpler situation of incompressibles flows [13].

**Remark 3.1** *Theoretical calculations and experimental measurements have shown that the viscosity ratio* $\kappa/\eta$ *is of order unity for polyatomic gases [2, 5, 8]. Volume viscosity also arise in dense gases and in liquids so that its absence in monatomic dilute gases is an exception rather than a rule [2, 8].*

**Remark 3.2** *The dimension* $n$ *appearing in the coefficient* $2/n$ *of the viscous tensor is normally the full spatial dimension, that is, the dimension* $n'$ *of the velocity phase space of the associated kinetic model. We may still assume that the spatial dimension of the model has been reduced, that is, the equations are considered in* $\mathbb{R}^n$ *with* $n < n'$*. The full size viscous tensor* $\Pi'$ *is then a matrix of order* $n'$*, and the corresponding coefficient must be* $2/n'$*. However, if we denote by* $\Pi$ *the upper left block of size* $n$ *of* $\Pi'$*, that is, the useful part of* $\Pi'$*, we may rewrite* $\Pi$ *in the form*

$$\Pi = -\big(\kappa + (\tfrac{2}{n} - \tfrac{2}{n'})\eta\big)\, \partial_x\!\cdot v\, I - \eta\big(\partial_x v + \partial_x v^t - \tfrac{2}{n}\partial_x\!\cdot v\, I\big), \tag{3.11}$$

*where* $I$ *is the unit tensor in* $n$ *dimensions. Therefore, using a smaller dimension* $n$ *instead of the full dimension* $n'$ *in the coefficient of the viscous tensor is equivalent to increasing the volume viscosity by the amount* $2\eta\,(n'-n)/nn'$*.*

## 3.3 Temperature dependence of transport coefficients

Previous analyses have shown that it is necessary to take into account the temperature dependence of transport coefficients [11, 12, 14]. The kinetic theory of polyatomic gases lead to the following simplified assumptions concerning the temperature dependence of the thermal conductivity $\lambda(T)$, the volume viscosity $\kappa(T)$, and the shear viscosity $\eta(T)$, away from small temperatures. We assume that $\lambda$, $\kappa$, and $\eta$ are $C^\infty(0,\infty)$ and such that there exist $\varkappa$, $\underline{\mathfrak{a}} > 0$, $\overline{\mathfrak{a}} > 0$, and $\overline{\mathfrak{a}}_\sigma > 0$ for any $\sigma \geq 1$, with

$$\underline{\mathfrak{a}}\, T^\varkappa \leq \lambda/c_v \leq \overline{\mathfrak{a}}\, T^\varkappa, \qquad \underline{\mathfrak{a}}\, T^\varkappa \leq \kappa \leq \overline{\mathfrak{a}}\, T^\varkappa, \qquad \underline{\mathfrak{a}}\, T^\varkappa \leq \eta \leq \overline{\mathfrak{a}}\, T^\varkappa, \tag{3.12}$$

$$T^\sigma\left(|\partial_T^\sigma \lambda| + |\partial_T^\sigma \kappa| + |\partial_T^\sigma \eta|\right) \leq \overline{\mathfrak{a}}_\sigma\, T^\varkappa. \tag{3.13}$$

Kinetic theory suggests that $1/2 \leq \varkappa \leq 1$ but the situations where $0 \leq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate from a mathematical point of view.

## 3.4 Rescaled higher order entropies

Letting $<\gamma^{[k]}> = <\rho r> = <\rho><r>$ we deduce after some algebra that the rescaled higher order entropy correctors $\widehat{\gamma}^{[k]} = \gamma^{[k]}/<\rho><r>$ are given by

$$\widehat{\gamma}^{[k]} = \epsilon_{\rm K}^{2k}\, \widehat{\rho}\left(\frac{1}{\widehat{\rho}\,\widehat{T}^{\frac{1}{2}-\varkappa}}\right)^{2k} \left\{ \Big|\frac{\partial_{\widehat{x}}^k \widehat{\rho}}{\widehat{\rho}}\Big|^2 + \epsilon_{\rm M}^2 \Big|\frac{\partial_{\widehat{x}}^k \widehat{v}}{\sqrt{\widehat{T}}}\Big|^2 + c_v \Big|\frac{\partial_{\widehat{x}}^k \widehat{T}}{\widehat{T}}\Big|^2 \right\}, \tag{3.14}$$

and we have recovered that the $(2k)^{\rm th}$ order entropy correctors are of order $\mathcal{O}(\epsilon_{\rm K}^{2k})$ as was expected from their formal construction in Section 2.

The mathematical fluid entropy $-S^{(0)}$ can be shown to be a strictly convex function of the conservative variables $U = (\rho, \rho v, E^{\rm tot})$ where the total energy per unit volume is $E^{\rm tot} = \rho(e + \frac{1}{2} v\cdot v)$ [10, 20]. We define $\gamma^{[0]} = \mathsf{C}_0 \psi^{[0]}$ where $\psi^{[0]}$ is the modified zeroth order entropy [18, 19]

$$\psi^{[0]} = -S^{(0)} + S^{(0)}_\infty + (\partial_\rho S^{(0)})_\infty(\rho - \rho_\infty) + (\partial_{E^{\rm tot}} S^{(0)})_\infty (E^{\rm tot} - E^{\rm tot}_\infty),$$

and $\mathsf{C}_0$ a positive constant to be determined later. The rescaled zeroth order term $\gamma^{[0]}$ is easily rewritten in the form

$$\widehat{\gamma}^{[0]}/\mathsf{C}_0 = \widehat{\rho}\log\big(\frac{\widehat{\rho}}{\widehat{\rho}_\infty}\big) - \big(\widehat{\rho} - \widehat{\rho}_\infty\big) + \epsilon_{\rm M}^2 \tfrac{1}{2}\widehat{\rho}\frac{\widehat{v}^2}{\widehat{T}_\infty} + \widehat{\rho}\widehat{c}_v\Big(\frac{\widehat{T} - \widehat{T}_\infty}{\widehat{T}_\infty} - \log\big(\frac{\widehat{T}}{\widehat{T}_\infty}\big)\Big). \tag{3.15}$$

## 3.5 Molecular coordinates

The proper framework required to investigate asymptotic expansions of higher order entropies involve the rescaled coordinates $(\widehat{t}, \widehat{x})$, the rescaled unknowns $(\widehat{\rho}, \widehat{v}, \widehat{T})$, and the rescaled governing equations (3.8)–(3.10) depending on the small parameters $\epsilon_{\rm K}$ and $\epsilon_{\rm M}$. In this framework, parameter dependent a priori estimates and entropic inequalities can directly be obtained from the rescaled fluid governing equations.

On the other hand, it is also possible to completely eliminate the parameters $\epsilon_{\rm K}$ and $\epsilon_{\rm M}$ from the governing equations and from higher order entropy expansions by using a set of coordinates associated with the molecular properties of the fluid. More specifically, let us introduce the new coordinates

$$\bar{x} = \frac{\widehat{x}}{\epsilon_{\rm K}}, \qquad \bar{t} = \frac{\widehat{t}}{\epsilon_{\rm K}\epsilon_{\rm M}}, \tag{3.16}$$

associated with the characteristic length $\epsilon_{\rm K}{<}x{>} = {<}l{>}$ and the characteristic time $\epsilon_{\rm K}\epsilon_{\rm M}{<}t{>} = {<}l{>}/{<}c{>}$ so that $\bar{x}$ is measured in units of the mean free path ${<}l{>}$ and $\bar{t}$ in units of the particle collision time ${<}l{>}/{<}c{>}$. We then have $\partial_{\bar{x}}^{\alpha} = \epsilon_{\rm K}^{|\alpha|}\partial_x^{\alpha}$ and $\partial_{\bar{t}}^{\sigma} = (\epsilon_{\rm K}\epsilon_{\rm M})^{\sigma}\partial_t^{\sigma}$ for any multiindex $\alpha \in \mathbb{N}^n$ and any $\sigma \in \mathbb{N}$. By using these molecular coordinates $(\bar{t}, \bar{x})$, and upon defining the velocity $\bar{v}$ by

$$\bar{v} = \epsilon_{\rm M}\, \widehat{v} = \epsilon_{\rm M}\, \frac{v}{{<}v{>}} = \frac{v}{{<}c{>}}, \tag{3.17}$$

we obtain after a little algebra the governing equations

$$\partial_{\bar{t}}\rho + \partial_{\bar{x}}\cdot(\rho\bar{v}) = 0, \tag{3.18}$$

$$\partial_{\bar{t}}(\rho\bar{v}) + \partial_{\bar{x}}\cdot(\rho\bar{v}\otimes\bar{v}) + \partial_{\bar{x}}\, p + \tfrac{1}{\rm Re}\partial_{\bar{x}}\cdot \overline{\mathit{\Pi}} = 0, \tag{3.19}$$

$$\partial_{\bar{t}}(\rho e) + \partial_{\bar{x}}\cdot(\rho\bar{v}e) + \tfrac{1}{\rm Re}\partial_{\bar{x}}\cdot \overline{\mathcal{Q}} = -\tfrac{1}{\rm Re}\overline{\mathit{\Pi}}{:}\partial_{\bar{x}}\bar{v} - p\partial_{\bar{x}}\cdot\bar{v}, \tag{3.20}$$

where $\overline{\mathit{\Pi}} = -\kappa\partial_{\bar{x}}\cdot\bar{v}I - \eta\bar{d}$, $\bar{d} = \partial_{\bar{x}}\bar{v} + \partial_{\bar{x}}\bar{v}^t - \frac{2}{n}\partial_{\bar{x}}\cdot\bar{v}I$ and $\overline{\mathcal{Q}} = -\lambda\partial_{\bar{x}}T$. In these equations, with a slight abuse of notation, we have denoted by the same letter the corresponding functions expressed in physical $x$ or molecular $\bar{x}$ coordinates. It is then remarkable that all small parameters have been eliminated from the system (3.18)–(3.20) so that we may use any result obtained in previous work [14]. A second method to investigate asymptotic expansions is therefore to use molecular coordinates and to map back these results to the unknowns $(\rho, v, T)$ written in terms of the macroscopic coordinates $(\widehat{t}, \widehat{x})$. These two methods are of course equivalent but working in physical coordinates is usually more instructive and more flexible.

# 4 Weighted inequalities

Higher order entropies naturally introduce weight factors in the form of powers of temperature or density when estimating Lebesgue norms of the flow variables derivatives. We restate here weighted estimates of derivatives [9, 12, 14, 25, 26] and further specify the scaling dependence of the corresponding estimating constants. We work with dimensionless quantities and omit hat accents for the sake of notational simplicity. We denote by $BMO$ the space of functions with bounded mean oscillations [17, 23, 25, 26], by $\mathcal{C}_0^0$ the space of continuous functions that vanish at infinity and by $H^k$ the usual Sobolev spaces [22, 23, 25].

## 4.1 Weighted products of derivatives

We investigate weighted products of derivatives of the rescaled unknowns $r$, $\tau$ and $w$, that will be taken to be $r = \log\rho$, $w = v/\sqrt{T}$, and $\tau = \log T$ in our applications. In the following Theorem, since in our applications $w$ and $\tau$ are parabolic variables, the total number of derivations $k$ can be left unchanged.

**Theorem 4.1** *Let $k \geq 1$ be an integer, $\bar{\theta} > 0$ be positive, $1 < p < \infty$, $\tau$ be such that $\tau - \tau_\infty \in H^k(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ for some constant $\tau_\infty$ and let $r \in BMO$. There exist scale invariant positive constants $\delta(n, p, \bar{\theta})$ and $c(k, n, p)$, only depending on $(n, p, \bar{\theta})$ and $(k, n, p)$, respectively, such that if $\|r\|_{BMO} + \|\tau\|_{BMO} < \delta$, then for any $a$, $b$ with $|a| + |b| \leq \bar{\theta}$, any integer $l \geq 1$, and any multiindices $\alpha^j$, $1 \leq j \leq l$, with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1\leq j\leq l} |\alpha^j| = k$, whenever $e^{a\tau+br}\partial^k\tau \in L^p(\mathbb{R}^n)$, the following inequality holds*

$$\Big\| e^{a\tau+br} \prod_{1\leq j\leq l} \partial^{\alpha^j}\tau \Big\|_{L^p} \leq \; c\big\|\tau\big\|_{BMO}^{l-1} \; \big\|e^{a\tau+br}\partial^k\tau\big\|_{L^p}, \tag{4.1}$$

*where we have defined for any smooth scalar function $\phi$*

$$|\partial^k\phi|^p = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\big(\partial^\alpha\phi\big)^p = \sum_{1\leq i_1,\dots,i_k\leq n} (\partial_{i_1}\cdots\partial_{i_k}\phi)^p. \tag{4.2}$$

*Further assuming $w \in H^k(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $e^{a\tau+br}\partial^k w \in L^p(\mathbb{R}^n)$, and $0 \le \bar{l} \le l$, then*

$$\Big\| e^{a\tau+br} \prod_{1\le j\le \bar{l}} \partial^{\alpha^j} w \prod_{\bar{l}+1\le j\le l} \partial^{\alpha^j}\tau \Big\|_{L^p} \le c\Big( \|w\|_{BMO} + \|\tau\|_{BMO} \Big)^{l-1} \times \Big( \big\|e^{a\tau+br}\partial^k w\big\|_{L^p} + \big\|e^{a\tau+br}\partial^k\tau\big\|_{L^p} \Big), \tag{4.3}$$

*where we have defined $\big|\partial^k w\big|^p = \sum_{1\le i\le n}\big|\partial^k w_i\big|^p$ and where, in the left hand member of (4.3), with a slight abuse of notation, we have denoted by $w$ any of its components $w_1, \dots, w_n$.*

Since in our applications $r = \log\rho$ will be a hyperbolic variable, the total number of derivations appearing in the estimates need to be decreased by using a weighted $L^\infty$ norms of the gradients [14]. We denote by by $\mathcal{C}_0^1$ the space of continuously differentiable functions that vanish at infinity with their gradients.

**Theorem 4.2** *Let $k \ge 2$ be an integer, $\bar\theta > 0$ be positive, $1 < p < \infty$, $\tau$, $r$, $w$ be such that $\tau - \tau_\infty, r - r_\infty, w \in H^{k-1}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ for some constants $\tau_\infty$ and $r_\infty$. Let $a$, $b$, $\bar{a}$, and $\bar{b}$ be constants with $|a| + |b| \le \bar\theta$, and $|\bar a| + |\bar b| \le \bar\theta$, where $\bar\theta > 0$ and define $g = \exp(a\tau + br)$ and $h = \exp(\bar a\tau + \bar b r)$. Let $l \ge 2$, let $\alpha^j$, $1 \le j \le l$, be multiindices with $|\alpha^j| \ge 1$, $1 \le j \le l$, and $\sum_{1\le j\le l}|\alpha^j| = k$. There exist scale invariant positive constants $\delta(k,n,p,\bar\theta)$ and $c(k,n,p,\bar\theta)$, only depending on $(k,n,p,\bar\theta)$, such that if $\|r\|_{BMO} + \|\tau\|_{BMO} < \delta$, then whenever $gh^{k-1}\partial^{k-1}r \in L^p(\mathbb{R}^n)$, $gh^{k-1}\partial^{k-1}w \in L^p(\mathbb{R}^n)$, $gh^{k-1}\partial^{k-1}\tau \in L^p(\mathbb{R}^n)$, $gh^{k-2}\partial^{k-2}r \in L^p(\mathbb{R}^n)$, $gh^{k-2}\partial^{k-2}w \in L^p(\mathbb{R}^n)$, $gh^{k-2}\partial^{k-2}\tau \in L^p(\mathbb{R}^n)$, and $1 \le \bar l \le \tilde l \le l$, we have the estimates*

$$\Big\| gh^k \prod_{1\le j\le \bar l} \partial^{\alpha^j} r \prod_{\bar l+1\le j\le \tilde l} \partial^{\alpha^j} w \prod_{\tilde l+1\le j\le l} \partial^{\alpha^j}\tau \Big\|_{L^p} \le c\|\mathrm{w}\|_{BMO}^{l-2} \|h\partial_x \mathrm{w}\|_{L^\infty} \|gh^{k-1}\partial^{k-1}\mathrm{w}\|_{L^p} + c\mathbf{1}_{k\ge 3}\|\mathrm{w}\|_{BMO}^{(l-3)^+} \|h\partial_x\mathrm{w}\|_{L^\infty}^2 \|gh^{k-2}\partial^{k-2}\mathrm{w}\|_{L^p}, \tag{4.4}$$

*where we have defined $\mathrm{w} = (r, w, \tau)$, $\|\mathrm{w}\|_{BMO} = \|r\|_{BMO} + \|w\|_{BMO} + \|\tau\|_{BMO}$, $\|h\partial_x\mathrm{w}\|_{L^\infty} = \|h\partial_x r\|_{L^\infty} + \|h\partial_x w\|_{L^\infty} + \|h\partial_x\tau\|_{L^\infty}$, $\|gh^m\partial^m\mathrm{w}\|_{L^p}^p = \|gh^m\partial^m r\|_{L^p}^p + \|gh^m\partial^m w\|_{L^p}^p + \|gh^m\partial^m\tau\|_{L^p}^p$ for any $m \in \mathbb{N}$, and where $\mathbf{1}_{k\ge3} = 0$ if $k < 3$ and $\mathbf{1}_{k\ge3} = 1$ if $k \ge 3$. In particular, in the situation $2 \le k \le 3$, the second term in the right hand side of in (4.4) is absent.*

## 4.2 Weighted products of renormalized derivatives

We now estimate products of derivatives of density, temperature and velocity components rescaled by the proper renormalizing factors. Theorems 4.3 and 4.4 are essentially consequences of Theorems 4.1 and 4.2 and of differential identities [12, 14].

**Theorem 4.3** *Let $k \ge 1$ be an integer, $\bar\theta > 0$ be positive , $1 < p < \infty$, $T$ be such that $T \ge T_{\min} > 0$ and $T - T_\infty \in H^k(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ for some positive constant $T_\infty$ and $\rho$ be positive such that $r = \log\rho \in BMO$. There exist scale invariant positive constants $\delta(n,p,\bar\theta)$ and $c(k,n,p)$, only depending on $(n,p,\bar\theta)$ and $(k,n,p)$, respectively, such that if $\|\log\rho\|_{BMO} + \|\log T\|_{BMO} < \delta$, then for any real $a$ and $b$ such that $|a| + |b| \le \bar\theta$, any integer $l \ge 1$, and any multiindices $\alpha^j$, $1 \le j \le l$, with $|\alpha^j| \ge 1$, $1 \le j \le l$, and $\sum_{1\le j\le l}|\alpha^j| = k$, whenever $T^a\rho^b(\partial^k T)/T \in L^p(\mathbb{R}^n)$, we have the estimates*

$$\Big\| T^a\rho^b \prod_{1\le j\le l} \frac{\partial^{\alpha^j} T}{T} \Big\|_{L^p} \le c\|\log T\|_{BMO}^{l-1} \Big\| T^a\rho^b \frac{\partial^k T}{T} \Big\|_{L^p}. \tag{4.5}$$

*Assuming $v \in H^k(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $\|\log\rho\|_{BMO} + \|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta$, whenever $T^a\rho^b(\partial^k v)/\sqrt{T} \in L^p(\mathbb{R}^n)$, we have for $0 \le \bar l \le l$*

$$\Big\| T^a\rho^b \prod_{1\le j\le \bar l} \frac{\partial^{\alpha^j} v}{\sqrt{T}} \prod_{\bar l+1\le j\le l} \frac{\partial^{\alpha^j} T}{T} \Big\|_{L^p} \le c\Big( \Big\|\frac{v}{\sqrt{T}}\Big\|_{L^\infty} + \|\log T\|_{BMO} \Big)^{l-1} \times \Big( \Big\|T^a\rho^b \frac{\partial^k v}{\sqrt{T}}\Big\|_{L^p} + \Big\|T^a\rho^b\frac{\partial^k T}{T}\Big\|_{L^p} \Big), \tag{4.6}$$

*where, in the left hand member, with a slight abuse of notation, we have denoted by $v$ any of its components $v_1, \dots, v_n$.*

**Theorem 4.4** *Let $k \geq 2$ be an integer, $\bar{\theta} > 0$ be positive , $1 < p < \infty$, $\rho$, $v$, $T$, be such that $\rho \geq \rho_{\min}$, $T \geq T_{\min}$, and $\rho - \rho_\infty, v, T - T_\infty \in H^{k-1}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ for positive constants $\rho_\infty$, $\rho_{\min}$, $T_\infty$ and $T_{\min}$. Let $a$, $b$ $\bar{a}$, and $\bar{b}$ be constants with $|a| + |b| \leq \bar{\theta}$, and $|\bar{a}| + |\bar{b}| \leq \bar{\theta}$, where $\bar{\theta} > 0$, and $g = T^a \rho^b$, $h = T^{\bar{a}} \rho^{\bar{b}}$. Let $l \geq 2$, let $\alpha^j$, $1 \leq j \leq l$, be multiindices with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1 \leq j \leq l} |\alpha^j| = k$. There exist scale invariant positive constants $\delta(k, n, p, \bar{\theta})$ and $c(k, n, p, \bar{\theta})$, only depending on $(k, n, p, \bar{\theta})$, such that if $\| \log \rho \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty} + \| \log T \|_{BMO} < \delta$, then whenever $gh^{k-1}\partial^{k-1}\rho/\rho \in L^p(\mathbb{R}^n)$, $gh^{k-1}\partial^{k-1}v/\sqrt{T} \in L^p(\mathbb{R}^n)$, $gh^{k-1}\partial^{k-1}T/T \in L^p(\mathbb{R}^n)$, $gh^{k-2}\partial^{k-2}\rho/\rho \in L^p(\mathbb{R}^n)$, $gh^{k-2}\partial^{k-2}v/\sqrt{T} \in L^p(\mathbb{R}^n)$, $gh^{k-2}\partial^{k-2}T/T \in L^p(\mathbb{R}^n)$, we have for $0 \leq \bar{l} \leq \tilde{l} \leq l$*

$$\Big\| gh^k \prod_{1 \leq j \leq \bar{l}} \frac{\partial^{\alpha^j}\rho}{\rho} \prod_{\bar{l}+1 \leq j \leq \tilde{l}} \frac{\partial^{\alpha^j} v}{\sqrt{T}} \prod_{\tilde{l}+1 \leq j \leq l} \frac{\partial^{\alpha^j} T}{T} \Big\|_{L^p} \leq c \|\mathrm{w}\|'^{\,l-2}_{BMO} \|h\partial_x \mathrm{w}\|'_{L^\infty} \|gh^{k-1}\partial^{k-1}\mathrm{w}\|'_{L^p}$$
$$+ c\mathbf{1}_{k\geq 3} \|\mathrm{w}\|'^{\,(l-3)^+}_{BMO} \|h\partial_x \mathrm{w}\|'^{\,2}_{L^\infty} \|gh^{k-2}\partial^{k-2}\mathrm{w}\|'_{L^p}, \tag{4.7}$$

*where, in the left hand member, with a slight abuse of notation, we have denoted by $v$ any of its components $v_1, \ldots, v_n$ and where we have denoted $\mathrm{w} = (r, w, \tau)$ and $\|\mathrm{w}\|'_{BMO} = \| \log \rho \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty} + \| \log T \|_{BMO}$, $\|h\partial_x \mathrm{w}\|'_{L^\infty} = \|h \frac{\partial_x \rho}{\rho}\|_{L^\infty} + \|h \frac{\partial_x v}{\sqrt{T}}\|_{L^\infty} + \|h \frac{\partial_x T}{T}\|_{L^\infty}$, $\|gh^m \partial^m \mathrm{w}\|'^{p}_{L^p} = \|gh^m \frac{\partial^m \rho}{\rho}\|^p_{L^p} + \|gh^m \frac{\partial^m v}{\sqrt{T}}\|^p_{L^p} + \|gh^m \frac{\partial^m T}{T}\|^p_{L^p}$, for any $m \in \mathbb{N}$. In particular, in the situation where $2 \leq k \leq 3$, the second term in the right hand side of (4.7) is absent. Note that there is a $L^\infty$ norm for the renormalized velocity $v/\sqrt{T}$ term in $\|\mathrm{w}\|'_{BMO}$.*

# 5 Asymptotics of Higher order entropy estimates

We investigate in this section parameter dependent higher order entropic estimates for compressible flows. We investigate solutions of the compressible equations in reduced form (3.8)–(3.10) rewritten by suppressing hat accents

$$\partial_t \rho + \partial_x \cdot (\rho v) = 0, \tag{5.1}$$
$$\partial_t (\rho v) + \partial_x \cdot (\rho v \otimes v) + \tfrac{1}{\epsilon_{\mathrm{M}}^2} \partial_x p - \tfrac{1}{\mathrm{Re}} \partial_x \cdot \big( \kappa \partial_x \cdot v I + \eta\, d \big) = 0, \tag{5.2}$$
$$\partial_t (\rho c_v T) + \partial_x \cdot (\rho v c_v T) - \tfrac{1}{\mathrm{Re}} \partial_x \cdot \big( \lambda\, \partial_x T \big) = \tfrac{\epsilon_{\mathrm{M}}^2}{\mathrm{Re}} \big( \kappa (\partial_x \cdot v)^2 + \tfrac{1}{2} \eta d{:}d \big) - p \partial_x \cdot v, \tag{5.3}$$

where $\rho$ is the density, $v$ the velocity, $p$ the pressure, $\kappa(T)$ the volume viscosity, $d = \partial_x v + \partial_x v^t - \frac{2}{n} \partial_x \cdot v I$ the deviatoric part of the strain rate tensor, $\eta(T)$ the viscosity, $c_v$ the constant volume heat capacity, $T$ the absolute temperature and $\lambda(T)$ the thermal conductivity. All these quantities are dimensionless and we have assumed for the sake of simplicity that the internal energy is proportional to temperature $e = c_v T$ where $c_v$ is a constant.

The relevant assumptions on the thermal conductivity $\lambda$, the volume viscosity $\kappa$ and the shear viscosity $\eta$ are derived from the kinetic theory of gases as discussed in Section 3.3 [8, 10, 12].

We consider the case of functions defined on $\mathbb{R}^n$ with $n \geq 2$, that are 'constant at infinity', and we only consider smooth solutions such that

$$\rho - \rho_\infty \in C\big([0, \bar{t}], H^l\big) \cap C^1\big([0, \bar{t}], H^{l-1}\big), \quad \partial_x \rho \in L^2\big((0, \bar{t}), H^{l-1}\big) \tag{5.4}$$
$$v, T - T_\infty \in C\big([0, \bar{t}], H^l\big) \cap C^1\big([0, \bar{t}], H^{l-2}\big), \quad \partial_x v, \partial_x T \in L^2\big((0, \bar{t}), H^l\big), \tag{5.5}$$

where $l$ is an integer such that $l \geq [n/2] + 3$, that is, $l > n/2 + 2$, $\bar{t}$ is some positive time, $\rho_\infty > 0$ a fixed positive density and $T_\infty > 0$ a fixed positive temperature. We also assume that $\rho$ and $T$ are such that $\rho \geq \rho_{\min}$ and $T \geq T_{\min}$ where $\rho_{\min} > 0$ and $T_{\min} > 0$ are fixed positive constants. Such smooth solutions are known to exist either locally in time or globally when the initial state is close to the constant state $(\rho_\infty, 0, T_\infty)$ [10, 18, 19, 21, 23, 24, 27].

**Remark 5.1** *In the special case where $\lambda = \mathfrak{a}_\lambda T^\varkappa$, $\eta = \mathfrak{a}_\eta T^\varkappa$, $\kappa = \mathfrak{a}_\kappa T^\varkappa$, and $c_v$ is constant, if $\big(\rho(t,x), v(t,x), T(t,x)\big)$ is a solution of the Navier-Stokes equations (5.1)–(5.3), then*

$$\Big( \xi^{2\varkappa-1}\, \zeta\, \rho(\xi\zeta t, \zeta x), \quad \xi\, v(\xi\zeta\, t, \zeta\, x), \quad \xi^2\, T(\xi\zeta\, t, \zeta\, x) \Big), \tag{5.6}$$

*is a solution for any positive $\xi$ and $\zeta$. For arbitrary transport coefficients, the one parameter family obtained by letting $\xi = 1$ is also a family of solutions. These scaling properties hold similarly for the systems (3.3)–(3.5), (3.8)–(3.10), (3.18)–(3.20), and the scaling properties of the incompressible case [12] can be recovered from (5.6) by letting $\zeta = \xi^{(1-2\varkappa)}$.*

## 5.1 Higher order entropies

Following the physical *ansatz* (2.5)(3.14) we define the $(2k)^{\rm th}$ order rescaled kinetic entropy corrector $\gamma^{[k]}$ by

$$\gamma^{[k]} = \rho\epsilon_{\rm K}^{2k} h^{2k} \Big( \frac{|\partial_x^k \rho|^2}{\rho^2} + \epsilon_{\rm M}^2 \frac{|\partial_x^k v|^2}{T} + c_v \frac{|\partial_x^k T|^2}{T^2} \Big), \tag{5.7}$$

where $h = 1/\rho T^{\frac12-\varkappa}$, $|\partial_x^k v|^2 = \sum_{1\leq i\leq n} |\partial_x^k v_i|^2$ and $|\partial_x^k \phi|^2 = \sum_{|\alpha|=k} (k!/\alpha!)(\partial_x^\alpha \phi)^2$ for any smooth scalar function $\phi$ like $\rho$, $T$, $v_i$, $1\leq i\leq n$, and where $k!/\alpha!$ are the multinomial coefficients, keeping in mind that hat accents are now omitted.

This choice of $\gamma^{[k]}$, with the coefficients $c_v$ in front of temperature derivatives, yields more convenient higher order entropic estimates. It eliminates various quadratic terms associated with hyperbolic variables thanks to symmetry properties. This choice can also be associated with symmetrized forms of the system of partial differential equations. Let us denote $\mathrm{U} = \big(\rho, \rho v, \rho(e+\frac12|v|^2)\big)$ the conservative variable, $\mathrm{V} = -(\partial_U S^{(0)})^t$ the entropic variable, and $\mathrm{Z} = \big(\rho, v, T\big)^t$ the natural variable, which is also a normal variable [15, 20]. Defining the matrix $\overline{A}_0 = (\partial_{\mathrm{Z}}\mathrm{V})^t \partial_{\mathrm{U}}\mathrm{V}(\partial_{\mathrm{Z}}\mathrm{V})$, which is associated with normal forms [15, 20], one can then write the higher order entropy correctors in the form $\gamma^{[k]} = \epsilon_{\rm K}^{2k} h^{2k} {<}\partial^k \mathrm{Z}, \overline{A}_0 \partial^k \mathrm{Z}{>}$, where $h$ is the weight associated with the dependence of the mean free path on density and temperature $h = l/{<}l{>}$. This choice can also be associated with a 'spatial gradient' Fisher information with for instance $\gamma^{[1]} = \epsilon_{\rm K}^2 h^2 \sum_{\mathrm{I}\in\mathcal{I}} k_{\rm B} \int_{\mathbb{R}^n} |\partial_x \log f^{(0)}|^2 f^{(0)}\, dc$. The weight $h$ is such that the spatial derivative operator $h\partial_x$ is invariant by the changes of scales (5.6) naturally associated with the Navier-Stokes equations.

Thanks to the fact that $v$ and $T$ are parabolic variables, we can expect source terms in the form $|\partial_x^{k+1} T/T|^2$ and $|\partial_x^{k+1} v/\sqrt{T}|^2$ to appear in the governing equation for $\gamma^{[k]}$—up to weight factors. However, since $\rho$ is a hyperbolic variable, there will be no such corresponding source term $|\partial_x^{k+1}\rho/\rho|^2$ for density. A priori estimates for density derivatives and more generally of hyperbolic variables derivatives indeed require to introduce extra entropic corrector terms. These extra corrector terms will yield source terms in the form $|\partial_x^k \rho/\rho|^2$. These terms are similar in spirit to the perturbed quadratic terms introduced by Kawashima in order to obtain hyperbolic variable derivatives estimates for linearized equations around equilibrium states and decay estimates [19]. They are used here with renormalized variables as well as with powers of $h$ as weights factors in order to obtain higher order entropic inequalities. We define the quantity $\gamma^{[k-\frac12]}$ by

$$\gamma^{[k-\frac12]} = \rho\epsilon_{\rm K}^{2k-1} h^{2k-1} \epsilon_{\rm M} \frac{\partial_x^{k-1} v}{\sqrt{T}} \cdot \frac{\partial_x^{k-1}\partial_x \rho}{\rho}, \tag{5.8}$$

where we have set for convenience

$$\partial_x^k \phi\, \partial_x^k \psi = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial_x^\alpha \phi\, \partial_x^\alpha \psi, \qquad \partial_x^k v \cdot \partial_x^k \partial_x \rho = \sum_{\substack{|\alpha|=k\\ 1\leq i\leq n}} \frac{k!}{\alpha!} \partial_x^\alpha v_i\, \partial_x^\alpha \partial_i \rho, \tag{5.9}$$

and we will see that in the $\gamma^{[k-\frac12]}$ governing equation there is a source term in the form $|\partial_x^k \rho/\rho|^2$—up to weight factors. From a physical point of view, we also note that $\gamma^{[k-\frac12]}$ is of the general form (2.4) for $S^{(2k-1)}$. Finally, we define the $(2k)^{\rm th}$ order kinetic entropy estimator by

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1\leq i\leq k} (\gamma^{[i]} + a\gamma^{[i-\frac12]}), \qquad k\geq 0. \tag{5.10}$$

Note that the quantities $\gamma^{[i-\frac12]}$, $1\leq i\leq k$, are multiplied by the small factor $a$ in (5.10) so as to not modify the majorizing properties of the correctors $\gamma^{[i]}$, $0\leq i\leq k$.

Following the physical *ansatz* (2.6), we define similarly the modified $(2k)^{\rm th}$ order kinetic entropy corrector $\tilde\gamma^{[k]}$ by

$$\tilde\gamma^{[k]} = \rho\epsilon_{\rm K}^{2k} h^{2k} \big( |\partial_x^k r|^2 + \epsilon_{\rm M}^2 |\partial_x^k w|^2 + c_v |\partial_x^k \tau|^2 \big), \tag{5.11}$$

where $r = \log\rho$, $w = v/\sqrt{T}$, and $\tau = \log T$, as well as

$$\tilde\gamma^{[k-\frac12]} = \rho\epsilon_{\rm K}^{2k-1} h^{2k-1} \epsilon_{\rm M} \partial_x^{k-1} w \cdot \partial_x^{k-1} \partial_x r, \tag{5.12}$$

$\tilde\gamma^{[0]} = \gamma^{[0]}$, and the modified $(2k)^{\rm th}$ order kinetic entropy estimators

$$\widetilde\Gamma^{[k]} = \tilde\gamma^{[0]} + \sum_{1\leq i\leq k} (\tilde\gamma^{[i]} + a\tilde\gamma^{[i-\frac12]}), \qquad k\geq 0. \tag{5.13}$$

The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$, as well as the estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$, will be shown to have similar properties and both may be used to derive a priori estimates. Strictly speaking, we should term $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ "$(2k)^{\rm th}$ order kinetic entropy correctors" or "$(2k)^{\rm th}$ order kinetic entropy deviation estimators" and $\gamma^{[k-\frac{1}{2}]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$ "$(2k-1)^{\rm th}$ order kinetic entropy correctors", and $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "$(2k)^{\rm th}$ order kinetic entropy estimators". However, we will often informally term $\gamma^{[k]}$, $\tilde{\gamma}^{[k]}$, $\gamma^{[k-\frac{1}{2}]}$, $\tilde{\gamma}^{[k-\frac{1}{2}]}$, $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "higher order entropies".

## 5.2 Balance equations

We present in this section the parameter dependent balance equations for $\gamma^{[k]}$ and $\gamma^{[k-\frac{1}{2}]}$. Similar equations can be derived for $\tilde{\gamma}^{[k]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$ but are omitted.

**Proposition 5.2** *Let $(\rho, v, T)$ be a smooth solution of the compressible Navier-Stokes equations* (5.1)–(5.3) *with regularity* (5.4)(5.5) *and let $1 \leq k \leq l$. Then the following balance equation holds in $\mathcal{D}'\big((0,\bar{t}\,) \times \mathbb{R}^n\big)$*

$$\partial_t \gamma^{[k]} + \partial_x\!\cdot\!(v\gamma^{[k]}) + \partial_x\!\cdot\!\varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0, \tag{5.14}$$

*where $\varphi_\gamma^{[k]} \in L^1\big((0,\bar{t}\,), L^1(\mathbb{R}^n)\big)$ is a flux and $\pi_\gamma^{[k]}$, $\Sigma_\gamma^{[k]}$, $\omega_\gamma^{[k]} \in L^1\big((0,\bar{t}\,), L^1(\mathbb{R}^n)\big)$ are source terms. The term $\pi_\gamma^{[k]}$ is given by*

$$\pi_\gamma^{[k]} = \frac{2g^2\epsilon_{\rm K}^{2k}h^{2(k+1)}}{\rm Re}\Big(\frac{\lambda}{T^{\varkappa}}\frac{|\partial_x^{k+1}T|^2}{T^2} + \epsilon_{\rm M}^2\frac{\eta}{T^{\varkappa}}\frac{|\partial_x^{k+1}v|^2}{T} + \epsilon_{\rm M}^2\frac{\frac{1}{3}\eta+\kappa}{T^{\varkappa}}\frac{|\partial_x^{k}(\partial_x\!\cdot v)|^2}{T}\Big), \tag{5.15}$$

*where $g = \rho T^{\frac{1}{2}(1-\varkappa)}$ and $h = 1/\rho T^{\frac{1}{2}-\varkappa}$, so that $\pi_\gamma^{[k]}$ only contains the temperature and velocity $(k+1)^{\rm th}$ derivatives squared as expected from the hyperbolic–parabolic structure of system of partial differential equations. The term $\Sigma_\gamma^{[k]}$ is in the form*

$$\Sigma_\gamma^{[k]} = \sum_{\sigma\nu\mu\phi}\frac{\epsilon_{\rm K}^{2k}c_{\sigma\nu\mu\phi}}{\rm Re}T^{\sigma-\varkappa}\partial_T^\sigma\phi\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{c_vT^{\varkappa}}\frac{g^2\epsilon_{\rm K}^{2k}h^{2(k+1)}}{\rm Re}\frac{|\partial_x^k\rho|^2}{\rho^2}\frac{\Delta_xT}{T}, \tag{5.16}$$

*where $c_{\sigma\nu\mu\phi}$ are constants and the sums are over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by*

$$\Pi_\nu^{(k+1)} = g\,h^{k+1}\prod_{1\leq|\alpha|\leq k+1}\Big(\frac{\partial_x^\alpha\rho}{\rho}\Big)^{\nu_\alpha}\,\Big(\epsilon_{\rm M}\frac{\partial_x^\alpha v}{\sqrt{T}}\Big)^{\nu'_\alpha}\,\Big(\frac{\partial_x^\alpha T}{T}\Big)^{\nu''_\alpha}, \tag{5.17}$$

*where $v$ denotes—with a slight abuse of notation—any of its components $v_1, \ldots, v_n$, and $\nu$ must be such that*

$$\sum_{1\leq|\alpha|\leq k+1}|\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1}\nu_\alpha = 0,$$

*so that there is a total number of $k+1$ derivations and there are no derivative of order $k+1$ of density. Moreover, there is at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ so that*

$$\sum_{|\alpha|=k+1}(\nu'_\alpha + \nu''_\alpha + \mu'_\alpha + \mu''_\alpha) \leq 1,$$

*and one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split between several derivative factors. Furthermore the term $\omega_\gamma^{[k]}$ is given by*

$$\omega_\gamma^{[k]} = \sum_{\nu\mu}\frac{\epsilon_{\rm K}^{2k-1}}{\rm Re}c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}, \tag{5.18}$$

*where $c_{\nu\mu}$ are constants and we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over*

$$\sum_{1\leq|\alpha|\leq k}|\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k, \qquad \sum_{1\leq|\alpha|\leq k}|\alpha|(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = k+1,$$

*so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = 0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$.*

**Proof.** The proof is lenghty and similar to the proof given in [14] for unscaled equations. □

**Remark 5.3** *Note that the velocity $\bar{v} = \epsilon_{\rm M} v$ naturally appears in the multilinear products* (5.17)*.*

**Proposition 5.4** *Let $(\rho, v, T)$ be a smooth solution of the compressible Navier-Stokes equations* (5.1)–(5.3) *with regularity* (5.4)(5.5) *and let $1 \leq k \leq l$. Then the following balance equation holds in $\mathcal{D}'\big((0,\bar{t}) \times \mathbb{R}^n\big)$*

$$\partial_t \gamma^{[k-\frac12]} + \partial_x\cdot(v\gamma^{[k-\frac12]}) + \partial_x\cdot\varphi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-\frac12]} + \Sigma_\gamma^{[k-\frac12]} + \omega_\gamma^{[k-\frac12]} = 0, \tag{5.19}$$

*where $\varphi_\gamma^{[k-\frac12]} \in L^1\big((0,\bar{t}), L^1(\mathbb{R}^n)\big)$, and $\pi_\gamma^{[k-\frac12]}$, $\Sigma_\gamma^{[k-\frac12]}$, $\omega_\gamma^{[k-\frac12]} \in L^1\big((0,\bar{t}), L^1(\mathbb{R}^n)\big)$ are source terms. The term $\pi_\gamma^{[k-\frac12]}$ is given by*

$$\pi_\gamma^{[k-\frac12]} = \frac{g^2 \epsilon_{\rm K}^{2(k-1)} h^{2k}}{\rm Re} \frac{|\partial_x^k \rho|^2}{\rho^2}, \tag{5.20}$$

*where $g = \rho T^{\frac12(1-\varkappa)}$, $h = 1/\rho T^{\frac12-\varkappa}$ so that $\pi_\gamma^{[k-\frac12]}$ will help to complete the missing gradient terms in $\pi_\gamma^{[k-1]}$. The term $\Sigma_\gamma^{[k-\frac12]}$ is in the form*

$$\begin{aligned}\Sigma_\gamma^{[k-\frac12]} = \sum_{\sigma\nu\mu\phi} & \frac{\epsilon_{\rm K}^{2k-1} c_{\sigma\nu\mu\phi}}{\rm Re} T^{\sigma-\varkappa} \partial_T^\sigma \phi \; \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} \\ & - \frac{\kappa + \frac43 \eta}{T^\varkappa} \frac{g^2 \epsilon_{\rm K}^{2k-1} h^{2k+1}}{\rm Re} \frac{\epsilon_{\rm M} \partial_x^k (\partial_x\cdot v)}{\sqrt{T}} \frac{\partial_x^k \rho}{\rho},\end{aligned} \tag{5.21}$$

*where the sums are over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leq|\alpha|\leq k}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k)}$ and $\Pi_\mu^{(k+1)}$ are defined as in the governing equation for $\gamma^{[k]}$ and $\Pi_\mu^{(k+1)}$ is always split between several derivative factors. Furthermore the term $\omega_\gamma^{[k-\frac12]}$ is given by*

$$\omega_\gamma^{[k-\frac12]} = \frac{\epsilon_{\rm K}^{2(k-1)}}{\rm Re} \Big( \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k)} + g^2 h^{2k} \frac{\partial_x^k T}{T} \frac{\partial_x^k \rho}{\rho} - g^2 h^{2k} \epsilon_{\rm M}^2 \frac{|\partial_x^{k-1}(\partial_x\cdot v)|^2}{T} \Big), \tag{5.22}$$

*and one of the products $\Pi_\nu^{(k)}$ or $\Pi_\mu^{(k)}$ is always split between several derivative factors.*

## 5.3 A priori estimates

Integrating the balance equation (5.14) for $\gamma^{[k]}$ and taking into account the smoothness properties with $1 \leq k \leq l$, we obtain that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx \leq \int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx + \int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx, \tag{5.23}$$

so that we have to estimate the integrals $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx$. Similarly, we obtain by integrating the balance equation (5.19) for $\gamma^{[k-\frac12]}$ that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k-\frac12]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[k-\frac12]}\, dx \leq \int_{\mathbb{R}^n} |\Sigma_\gamma^{[k-\frac12]}|\, dx + \int_{\mathbb{R}^n} |\omega_\gamma^{[k-\frac12]}|\, dx, \tag{5.24}$$

and we also have to estimate the integrals $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k-\frac12]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_\gamma^{[k-\frac12]}|\, dx$. Similar estimates can be conducted for the corresponding integrals $\int_{\mathbb{R}^n} |\Sigma_{\tilde\gamma}^{[k]}|\, dx$, $\int_{\mathbb{R}^n} |\omega_{\tilde\gamma}^{[k]}|\, dx$, $\int_{\mathbb{R}^n} |\Sigma_{\tilde\gamma}^{[k-\frac12]}|\, dx$, and $\int_{\mathbb{R}^n} |\omega_{\tilde\gamma}^{[k-\frac12]}|\, dx$ associated with the modified correctors but are omitted for brevity. We denote by $\chi$ the quantity

$$\begin{aligned}\chi = \| \log\rho \|_{BMO} + \epsilon_{\rm M} \| \frac{v}{\sqrt{T}} \|_{L^\infty} + \| \log T \|_{BMO} \\ + \epsilon_{\rm K} \| h \frac{\partial_x \rho}{\rho} \|_{L^\infty} + \epsilon_{\rm K} \epsilon_{\rm M} \| h \frac{\partial_x v}{\sqrt{T}} \|_{L^\infty} + \epsilon_{\rm K} \| h \frac{\partial_x T}{T} \|_{L^\infty} + \epsilon_{\rm K}^2 \| h^2 \frac{\partial_x^2 T}{T} \|_{L^\infty},\end{aligned} \tag{5.25}$$

and we will establish entropic type inequalities when $\chi$ is small enough. We could as well use the quantity

$$\widetilde{\chi} = \|r\|_{BMO} + \epsilon_{\rm M}\|w\|_{L^\infty} + \|\tau\|_{BMO} \\ + \epsilon_{\rm K}\|h\partial_x r\|_{L^\infty} + \epsilon_{\rm K}\epsilon_{\rm M}\|h\partial_x w\|_{L^\infty} + \epsilon_{\rm K}\|h\partial_x\tau\|_{L^\infty} + \epsilon_{\rm K}^2\|h^2\partial_x^2\tau\|_{L^\infty}, \tag{5.26}$$

but $\chi \leq \widetilde{\chi}(1+\widetilde{\chi})$ and $\widetilde{\chi} \leq \chi(1+\chi)$ so that $\chi$ and $\widetilde{\chi}$ are asymptotically equivalent in the neiborghood of zero. These quantities $\chi$ and $\widetilde{\chi}$ are invariant under the change of scales (5.6) described in Remark 5.1. They can also be interpreted as involving the natural variables $\log\rho$, $\log T$, and $v/\sqrt{r_{\rm g}T}$ appearing in Maxwellian distributions [4] and the natural scale $h$ associated with the mean free path. Since we have formally $v/\sqrt{r_{\rm g}T} = \mathcal{O}(\epsilon_{\rm M})$, $\log(T/T_\infty) = \mathcal{O}(\epsilon_{\rm M})$, and $\log(\rho/\rho_\infty) = \mathcal{O}(\epsilon_{\rm M})$, the constraint that $\chi$ and $\widetilde{\chi}$ remain small may also be interpreted as a small Mach number constraint, which is consistent with Enskog expansion [16]. In the following, all constants associated with a priori estimates and entropic inequalities may depend on the system parameters $\underline{\mathfrak{a}}$, $\overline{\mathfrak{a}}$, $\overline{\mathfrak{a}}_\sigma$, $\sigma \geq 1$, $\varkappa$, and $c_v$. However, these dependencies are made implicit in order to avoid notational complexities and only the dependence on $k$ and $n$ is made explicit.

**Proposition 5.5** *Let $(\rho, v, T)$ be a smooth solution of the compressible Navier-Stokes equations* (5.1)–(5.3) *with regularity* (5.4)(5.5) *and let $1 \leq k \leq l$. There exist positive constants $\delta(k,n)$ and $c_k = c(k,n)$—independent of $\epsilon_{\rm K}$ and $\epsilon_{\rm M}$—such that for $\chi < \delta$ we have*

$$\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\,dx \leq c_k\,\chi \int_{\mathbb{R}^n}\big(\pi_\gamma^{[k]} + \pi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-1]} + \mathbf{1}_{k\geq2}(\pi_\gamma^{[k-\frac32]} + \pi_\gamma^{[k-2]})\big)\,dx, \tag{5.27}$$

$$\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\,dx \leq c_k\,\chi \int_{\mathbb{R}^n}\big(\pi_\gamma^{[k]} + \pi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-1]} + \mathbf{1}_{k\geq2}(\pi_\gamma^{[k-\frac32]} + \pi_\gamma^{[k-2]})\big)\,dx. \tag{5.28}$$

**Proof.** In the expression (5.16) for $\Sigma_\gamma^{[k]}$, the last term is directly majorized as

$$\int_{\mathbb{R}^n} \frac{\lambda}{c_v T^\varkappa}\,\frac{g^2\epsilon_{\rm K}^{2k}h^{2(k+1)}}{\rm Re}\,\frac{|\partial_x^k\rho|^2}{\rho^2}\,\frac{|\Delta_x T|}{T}\,dx \leq c\epsilon_{\rm K}^2\|h^2\frac{\partial_x^2 T}{T}\|_{L^\infty}\int_{\mathbb{R}^n}\pi_\gamma^{[k-\frac12]}\,dx,$$

thanks to the properties of transport coefficients. On the other hand, since the quantities $T^{\sigma-\varkappa}\partial_T^\sigma\lambda$, $T^{\sigma-\varkappa}\partial_T^\sigma\kappa$ and $T^{\sigma-\varkappa}\partial_T^\sigma\eta$ are uniformly bounded from assumptions (3.12) we only have to estimate the $L^2$ norm of the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ in order to majorize the terms in the sum of (5.16). When $\Pi_\nu^{(k+1)}$ only contains derivatives of $v$ and $T$—in particular if there is a derivative of order $k+1$—we obtain from Theorem 4.3 applied to $(\overline{v}, T)$ with $k$ replaced by $k+1$, that when $\chi$ is small enough

$$\frac{\epsilon_{\rm K}^k}{\sqrt{\rm Re}}\,\|\Pi_\nu^{(k+1)}\|_{L^2} \leq c\Big(\|\log T\|_{BMO} + \epsilon_{\rm M}\|\frac{v}{\sqrt{T}}\|_{L^\infty}\Big)^{N_\nu-1}\left\{\int_{\mathbb{R}^n}\pi_\gamma^{[k]}\,dx\right\}^{\frac12},$$

where $N_\nu = \sum_{1\leq|\alpha|\leq k+1}(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = \sum_{1\leq|\alpha|\leq k+1}(\nu'_\alpha + \nu''_\alpha)$. However, if the product $\Pi_\nu^{(k+1)}$ is split—in particular if there is a derivative of density—we obtain from Theorem 4.4 applied to $(\rho, \overline{v}, T)$ with $k$ replaced by $k+1$, that when $\chi$ is small enough

$$\epsilon_{\rm K}^k\,\|\Pi_\nu^{(k+1)}\|_{L^2} \leq c\,\|{\rm w}\|_{BMO}^{\prime\,N_\nu-2}\;\epsilon_{\rm K}\|h\partial_x{\rm w}\|'_{L^\infty}\;\epsilon_{\rm K}^{k-1}\|gh^k\partial^k{\rm w}\|'_{L^2} \\ + c\,\mathbf{1}_{k\geq2}\|{\rm w}\|_{BMO}^{\prime\,(N_\nu-3)^+}\;\epsilon_{\rm K}^2\|h\partial_x{\rm w}\|^{\prime 2}_{L^\infty}\;\epsilon_{\rm K}^{k-2}\|gh^{k-1}\partial^{k-1}{\rm w}\|'_{L^2},$$

keeping the notation of Theorem 4.4 for $\|{\rm w}\|'_{BMO}$, $\|h\partial_x{\rm w}\|'_{L^\infty}$ and $\|gh^m\partial^m{\rm w}\|'_{L^2}$. Therefore, we obtain that

$$\frac{\epsilon_{\rm K}^k}{\sqrt{\rm Re}}\|\Pi_\nu^{(k+1)}\|_{L^2} \leq c\,\chi^{N_\nu-1}\left\{\int_{\mathbb{R}^n}\Big(\pi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-1]} + \mathbf{1}_{k\geq2}(\pi_\gamma^{[k-\frac32]} + \pi_\gamma^{[k-2]})\Big)\,dx\right\}^{\frac12},$$

where $N_\nu = \sum_{1\leq|\alpha|\leq k+1}(\nu_\alpha + \nu'_\alpha + \nu''_\alpha)$, using $\|{\rm w}\|'_{BMO} \leq \chi$, $\epsilon_{\rm K}\|h\partial_x{\rm w}\|'_{L^\infty} \leq \chi$, $\chi \leq 1$, and

$$\frac{\epsilon_{\rm K}^{2(i-1)}}{\rm Re}\,\|gh^i\partial^i{\rm w}\|^{\prime 2}_{L^2} \leq c\int_{\mathbb{R}^n}(\pi_\gamma^{[i-\frac12]} + \pi_\gamma^{[i-1]})\,dx.$$

Combining these estimates, we obtain for $\chi$ small enough

$$\frac{\epsilon_{\mathrm{K}}^{2k}}{\mathrm{Re}}\|\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}\|_{L^1} \le c\,\chi\int_{\mathbb{R}^n}\Big(\pi_\gamma^{[k]} + \pi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-1]} + \mathbf{1}_{k\ge 2}\big(\pi_\gamma^{[k-\frac32]} + \pi_\gamma^{[k-2]}\big)\Big)\,dx,$$

since at least one of the products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split into two or more derivative factors so that $N_\nu + N_\mu - 2 \ge 1$. The same type of estimates can be obtained for the convective contributions $\omega_\gamma^{[k]}$ since the corresponding products $\Pi_\mu^{(k+1)}$ are always split between several derivative factors. □

**Proposition 5.6** *Let $(\rho, v, T)$ be a smooth solution of the compressible Navier-Stokes equations* (5.1)–(5.3) *with regularity* (5.4)(5.5) *and let $1 \le k \le l$. There exist positive constants $\delta(k,n)$ and $c_k = c(k,n)$—independent of $\epsilon_{\mathrm{K}}$ and $\epsilon_{\mathrm{M}}$—such that for $\chi < \delta$ we have*

$$\int_{\mathbb{R}^n}|\Sigma_\gamma^{[k-\frac12]}|\,dx \le c_k\,\chi\int_{\mathbb{R}^n}\big(\pi_\gamma^{[k]} + \pi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-1]} + \mathbf{1}_{k\ge2}(\pi_\gamma^{[k-\frac32]} + \pi_\gamma^{[k-2]})\big)\,dx + \mathsf{c}_0\left\{\int_{\mathbb{R}^n}\pi_\gamma^{[k]}\,dx\right\}^{\frac12}\left\{\int_{\mathbb{R}^n}\pi_\gamma^{[k-\frac12]}\,dx\right\}^{\frac12}, \tag{5.29}$$

$$\int_{\mathbb{R}^n}|\omega_\gamma^{[k-\frac12]}|\,dx \le c_k\,\chi\int_{\mathbb{R}^n}\big(\pi_\gamma^{[k]} + \pi_\gamma^{[k-\frac12]} + \pi_\gamma^{[k-1]} + \mathbf{1}_{k\ge2}(\pi_\gamma^{[k-\frac32]} + \pi_\gamma^{[k-2]})\big)\,dx + \mathsf{c}_0'\int_{\mathbb{R}^n}\pi_\gamma^{[k-1]}\,dx + \mathsf{c}_0'\left\{\int_{\mathbb{R}^n}\pi_\gamma^{[k-\frac12]}\,dx\right\}^{\frac12}\left\{\int_{\mathbb{R}^n}\pi_\gamma^{[k-1]}\,dx\right\}^{\frac12}, \tag{5.30}$$

*where $\mathsf{c}_0$ and $\mathsf{c}_0'$ are constants independent of $k$, $n$, $\epsilon_{\mathrm{K}}$ and $\epsilon_{\mathrm{M}}$.*

**Proof.** All split terms of $\Sigma_\gamma^{[k-\frac12]}$ or $\omega_\gamma^{[k-\frac12]}$ are estimated as in the proof of the proposition 5.5 whereas the special terms are directly estimated in terms of $\pi_\gamma^{[k]}$, $\pi_\gamma^{[k-\frac12]}$ and $\pi_\gamma^{[k-1]}$ with constants independent of $k$, $n$, $\epsilon_{\mathrm{K}}$ and $\epsilon_{\mathrm{M}}$. □

## 5.4 Zeroth order estimates

The balance equation for

$$\gamma^{[0]}/\mathsf{C}_0 = \rho\log\big(\frac{\rho}{\rho_\infty}\big) - (\rho - \rho_\infty) + \epsilon_{\mathrm{M}}^2\tfrac12\rho\frac{v^2}{T_\infty} + \rho c_v\Big(\frac{T - T_\infty}{T_\infty} - \log\big(\frac{T}{T_\infty}\big)\Big). \tag{5.31}$$

can be written—after some algebra—in the form

$$\partial_t\frac{\gamma^{[0]}}{\mathsf{C}_0} + \partial_x\cdot\Big(\rho v\big(s_\infty - s + c_p\frac{T - T_\infty}{T_\infty} + \tfrac12\epsilon_{\mathrm{M}}^2\frac{|v|^2}{T_\infty}\big)\Big) + \frac{1}{\mathrm{Re}}\partial_x\cdot\Big(\frac{q}{T_\infty} - \frac{q}{T} + \epsilon_{\mathrm{M}}^2\frac{\Pi\cdot v}{T}\Big) + \frac{1}{\mathrm{Re}}\Big(\frac{\lambda|\partial_x T|^2}{T^2} + \epsilon_{\mathrm{M}}^2\frac{\eta|d|^2}{2T} + \epsilon_{\mathrm{M}}^2\frac{\kappa(\partial_x\cdot v)^2}{T}\Big)\,dx = 0. \tag{5.32}$$

**Proposition 5.7** *Let $\gamma^{[0]}$ be given by* (5.31). *Then $\gamma^{[0]} \ge 0$ and there exists positive constants $\mathsf{C}_0$ and $\delta_0 > 0$—independent of $\epsilon_{\mathrm{K}}$ and $\epsilon_{\mathrm{M}}$—such that for $\chi < \delta_0$ small enough*

$$\partial_t\int_{\mathbb{R}^n}\gamma^{[0]}\,dx + \int_{\mathbb{R}^n}\pi_\gamma^{[0]}\,dx \le 0, \tag{5.33}$$

*where we define from* (5.15)

$$\pi_\gamma^{[0]} = \frac{2g^2h^2}{\mathrm{Re}}\Big(\frac{\lambda}{T^\varkappa}\,\frac{|\partial_x^1 T|^2}{T^2} + \epsilon_{\mathrm{M}}^2\frac{\eta}{T^\varkappa}\frac{|\partial_x^1 v|^2}{T} + \epsilon_{\mathrm{M}}^2\frac{\frac13\eta + \kappa}{T^\varkappa}\frac{(\partial_x\cdot v)^2}{T}\Big).$$

**Proof.** The proof is similar to that of the unscaled case [14] and 5.33 is a direct consequence of 5.32. □

## 5.5 Higher order estimates

We have defined the $(2k)^{\rm th}$ order Boltzmann kinetic entropy estimators by $\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]} + a(\gamma^{[\frac12]} + \cdots + \gamma^{[k-\frac12]})$ and $\widetilde{\Gamma}^{[k]} = \tilde\gamma^{[0]} + \cdots + \tilde\gamma^{[k]} + a(\tilde\gamma^{[\frac12]} + \cdots + \tilde\gamma^{[k-\frac12]})$, for $k \geq 0$, so that

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1\leq l\leq k} \rho\epsilon_{\rm K}^{2l} h^{2l}\Big(\frac{|\partial_x^l\rho|^2}{\rho^2} + \epsilon_{\rm M}^2\,\frac{|\partial_x^l v|^2}{T} + c_v\,\frac{|\partial_x^l T|^2}{T^2}\Big) + \sum_{1\leq l\leq k} a\rho\epsilon_{\rm K}^{2l-1} h^{2l-1}\epsilon_{\rm M}\,\frac{\partial_x^{l-1}v}{\sqrt{T}}\cdot\frac{\partial_x^{l-1}\partial_x\rho}{\rho}, \tag{5.34}$$

$$\widetilde{\Gamma}^{[k]} = \tilde\gamma^{[0]} + \sum_{1\leq l\leq k} \rho\epsilon_{\rm K}^{2l} h^{2l}\Big(|\partial_x^l r|^2 + \epsilon_{\rm M}^2\,|\partial_x^l w|^2 + c_v|\partial_x^l\tau|^2\Big) + \sum_{1\leq l\leq k} \rho\epsilon_{\rm K}^{2l-1} h^{2l-1}\epsilon_{\rm M}\,\partial_x^{l-1}w\cdot\partial_x^{l-1}\partial_x r, \tag{5.35}$$

and we have to establish that these kinetic entropy estimators obey entropic principles for the solutions of the compressible fluid model (5.1)–(5.3).

**Lemma 5.8** *Let $(\rho, v, T)$ be a smooth solution of the compressible Navier-Stokes equations* (5.1)–(5.3) *with regularity* (5.4)(5.5)*, assume that $T \geq T_{\min}$. There exists $\mathsf{B}_0(\frac{T_{\min}}{T_\infty})$ such that for $\mathsf{C}_0 \geq \mathsf{B}_0$, $0 < a \leq 1$, and $0 \leq k \leq l$*

$$\tfrac12(\gamma^{[0]} + \cdots + \gamma^{[k]}) \leq \Gamma^{[k]} \leq \tfrac32(\gamma^{[0]} + \cdots + \gamma^{[k]}), \qquad 0 \leq k \leq l, \tag{5.36}$$

$$\tfrac12(\tilde\gamma^{[0]} + \cdots + \tilde\gamma^{[k]}) \leq \widetilde{\Gamma}^{[k]} \leq \tfrac32(\tilde\gamma^{[0]} + \cdots + \tilde\gamma^{[k]}), \qquad 0 \leq k \leq l. \tag{5.37}$$

*Moreover, assuming that $T \geq T_{\min}$ and $\rho \leq \rho_{\max}$, there exists exists $\mathsf{B}_0(\frac{T_{\min}}{T_\infty}, \frac{\rho_{\max}}{\rho_\infty})$ such that for $\mathsf{C}_0 \geq \mathsf{B}_0$,*

$$\rho\big(|r - r_\infty|^2 + \epsilon_{\rm M}^2|w|^2 + c_v|\tau - \tau_\infty|^2\big) \leq \gamma^{[0]}. \tag{5.38}$$

**Proof.** Using the Cauchy-Schwartz inequality, it is straightforward to check that for any $1 \leq i \leq k \leq l$

$$|\gamma^{[i-\frac12]}| \leq \Big\{\rho\epsilon_{\rm K}^{2(i-1)} h^{2(i-1)}\epsilon_{\rm M}^2\Big|\frac{\partial^{i-1}v}{\sqrt{T}}\Big|^2\Big\}^{\frac12}\Big\{\rho\epsilon_{\rm K}^{2i}h^{2i}\Big|\frac{\partial^i\rho}{\rho}\Big|^2\Big\}^{\frac12}$$
$$\leq \frac12\Big(\rho\epsilon_{\rm K}^{2(i-1)} h^{2(i-1)}\epsilon_{\rm M}^2\Big|\frac{\partial^{i-1}v}{\sqrt{T}}\Big|^2 + \rho\epsilon_{\rm K}^{2i}h^{2i}\Big|\frac{\partial^i\rho}{\rho}\Big|^2\Big).$$

Therefore, half of the density part of $\gamma^{[i]}$ and half of the velocity part of $\gamma^{[i-1]}$ compensate for $|\gamma^{[i-\frac12]}|$ provided we ensure that $\gamma^{[0]} \geq \rho\epsilon_{\rm M}^2|v/\sqrt{T}|^2$ but this is a consequence of $\mathsf{C}_0 \geq 2T_\infty/T_{\min}$. The same method also applies for the modified estimators $\tilde\gamma^{[i-\frac12]}$, $1 \leq i \leq k$, and this yields Inequalities (5.36) and (5.37) upon summing over $1 \leq i \leq k$. Inequality (5.38) is a consequence of

$$\frac{T_{\min}}{2T_\infty}\epsilon_{\rm M}^2|w|^2 \leq \epsilon_{\rm M}^2\frac{|v|^2}{2T_\infty},$$

$$\frac{T_{\min}}{2T_\infty}|\tau - \tau_\infty|^2 \leq \exp(\tau - \tau_\infty) - 1 - (\tau - \tau_\infty),$$

valid for $\tau_{\min} \leq \tau$, where $\tau_{\min} = \log T_{\min}$, $\tau_\infty = \log T_\infty$ and $T_{\min} \leq T_\infty$, and of

$$\frac{\rho_\infty}{2\rho_{\max}}|r - r_\infty|^2 \leq \exp(r_\infty - r) - 1 - (r_\infty - r),$$

valid for $r \leq r_{\max}$, where $r_{\max} = \log\rho_{\max}$, $r_\infty = \log\rho_\infty$ and $r_\infty \leq r_{\max}$ letting $\mathsf{B}_0 = \max(1, \frac{2T_\infty}{T_{\min}}, \frac{\rho_{\max}}{2\rho_\infty})$ and $\mathsf{C}_0 \geq \mathsf{B}_0$. □

We can now combine the estimates of Propositions 5.5, 5.6, and 5.7, and the differential inequalities (5.23) and (5.24) in order to obtain entropic estimates.

**Theorem 5.9** *Let $(\rho, v, T)$ be a smooth solution of the compressible Navier-Stokes equations (5.1)–(5.3) with regularity (5.4)(5.5) and let $1 \leq k \leq l$. There exists positive constants $b$, $\bar{a}$, and $\delta(k, n)$ such that for the fixed value $a = \bar{a}$ and $\chi < \delta$ we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + \frac{b}{\epsilon_{\mathrm{K}}^2 \mathrm{Re}} \int_{\mathbb{R}^n} \rho T^{1-\varkappa} (\gamma^{[1]} + \cdots + \gamma^{[k]})\, dx \leq 0, \tag{5.39}$$

*with similar results for the modified higher order entropies*

$$\partial_t \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx + \frac{b}{\epsilon_{\mathrm{K}}^2 \mathrm{Re}} \int_{\mathbb{R}^n} \rho T^{1-\varkappa} (\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]})\, dx \leq 0. \tag{5.40}$$

**Proof.** The inequality (5.39) is a consequence of

$$\partial_t \int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + \frac{1}{5} \int_{\mathbb{R}^n} \big( \pi_\gamma^{[0]} + \cdots + \pi_\gamma^{[k]} + a(\pi_\gamma^{[\frac{1}{2}]} + \cdots \pi_\gamma^{[k-\frac{1}{2}]}) \big)\, dx \leq 0, \tag{5.41}$$

valid for $a$ small enough and $\chi/a$ small enough. This inequality (5.41) is established from the estimates of Propositions 5.5, and 5.7. The proof is similar to that of the unscaled case [14] since the estimating constants have been shown to be independent of $\epsilon_{\mathrm{K}}$ and $\epsilon_{\mathrm{M}}$, and the proof of (5.40) is similar. □

By integrating these differential inequalities, be obtain in particular that

$$\int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + \frac{b}{\epsilon_{\mathrm{K}}^2 \mathrm{Re}} \int_0^t \!\! \int_{\mathbb{R}^n} \rho T^{1-\varkappa} (\gamma^{[1]} + \cdots + \gamma^{[k]})\, dx dt \leq \int_{\mathbb{R}^n} \Gamma_0^{[k]}\, dx, \tag{5.42}$$

where the subscript 0 indicates that the functionals are estimated at the initial time $t = 0$, with similar results for the modified entropies.

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx + \frac{b}{\epsilon_{\mathrm{K}}^2 \mathrm{Re}} \int_0^t \!\! \int_{\mathbb{R}^n} \rho T^{1-\varkappa} (\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]})\, dx dt \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[k]}\, dx. \tag{5.43}$$

One can also obtain the following exponential estimates of higher order entropy estimators.

**Corollary 5.10** *Keep the assumptions of Theorem 5.9, assume that $T \geq T_{\min}$, $\rho_{\min} \leq \rho \leq \rho_{\max}$, $0 \leq \varkappa \leq 1$, and that $\mathsf{C}_0 \geq \mathsf{B}_0(\frac{T_{\min}}{T_\infty}, \frac{\rho_{\max}}{\rho_\infty})$ as in Lemma 5.8. Then we have estimates in the form*

$$\int_{\mathbb{R}^n} \Gamma^{[k]}\, dx \leq \exp(-\mathsf{a}t) \int_{\mathbb{R}^n} \Gamma_0^{[k]}\, dx + 2\big(1 - \exp(-\mathsf{a}t)\big) \int_{\mathbb{R}^n} \gamma_0^{[0]}\, dx,$$

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx \leq \exp(-\mathsf{a}t) \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[k]}\, dx + 2\big(1 - \exp(-\mathsf{a}t)\big) \int_{\mathbb{R}^n} \tilde{\gamma}_0^{[0]}\, dx.$$

*where $\mathsf{a} = b \rho_{\min} T_{\min}^{1-\varkappa} / 2\epsilon_{\mathrm{K}}^2 \mathrm{Re}$.*

**Proof.** This results from the differential inequality

$$\partial_t \int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + 2\,\mathsf{a} \int_{\mathbb{R}^n} (\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]})\, dx \leq 2\,\mathsf{a} \int_{\mathbb{R}^n} \gamma_0^{[0]}\, dx, \tag{5.44}$$

and from Inequality 5.36 upon time integration. The proof for the modified entropy estimators is similar. □

Theorem 5.9 shows that the $(2k)^{\mathrm{th}}$ order kinetic entropy estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ effectively obey entropic principles. Upon integrating inequalities (5.39) or (5.40), a priori estimates are obtained for the solutions of the compressible Navier-Stokes equations. These entropic inequalities and the related a priori estimates are invariant—up to a multiplicative factor—by the change of scales (5.6) described in Remark 5.1 and naturally associated to the Navier-Stokes equations. Since we have formally $v/\sqrt{r_{\mathrm{g}} T} = \mathcal{O}(\epsilon_{\mathrm{M}})$, $\log(T/T_\infty) = \mathcal{O}(\epsilon_{\mathrm{M}})$, and $\log(\rho/\rho_\infty) = \mathcal{O}(\epsilon_{\mathrm{M}})$, the constraint that $\chi$ or $\widetilde{\chi}$ remain small may be interpreted as a small Mach number constraint, which is consistent with Enskog expansion [16]. These estimates also provide a thermodynamic interpretation of the corresponding weighted Sobolev norms involving renormalized variables as well the dependence on density and temperature of the local mean free path through the factor $h$ which ensures that the operator $h\partial_x$ is scale invariant.

**Remark 5.11** *In the special situation $\varkappa = 1/2$ the weight $h$ does not depend anymore on temperature and consequently the control over second order derivatives of temperature is not neede in $\chi$ or $\widetilde{\chi}$. A value $\varkappa = 1/2$ corresponds to an infinite interaction potential at small interparticle distances.*

### 5.6 Sobolev norms in molecular coordinates

Higher order entropic inequalities yield a priori estimates as soon as the quantity $\chi$ is small enough. It is possible to rewrite $\chi$ in terms of the molecular coordinates $(\bar{t}, \bar{x})$ introduced in Section 3.5

$$\begin{aligned} \chi = & \| \log T\|_{BMO} + \| \log \rho\|_{BMO} + \|\bar{v}/\sqrt{T}\|_{L^\infty} \\ & + \|h\partial_{\bar{x}} \log T\|_{L^\infty} + \|h\partial_{\bar{x}} \log \rho\|_{L^\infty} + \|h\partial_{\bar{x}}\bar{v}/\sqrt{T}\|_{L^\infty} + \|h^2\partial_{\bar{x}}^2 \log T\|_{L^\infty}, \end{aligned}$$

in such a way that the reduced velocity $\bar{v}$, the variations of $\log \rho$ and $\log T$, and their derivatives in molecular coordinates have to be small.

Higher order entropic inequalities (5.40) directly yield estimates for $r$, $\overline{w} = \epsilon_{\rm M} v/\sqrt{T}$, and $\tau = \log T$, through the integrals

$$\int_{\mathbb{R}^n} \big(\tilde{\gamma}^{[0]} + \sum_{1\leq l\leq k} \rho\epsilon_{\rm K}^{2l} h^{2l} (|\partial_x^l r|^2 + |\partial_x^l \overline{w}|^2 + c_v|\partial_x^l \tau|^2)\big)\, dx, \tag{5.45}$$

that can be written similarly in terms of molecular coordinates. In particular, when $\mathsf{C}_0 \geq \mathsf{B}_0(\frac{T_{\min}}{T_\infty}, \frac{\rho_{\max}}{\rho_\infty})$ as in Lemma 5.8, we have

$$\sum_{0\leq l\leq k} \rho h^{2l} \big(|\partial_{\bar{x}}^l (r-r_\infty)|^2 + |\partial_{\bar{x}}^l \overline{w}|^2 + c_v|\partial_{\bar{x}}^l (\tau-\tau_\infty)|^2\big) \leq 2\widetilde{\Gamma}^{[k]}, \tag{5.46}$$

so that $\epsilon_{\rm K}^{-n} \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx = \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, d\bar{x}$ essentially represents a Sobolev norm of $\overline{\rm w} - \overline{\rm w}_\infty = (r-r_\infty, \overline{w}, \tau-\tau_\infty)$ in weighted molecular coordinates $\bar{x}$ which take into account the local change in the mean free path $l = \,<\! l \!>\! h$ due to the variations of density and temperature.

**Remark 5.12** *The weight $h$ can be minorized independently of the maximum tempererature only when $\varkappa \geq 1/2$ as given by the kinetic theory of gases.*

**Remark 5.13** *It is also possible to investigate a priori estimates of entropic correctors integrals $\int_{\mathbb{R}^n} \gamma^{[k]}\, dx$ in terms of powers of the knudsen numbers $\mathcal{O}(\epsilon_{\rm K}^{2k+2})$.*

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