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HIGHER ORDER ENTROPIES FOR COMPRESSIBLE FLUID MODELS

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We investigate higher order entropies for compressible fluid models and related *a priori* estimates. Higher order entropies are kinetic entropy estimators suggested by Enskog expansion of Boltzmann entropy. These quantities are quadratic in the density ρ, velocity v, and temperature T renormalized derivatives. We investigate governing equations of higher order entropy correctors and related differential inequalities in the natural situation where the volume viscosity, the shear viscosity, and the thermal conductivity depend on temperature, essentially in the form $T^{\varkappa}$, as given by the kinetic theory of gases. Entropic inequalities are established when $\|\log\rho\|_{\mathrm{BMO}}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{\mathrm{BMO}}$, $\|h\partial_x\rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$ are small enough, where $h = 1/(\rho T^{\frac{1}{2}-\varkappa})$ is a weight associated with the dependence of the local mean free path on density and temperature. As an example of application, we investigate global existence of solutions when the initial values $\log(\rho_0/\rho_\infty)$, $v_0/\sqrt{T_0}$, and $\log(T_0/T_\infty)$ are small enough in appropriate spaces.

Keywords: Entropy; kinetic; fluid; higher order; Enskog expansion; Boltzmann; inequalities.

AMS Subject Classification: 35Q30, 76N10, 82B40

1. Introduction

The notion of entropy has been shown to be of fundamental importance in fluid modeling from both physical and mathematical points of view.[4,6–12,20–25,27,34,42–44,49–51,58,59,62–64] We have introduced in previous work[28–30] a notion of kinetic entropy estimators for fluid models, suggested by Enskog expansion of Boltzmann kinetic entropy. Conditional higher order entropic inequalities have been established in the situation of incompressible flows spanning the whole space.[28–30] In this paper, we investigate higher order entropies for compressible fluid models and related *a priori* estimates.

We consider compressible flows spanning the whole space with temperature-dependent thermal conductivity, shear viscosity and volume viscosity. We only consider smooth solutions defined on $\mathbb{R}^n$ that are "constant at infinity". The compressible Navier–Stokes equations can be written in the form

$$\begin{aligned}
&\partial_t \rho + \partial_x\cdot(\rho v) = 0,\\
&\partial_t(\rho v) + \partial_x\cdot(\rho v\otimes v + pI) - \partial_x\cdot\left(\kappa\partial_x\cdot vI + \eta\left(\partial_x v + \partial_x v^t - \frac{2}{n}\partial_x\cdot vI\right)\right) = 0,\\
&\partial_t(\rho e) + \partial_x\cdot(\rho e v) - \partial_x\cdot(\lambda\partial_x T) = \left(\kappa - \frac{2}{n}\eta\right)(\partial_x\cdot v)^2 + \frac{1}{2}\eta|\partial_x v + \partial_x v^t|^2 - p\partial_x\cdot v,
\end{aligned}$$

where t denotes time, x the n-dimensional Cartesian coordinate, ρ the density, v the velocity, p the pressure, and e the internal energy per unit mass. We assume for the sake of notational simplicity that $p = \rho T$ and $e = c_v T$, where T is the temperature and c_v is a constant. The transport coefficients κ, η and λ are smooth functions of temperature and essentially behave — away from small temperatures — like a power of temperature $T^{\varkappa}$ as given by the kinetic theory of gases.

We only consider smooth solutions such that

$$\begin{gathered}
\rho - \rho_\infty \in C([0,\bar{t}], W^{l,2}) \cap C^1([0,\bar{t}], W^{l-1,2}),\\
v, T - T_\infty \in C([0,\bar{t}], W^{l,2}) \cap C^1([0,\bar{t}], W^{l-2,2}) \cap L^2((0,\bar{t}), W^{l+1,2}),
\end{gathered}$$

where l is an integer such that $l > n/2 + 2$, $\bar{t}$ is some positive time, $\rho_\infty > 0$ a fixed positive density and $T_\infty > 0$ a fixed positive temperature. We also assume that ρ and T are such that $\rho \geq \rho_{\min}$ and $T \geq T_{\min}$ where $\rho_{\min} > 0$ and $T_{\min} > 0$ are fixed positive constants.

Higher order entropy correctors are first suggested by Enskog expansion of Boltzmann kinetic entropy. The corresponding balance equations may also be seen as a generalization of Bernstein equations to systems of partial differential equations but expressed with renormalized variables. Higher order entropy correctors are quadratic with respect to the density, velocity, and temperature renormalized derivatives and are taken in the form

$$\gamma^{[k]} = \rho h^{2k}\left(\frac{|\partial^k\rho|^2}{\rho^2} + \frac{|\partial^k v|^2}{T} + c_v\frac{|\partial^k T|^2}{T^2}\right),$$

where $h = 1/(T^{\frac{1}{2}-\varkappa}\rho)$ is a weight associated with the dependence of the local mean free path on density and temperature. The square of kth derivatives of a scalar function ϕ, like T, ρ, or v_i, $1 \leq i \leq n$, is defined by $|\partial^k\phi|^2 = \sum_{|\alpha|=k}(k!/\alpha!)(\partial^\alpha\phi)^2$ and we set $|\partial^k v|^2 = \sum_{1\leq i\leq n}|\partial^k v_i|^2$. We derive balance equations of higher order entropy correctors for compressible fluid models with temperature-dependent viscosities and thermal conductivity.

Higher order kinetic entropy estimators are obtained upon summing a zeroth order fluid entropy $\gamma^{[0]}$ written in the form

$$\gamma^{[0]}/\mathsf{C}_0 = \left(\rho \log\left(\frac{\rho}{\rho_\infty}\right) - (\rho - \rho_\infty)\right) + \frac{1}{2}\rho\frac{v^2}{T_\infty} + \rho c_v\left(\frac{T-T_\infty}{T_\infty} - \log\left(\frac{T}{T_\infty}\right)\right),$$

with higher order entropy correctors $\gamma^{[i]}$, $1 \leq i \leq k$. These kinetic entropy estimators $\gamma^{[0]} + \cdots + \gamma^{[k]}$ may also be interpreted as kinetic Fisher information estimators.[29] The hyperbolic–parabolic nature of the system of partial differential equations governing compressible fluids further imposes to consider extra correctors associated with density which is a hyperbolic variable. These extra correctors are in the form

$$\gamma^{[k-\frac{1}{2}]} = \rho h^{2k-1}\frac{\partial^{k-1}v}{\sqrt{T}}\cdot\frac{\partial^{k-1}\partial_x\rho}{\rho},$$

where $\partial^k u \cdot \partial^k \partial_x \rho$ is defined as $\sum_{i,|\alpha|=k}(k!/\alpha!)\partial^\alpha u_i \partial^k \partial_i \rho$. These terms are similar to the perturbed quadratic terms introduced by Kawashima[42] in order to obtain hyperbolic variable derivative estimates for linearized equations around equilibrium states and decay estimates and are used here with renormalized variables as well as with powers of h as extra weight factors.

We also establish weighted inequalities in Sobolev and Lebesgue spaces. These inequalities are required in order to establish *a priori* estimates since we are using renormalized variables with powers of temperature and density as weights and since we also consider flows with temperature-dependent thermal conductivity and viscosities. These inequalities assume that a weighted L^∞ norm of the gradients is finite in addition to the L^∞ or BMO norm of the functions. They differ from previous inequalities established for incompressible flows[29] where only the L^∞ or BMO norm of the functions were assumed to be finite. A weighted L^∞ norm of the gradients is required in order to reduce the number of derivation of hyperbolic variables in *a priori* estimates.

Entropic estimates are derived by combining higher order entropy correctors balance equations with weighted inequalities. We obtain differential inequalities for higher order entropy correctors when the quantity

$$\begin{aligned}\chi_\gamma = \|\log\rho\|_{\mathrm{BMO}} + \left\|\frac{v}{\sqrt{T}}\right\|_{L^\infty} + \|\log T\|_{\mathrm{BMO}} \\ + \left\|h\frac{\partial_x\rho}{\rho}\right\|_{L^\infty} + \left\|h\frac{\partial_x v}{\sqrt{T}}\right\|_{L^\infty} + \left\|h\frac{\partial_x T}{T}\right\|_{L^\infty} + \left\|h^2\frac{\partial_x^2 T}{T}\right\|_{L^\infty}\end{aligned}$$

is small enough. As a consequence, we establish that higher order kinetic entropy estimators — obtained by summing up a zeroth order entropy with kinetic entropy correctors — obey conditional entropic principles typically in the following form.

Theorem 1.1. *Let (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations and let $1 \leq k \leq l$. There exist positive constants C_0, $\bar{a} \leq 1$, $\underline{b}$, and δ'_{N} such* *that when* $\chi_\gamma < \delta'_{\mathrm{N}}$ *we have* $|\gamma^{[\frac12]} + \cdots + \gamma^{[k-\frac12]}| \le \frac12(\gamma^{[0]} + \cdots + \gamma^{[k]})$, *and*

$$\partial_t \int_{\mathbb{R}^n} (\gamma^{[0]} + \cdots + \gamma^{[k]} + \bar{a}(\gamma^{[\frac12]} + \cdots + \gamma^{[k-\frac12]}))\, dx$$
$$+ \underline{b} \int_{\mathbb{R}^n} \rho T^{1-\varkappa}(\gamma^{[1]} + \cdots + \gamma^{[k]})\, dx \le 0.$$

These inequalities are investigated in Sec. 5.4 where more precise statements are established. Similar estimates are also obtained with the modified higher order entropy correctors $\tilde{\gamma}^{[k]} = \rho h^{2k}(|\partial^k r|^2 + |\partial^k w|^2 + c_v|\partial^k \tau|^2)$, $k \ge 1$, where $r = \log\rho$, $w = v/\sqrt{T}$, $\tau = \log T$, and with $\tilde{\gamma}^{[k-\frac12]} = \rho h^{2k-1}\partial^{k-1} w{\cdot}\partial^{k-1}\partial_x r$, $k \ge 1$.

Upon integrating the corresponding differential inequalities, *a priori* estimates are obtained for the solutions of the compressible Navier–Stokes equations. These entropic inequalities and the related *a priori* estimates are also scaling invariant. More specifically, in the special case where $\lambda = \mathfrak{a}_\lambda T^\varkappa$, $\eta = \mathfrak{a}_\eta T^\varkappa$, $\kappa = \mathfrak{a}_\kappa T^\varkappa$, and c_v is constant, if $(\rho(t,x), v(t,x), T(t,x))$ is a solution then $(\xi^{2\varkappa-1}\ \zeta\ \rho(\xi\zeta t, \zeta x),\ \xi\ v(\xi\zeta\ t, \zeta\ x),\ \xi^2\ T(\xi\zeta\ t, \zeta\ x))$ is also a solution for any positive ξ and ζ. The higher order entropy estimates are then invariant — up to a multiplicative factor — by these two parameters family of transformations.

Since we have formally $v/\sqrt{T} = \mathcal{O}(\mathrm{Ma})$, $\log(T/T_\infty) = \mathcal{O}(\mathrm{Ma})$, and $\log(\rho/\rho_\infty) = \mathcal{O}(\mathrm{Ma})$, where Ma denotes the Mach number, the constraint that χ_γ remains small may be interpreted as a small Mach number constraint, which is consistent with Enskog expansion.[34] These estimates also provide a thermodynamic interpretation of the corresponding weighted Sobolev norms involving either renormalized derivatives — or derivatives of the renormalized variable — and involving as well the dependence on density and temperature of the local mean free path through the factor h. This factor h ensures in particular that the operator $h\partial_x$ is scale invariant.

Many results have been devoted to the existence of solutions for the compressible Navier–Stokes equations.[24,42,49,58] We mention in particular the local existence result of Nash[57] and the global existence result around equilibrium states of Matsumura and Nishida.[51] More recently, Danchin[15,16] has established global existence of solutions in critical hybrid Besov spaces with minimum regularity for the isentropic as well as the full compressible model around constant equilibrium states, and Hoff[37,38] has also investigated discontinuous solutions with small data. Alazard[2] has further investigated the limit of small Mach number flows for inviscid as well as viscous compressible flows with large temperature variations. For general hyperbolic systems we mention the results of Benzoni and Serre[4] and Serre,[59] and for composite hyperbolic–parabolic systems, the fundamental results of Kawashima.[42,43] With respect to weak solutions, we mention the pioneering work of Lions[49] as well as the fundamental results of Feireisl,[23,24] Bresch and Desjardins,[8,9] Bresch, Desjardins, and Vallet,[10] Mellet and Vasseur,[50] and Feireisl and Novotný.[25]

Various aspects of the *a priori* estimates obtained by these authors are discussed in Sec. 5.4. Estimates for smooth solutions are generally obtained upon deriving the governing equations, multiplying by the solution derivatives, and integrating in space and time, whereas estimates for weak solutions are usually derived from energy and zeroth order entropy estimates as well as by using renormalized equations. The estimates that are closest in spirit to higher order entropy inequalities are the estimates of Bresch and Desjardins.[8,9] Indeed, upon assuming that the volume and shear viscosity only depend on density $\kappa(\rho)$ and $\eta(\rho)$ and are constrained by the relation $\kappa(s) - \frac{2}{n}\eta(s) = 2(s\eta'(s) - \eta(s))$ (in our notation), Bresch and Desjardin have introduced a new entropy in the form (in our notation)

$$\rho T \left| \frac{v}{\sqrt{T}} + 2\frac{\eta'(\rho)}{\sqrt{T}} \frac{\partial_x \rho}{\rho} \right|^2,$$

which present many similarities with the higher entropy correctors $\gamma^{[1]}$ and $\gamma^{[\frac{1}{2}]}$.

Finally, as an example of application of higher order entropic estimates, we establish a global existence theorem around constant equilibrium states provided that $\log(\rho_0/\rho_\infty)$, $\log(T_0/T_\infty)$, and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces, which may be interpreted heuristically as an existence theorem for small Mach number flows. We do not claim originality in these existence results since it is well known that such smooth solution exists, but in its variant proof since it illustrates the use of higher order entropic estimates and the results are formulated in terms of higher order entropy estimators.

In Sec. 2, we discuss the concept of higher order entropies. In Sec. 3, we derive higher order entropies governing equations and in Sec. 4, we establish various weighted inequalities. In Sec. 5, the core of the paper, we establish that higher order entropies satisfy conditional entropic inequalities. Finally, in Sec. 6, as an example of application, we concentrate on global solutions.

2. Higher Order Entropies

In this section we briefly motivate the introduction of higher order entropies by discussing Bernstein equations and Enskog expansion of kinetic entropy.[28,29]

2.1. *A thermodynamic interpretation of Bernstein equations*

For parabolic — or elliptic — scalar equations, *a priori* estimates for derivatives can be obtained by using Bernstein method.[5,47] More specifically, consider — as a simple exemple — the heat equation

$$\partial_t u - \Delta u = 0.$$

Defining $\zeta^{[k]} = |\partial^k u|^2 = \sum_{1\le i_1,\ldots,i_k\le n}(\partial_{i_1}\cdots\partial_{i_k}u)^2$, Bernstein equation for the kth derivative can be written in the form

$$\partial_t \zeta^{[k]} - \Delta\zeta^{[k]} + 2|\partial^{k+1}u|^2 = 0, \tag{2.1}$$

and more generally, for equations with variables coefficients, Bernstein equations are associated with sums of squares of derivatives.[47] With Bernstein method, the higher order derivatives source term $|\partial^{k+1}u|^2$ is discarded, Eq. (2.1) then yields $\partial_t\zeta^{[k]} - \Delta\zeta^{[k]} \leq 0$, and the maximum principle can be used.[5,47] However, one may also directly integrate Bernstein equations to get estimates of the integrals $\int_{\mathbb{R}^n} \zeta^{[k]}\, dx$, and this method is still valid if the flux term $\partial_x{\cdot}(\partial_x\zeta^{[k]})$ is simply a term in divergence form $\partial_x{\cdot}\varphi^{[k]}$ as may be expected for balance equations associated with squares of derivatives of solutions of a system of partial differential equations. We may therefore try to derive equations similar to that of Bernstein for systems of partial differential equations, with non-negative source terms. In this perspective, the structure of (2.1) appears to be formally similar to that of an entropy balance, where $\zeta^{[k]}$, $k \geq 1$, play the rôle of generalized entropies, even though there also exist zeroth order entropies like u^2. In the next section, we introduce a kinetic framework supporting this entropic interpretation.

2.2. *Enskog expansion of Boltzmann kinetic entropy*

In a semi-quantum framework, the state of a polyatomic gas is described by a particle distribution function $f(t,x,c,\textsc{i})$ — governed by Boltzmann equation — where t denotes time, x the n-dimensional cartesian coordinate, c the particle velocity, $\textsc{i}$ the index of the particle quantum state, and $\mathcal{I}$ is the corresponding indexing set.[11,20,22,27] Approximate solutions of Boltzmann's equation can be obtained from a first-order Enskog expansion $f = f^{(0)}(1 + \varepsilon\phi^{(1)} + \mathcal{O}(\varepsilon^2))$ where $f^{(0)}$ is the local Maxwellian distribution, $\phi^{(1)}$ the perturbation associated with the Navier–Stokes regime and ε the usual Enskog formal expansion parameter. The compressible Navier–Stokes equations for polyatomic gases can then be obtained upon taking moments of Boltzmann's equation.[12,22,27]

The kinetic entropy $S^{\mathrm{kin}} = -k_{\mathrm{B}} \sum_{\textsc{i}\in\mathcal{I}} \int_{\mathbb{R}^n} f(\log f - 1)dc$, where k_{B} denotes Boltzmann constant, satisfies the H theorem, i.e. the second principle of thermodynamics. Enskog expansion $f/f^{(0)} = 1 + \varepsilon\phi^{(1)} + \cdots + \epsilon^{2k}\phi^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$ then induces expansions for S^{kin} in the form

$$S^{\mathrm{kin}} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots + \varepsilon^{2k} S^{(2k)} + \mathcal{O}(\varepsilon^{2k+1}), \tag{2.2}$$

where $S^{(0)}$ is the usual zeroth-order fluid entropy evaluated from the Maxwellian distribution $f^{(0)}$ and where $S^{(l)}$ is a sum of terms in the form $k_{\mathrm{B}} \sum_{\textsc{i}\in\mathcal{I}} \int_{\mathbb{R}^n} \prod_{1\leq i\leq l} (\phi^{(i)})^{\nu_i} f^{(0)}\, dc$ with non-negative integers $\nu_i \geq 0$, $1 \leq i \leq l$, such that $l = \sum_{1\leq i\leq l} i\nu_i$. For compressible polyatomic gases after detailed calculations, one can establish that

$$-\rho S^{(2)} = \overline{\lambda}|\partial_x T|^2 + \overline{\kappa}(\partial_x{\cdot} v)^2 + \frac{1}{2}\overline{\eta}|d|^2, \tag{2.3}$$

where T denotes the absolute temperature, ρ the density, v the gas velocity, $d = \partial_x v + \partial_x v^t - \frac{2}{n}(\partial_x{\cdot} v)I$ the nonisotropic part of the strain rate tensor, $|d|^2$ the sum $|d|^2 = \sum_{ij} d_{ij}^2$, and where the scalar coefficients $\overline{\lambda}$, $\overline{\kappa}$, and $\overline{\eta}$ only depend on temperature. In a first approximation, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that

$\overline{\lambda} = (1/2r_{\rm g}c_p)\lambda^2/T^3$, $\overline{\kappa} = (3c_v/4r_{\rm g}c_{\rm int})\kappa^2/T^2$, and $\overline{\eta} = (1/2r_{\rm g})\eta^2/T^2$ where c_p is the constant pressure specific heat per unit mass, c_v the constant volume specific heat per unit mass, $r_{\rm g}$ the gas constant per unit mass, $c_{\rm int}$ the internal specific heat per unit mass, λ the thermal conductivity, η the shear viscosity, κ the volume viscosity, and the actual values of the numerical factors in front of $\overline{\lambda}$, $\overline{\kappa}$, and $\overline{\eta}$ are evaluated here for $n = 3$.

More generally, from the general expression of $\phi^{(l)}$ in the absence of external forces acting on the particles,[29] one can establish that for any $j \geq 2$

$$S^{(j)} = \rho r_{\rm g} \left(\frac{\eta}{\rho\sqrt{r_{\rm g}T}} \right)^j \sum_{\nu} c_\nu \prod_{1\leq|\alpha|\leq j} \left(\frac{\partial_x^\alpha \rho}{\rho} \right)^{\nu_\alpha} \left(\frac{\partial_x^\alpha v}{\sqrt{r_{\rm g}T}} \right)^{\nu'_\alpha} \left(\frac{\partial_x^\alpha T}{T} \right)^{\nu''_\alpha}, \qquad (2.4)$$

where $\nu_\alpha, \nu'_\alpha, \nu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leq|\alpha|\leq j}$ must be such that $\sum_{1\leq|\alpha|\leq j} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = j$ and where the coefficients c_ν are smooth scalar functions of $\log T$ of order unity. In the even case $j = 2k$, after integrations by parts in the integral $\int_{\mathbb{R}^n} S^{(2k)}\, dx$, in order to eliminate spatial derivatives of order strictly greater than k, and by using interpolation inequalities, one obtains that $|\int_{\mathbb{R}^n} S^{(2k)}\, dx|$ is essentially controlled by the integral of

$$\gamma^{[k]} = \rho r_{\rm g} \left(\frac{\eta}{\rho\sqrt{r_{\rm g}T}} \right)^{2k} \left(\left| \frac{\partial_x^k \rho}{\rho} \right|^2 + \left| \frac{\partial_x^k v}{\sqrt{r_{\rm g}T}} \right|^2 + \frac{c_v}{r_{\rm g}} \left| \frac{\partial_x^k T}{T} \right|^2 \right), \qquad (2.5)$$

or equivalently of

$$\widetilde{\gamma}^{[k]} = \rho r_{\rm g} \left(\frac{\eta}{\rho\sqrt{r_{\rm g}T}} \right)^{2k} \left(|\partial_x^k \log\rho|^2 + |\partial_x^k (v/\sqrt{r_{\rm g}T})|^2 + \frac{c_v}{r_{\rm g}} |\partial_x^k \log T|^2 \right), \qquad (2.6)$$

and, in the odd case $j = 2k-1$, $|\int_{\mathbb{R}^n} S^{(2k-1)}\, dx|$ is also controlled by $\int_{\mathbb{R}^n} \gamma^{[k]}\, dx$ and $\int_{\mathbb{R}^n} \gamma^{[k-1]}\, dx$. This suggests quantities in the form $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as $(2k)$th order kinetic entropy correctors — or kinetic entropy deviation estimators.[29] Note that, at variance with $S^{(2)}$, it is not clear that $S^{(2k)}$ has a sign, and this is a motivation for using quantities like $\gamma^{[k]}$ and $\widetilde{\gamma}^{[k]}$ rather than $S^{(2k)}$, beyond simplicity. We are therefore looking for *majorizing entropic correctors* that we are free to modify for convenience, e.g. by multiplying the temperature derivatives by the factor $c_v/r_{\rm g}$. These correctors may also be rescaled by mutiplicative constants depending on k and their temperature dependence may be simplified in accordance with that of transport coefficients. Finally, a similar analysis can also be conducted for the Fisher information and suggests the same quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as higher order kinetic information correctors.

2.3. *Persistence of kinetic entropy*

Denoting by $\gamma^{[0]}$ a non-negative quantity associated with the zeroth order entropy $S^{(0)}$, we investigate kinetic entropy estimators in the form $\gamma^{[0]} + \cdots + \gamma^{[k]}$, with

$0 \leq k \leq l$, for the solutions of a second-order system of partial differential equations modeling a compressible fluid. For this system, the zeroth order entropy $S^{(0)}$ is already of fundamental importance as imposed by its hyperbolic–parabolic structure and the corresponding symmetrizing properties.[27,33,42,44,59] Therefore, we only consider the quantities $\gamma^{[0]}+\cdots+\gamma^{[k]}$, $0 \leq k \leq l$, as a family of mathematical entropy estimators — of kinetic origin — and we will establish that they indeed satisfy conditional entropic inequalities for solutions of compressible fluid equations. This will yield incidentally a thermodynamic interpretation of the corresponding weighted Sobolev norms.

This point of view differs from that of thermodynamic theories that have already considered entropies differing from that of zeroth order, that is, entropies depending on transport fluxes or on macroscopic variable gradients. These generalized entropies have been associated notably with Burnett type equations[12,22] or extended thermodynamics.[56] In both situations, new macroscopic equations are correspondingly obtained, that is, "extended fluid models," which are systems of partial differential equations of higher orders than Navier–Stokes type equations.

3. Higher Order Entropies Governing Equations

We first present the equations governing compressible fluids and then discuss the temperature dependence of transport coefficients as obtained from the kinetic theory of gases. We then derive governing equations for kinetic entropy correctors of arbitrary order.

3.1. *Fluid governing equations*

The conservation equations governing compressible fluids can be written[27,49]

$$\partial_t \rho + \partial_x \cdot (\rho v) = 0, \tag{3.1}$$

$$\partial_t (\rho v) + \partial_x \cdot (\rho v \otimes v + pI) + \partial_x \cdot \Pi = 0, \tag{3.2}$$

$$\partial_t (\rho e) + \partial_x \cdot (\rho e v) + \partial_x \cdot Q = -\Pi : \partial_x v - p \partial_x \cdot v, \tag{3.3}$$

where t denotes time, x the n-dimensional Cartesian coordinate, ρ the density, v the velocity, p the pressure, I the unit tensor, Π the viscous tensor, e the internal energy per unit mass, and Q the heat flux. In these equation, ∂_t denotes partial derivation with respect to time, $\partial_x = (\partial_1, \ldots, \partial_n)^t$ the usual spatial differential operator, and ^{t} the transposition operator. We assume for the sake of notational simplicity that these governing equations are in reduced form in such a way that the specific gas constant r_{g} is taken to be unity. The pressure is given by the state law $p = \rho T$ where T is the temperature and the energy per unit mass e is taken for simplicity in the form $e = c_v T$ where c_v is a constant.

The viscous tensor and the heat flux can be obtained from the kinetic theory of gases and written in the form

$$\Pi = -\kappa(T)\partial_x\cdot vI - \eta(T)\left(\partial_x v + \partial_x v^t - \frac{2}{n}\partial_x\cdot vI\right), \tag{3.4}$$

$$Q = -\lambda(T)\partial_x T, \tag{3.5}$$

where $\kappa(T)$ denotes the volume viscosity, $\eta(T)$ the shear viscosity, and $\lambda(T)$ the thermal conductivity. We will denote by $d = \partial_x v + \partial_x v^t - \frac{2}{n}\partial_x\cdot vI$ the non-isotropic part of the strain rate tensor so that $\Pi = -\kappa\partial_x\cdot vI - \eta d$. The assumptions on the transport coefficients κ, η, and λ — which are smooth functions of temperature — are specified in Sec. 3.2.

Our aim is not to study various boundary conditions and we only consider the case of functions defined on $\mathbb{R}^n$ that are "constant at infinity". From Galilean invariance, we can also choose that v vanishes at infinity. Therefore we only consider smooth solutions such that

$$\rho - \rho_\infty \in C([0,\bar{t}], W^{l,2}) \cap C^1([0,\bar{t}], W^{l-1,2}), \tag{3.6}$$

$$v, T - T_\infty \in C([0,\bar{t}], W^{l,2}) \cap C^1([0,\bar{t}], W^{l-2,2}) \cap L^2((0,\bar{t}), W^{l+1,2}), \tag{3.7}$$

where l is an integer such that $l \geq [n/2] + 3$, i.e. $l > n/2 + 2$, $\bar{t}$ is some positive time, $\rho_\infty > 0$ a fixed positive density and $T_\infty > 0$ a fixed positive temperature. We also assume that ρ and T are such that $\rho \geq \rho_{\min}$ and $T \geq T_{\min}$ where $\rho_{\min} > 0$ and $T_{\min} > 0$ are fixed positive constants. Such smooth solutions are known to exist[27,41–45,51,57,63] either locally in time or globally when the initial state is close to the constant state $(\rho_\infty, 0, T_\infty)$. We use classical notation for functional spaces[1,65] as for instance $W^{k,p} = W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$ for the usual Sobolev space with $k \geq 0$ and $1 \leq p < \infty$, and $W^{-k,p'}$ for its dual where $p' = p/(p-1)$.

Remark 3.1. In the special case where $\lambda = \mathfrak{a}_\lambda T^\varkappa$, $\eta = \mathfrak{a}_\eta T^\varkappa$, $\kappa = \mathfrak{a}_\kappa T^\varkappa$, and c_v is constant, if $(\rho(t,x), v(t,x), T(t,x))$ is a solution of the Navier–Stokes equations (3.1)–(3.3), then

$$(\xi^{2\varkappa-1}\zeta\rho(\xi\zeta t, \zeta x), \quad \xi v(\xi\zeta t, \zeta x), \quad \xi^2 T(\xi\zeta t, \zeta x)), \tag{3.8}$$

is also a solution for any positive ξ and ζ. For arbitrary transport coefficients, the one-parameter family obtained by letting $\xi = 1$ is still a family of solutions. The scaling properties of the incompressible case[29] can also be recovered from (3.8) by letting $\zeta = \xi^{1-2\varkappa}$.

Remark 3.2. All the results obtained in this paper are also valid if the internal energy e per unit mass is taken to be $e = e_0 + \int_0^T c_v(s)\,ds$ with a heat capacity coefficient c_v depending on temperature in such a way that

$$\underline{c} \leq c_v \leq \overline{c}, \quad T^\sigma|\partial_T^\sigma c_v| \leq \overline{c}_\sigma, \quad \sigma \geq 1,$$

where $\underline{c} > 0$, $\overline{c} > 0$, and $\overline{c}_\sigma > 0$, $\sigma \geq 1$, are positive constants. We will not explicit the corresponding results for the sake of simplicity.

Remark 3.3. The dimension n appearing in the coefficient $2/n$ of the viscous tensor (3.4) is normally the full spatial dimension, that is, the dimension n' of the velocity phase space of the associated kinetic model. We may still assume that the spatial dimension of the model has been reduced, that is, the equations are considered in $\mathbb{R}^n$ with $n \leq n'$. The full size viscous tensor Π' is then a matrix of order n', and the corresponding coefficient is $2/n'$. However, if we denote by Π the upper left block of size n of Π', that is, the useful part of Π', we may rewrite Π in the form

$$\Pi = -\left(\kappa + \left(\frac{2}{n} - \frac{2}{n'}\right)\eta\right)\partial_x\cdot vI - \eta\left(\partial_x v + \partial_x v^t - \frac{2}{n}\partial_x\cdot vI\right), \tag{3.9}$$

where I is the unit tensor in n dimensions. Therefore, using a smaller dimension n instead of the full dimension n' in the coefficient of the viscous tensor is equivalent to increasing the volume viscosity by the amount $2\eta(n'-n)/nn'$. As a practical example, we have $n' = 3$ in our physical world, but we may still consider a fluid model with $n = 2$, and upon modifying the volume viscosity, the coefficient $2/3$ in Π can be transformed into 1.

Remark 3.4. The fluid governing equations have been derived[3,19] by Navier in 1822, Cauchy in 1823, Poisson in 1831, Saint-Venant in 1843 — from an unpublished work of 1837 — and Stokes in 1845.

3.2. *Temperature dependent transport coefficients*

We discuss in this section the temperature dependence of transport coefficients in a dilute gas. The situation of a dense gas will be addressed in Sec. 3.7 for completeness. Only the assumptions on transport coefficients associated with a dilute gas — as derived from the kinetic theory of gases — will be used in this paper.

Thermal conductivity, shear viscosity, and volume viscosity of a polyatomic dilute gas depend on temperature

$$\lambda = \lambda(T), \quad \eta = \eta(T), \quad \kappa = \kappa(T), \tag{3.10}$$

as shown by the kinetic theory of gases.[12,22,27] When one term Sonine–Wang–Chang–Uhlenbeck polynomial expansions are used to evaluate perturbed distribution functions, the coefficients λ/c_v, η and κ are found in the form $\lambda/c_v = \mathfrak{a}_\lambda T^{1/2}/\Omega^{(2,2)*}$, $\eta = \mathfrak{a}_\eta T^{1/2}/\Omega^{(2,2)*}$, and $\kappa/\eta = \mathfrak{a}_\kappa c^{\mathrm{int}}\xi^{\mathrm{int}}/c_v^2$ where $\mathfrak{a}_\lambda$, $\mathfrak{a}_\eta$ and $\mathfrak{a}_\kappa$ are constants, $\Omega^{(2,2)*}$ a reduced collision integral, c^{int} the internal heat capacity per unit mass, and ξ^{int} a collision number associated with internal energy relaxation. Note in particular that the ratios $\lambda/c_v\eta$ and κ/η are bounded. For the *rough rigid sphere* model for instance, we have exactly[12,22] $\lambda/c_v = \mathfrak{a}_\lambda T^{1/2}$, $\eta = \mathfrak{a}_\eta T^{1/2}$ and $\kappa = \mathfrak{a}_\kappa T^{1/2}$. Similarly, for particles interacting as point centers of repulsion with an interaction potential $V = c/\mathrm{r}^\nu$, where r is the distance between two particles, one establishes[12,22] that $\Omega^{(2,2)*}$ is proportional to $T^{-2/\nu}$ so that we have $\lambda/c_v = \mathfrak{a}_\lambda T^\varkappa$, and $\eta = \mathfrak{a}_\eta T^\varkappa$ with $\varkappa = 1/2 + 2/\nu$, and κ inherits the same scaling $\kappa = \mathfrak{a}_\kappa T^\varkappa$

if we assume that c^{int}, ξ^{int} , and c_v are constants. The temperature exponent $\varkappa$ then varies from $\varkappa = 1/2$ for rigid spheres with $\nu = \infty$ up to $\varkappa = 1$ for Maxwell molecules with $\nu = 4$.

More generally, consider particles interacting with a Lennard–Jones ν–ν' potential $V = 4\varepsilon((\sigma/\mathrm{r})^{\nu} - (\sigma/\mathrm{r})^{\nu'})$ where σ denotes the collision diameter, ε the potential well depth, and ν, ν' are intergers with $\nu > \nu'$ and typical values $\nu = 12$, $\nu' = 6$.[12,22] Collision integrals like $\Omega^{(2,2)*}$ then only depend on the reduced temperature $k_{\mathrm{B}}T/\varepsilon$, and, when $k_{\mathrm{B}}T/\varepsilon$ is large, the repulsive part $\mathrm{r}^{-\nu}$ is dominant[12] so that collision integrals behave like T^s with $s = 1/2 + 2/\nu$ for large T. In particular, the logarithm $\log \Omega^{(2,2)*}$ has linear asymptotes as function of $\log T$, and $d^k \log \Omega^{(2,2)*}/d(\log T)^k$ is bounded for any $k \geq 1$. In addition, classical models indicate that c^{int}, ξ^{int} and c_v converge towards constants for large temperatures.[27] As a consequence, $\log\lambda$, $\log\eta$, and $\log\kappa$ have parallel linear asymptotes as function of $\log T$, and $d^k \log\lambda/d(\log T)^k$, $d^k \log\eta/d(\log T)^k$, and $d^k \log\kappa/d(\log T)^k$ are bounded for any $k \geq 1$, or equivalently, $(1/\lambda)T^k d^k\lambda/dT^k$, $(1/\eta)T^k d^k\eta/dT^k$, and $(1/\kappa)T^k d^k\kappa/dT^k$ are bounded for any $k \geq 1$.

Similar results are also obtained when more than one term are taken into account in orthogonal polynomial expansions of perturbed distribution functions. Indeed, all collision integrals $\Omega^{(i,j)*}$, $i, j \geq 1$, have a common temperature behavior, that is, all ratios of collision integrals are bounded, as for instance for Lennard–Jones or Stockmayer potentials.[22,27] These collision integrals are then used to define the coefficients of the transport linear systems which thus share a common temperature scaling. As a consequence, the transport coefficients, which are obtained through solutions of transport linear systems, inherit a common temperature scaling.[27]

On the other hand, in our particular application, we are only interested in solutions such that $T \geq T_{\min}$, where $T_{\min}$ is fixed and positive. In this situation, the behavior of transport coefficients for small temperatures is not relevant and only the repulsive part of the interaction potential between particles plays a role. Therefore, from a mathematical point of view, since we are not interested in small temperatures, we assume that λ, η, and κ are $C^{\infty}(0,\infty)$, that there exist $\varkappa$, $\underline{\mathfrak{a}} > 0$, and $\overline{\mathfrak{a}} > 0$ with

$$\underline{\mathfrak{a}}T^{\varkappa} \leq \lambda/c_v \leq \overline{\mathfrak{a}}T^{\varkappa}, \quad \underline{\mathfrak{a}}T^{\varkappa} \leq \eta \leq \overline{\mathfrak{a}}T^{\varkappa}, \quad \underline{\mathfrak{a}}T^{\varkappa} \leq \kappa \leq \overline{\mathfrak{a}}T^{\varkappa}, \tag{3.11}$$

and that, for any integer $\sigma \geq 1$, there exists $\overline{\mathfrak{a}}_{\sigma} > 0$ with

$$T^{\sigma}(|\partial_T^{\sigma}\lambda| + |\partial_T^{\sigma}\eta| + |\partial_T^{\sigma}\kappa|) \leq \overline{\mathfrak{a}}_{\sigma}T^{\varkappa}. \tag{3.12}$$

Kinetic theory suggests that $1/2 \leq \varkappa \leq 1$ but the situations where $0 \leq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate from a mathematical point of view.

Remark 3.5. Theoretical calculations and experimental measurements have shown that the viscosity ratio κ/η is of order unity for polyatomic gases.[6,12,22] Using a one or two terms expansion in Sonine–Wang–Chang–Uhlenbeck polynomials for the perturbed distribution associated with volume viscosity, it is established for instance that $\kappa/\eta = \frac{\pi}{4} r_{\mathrm{g}} c^{\mathrm{int}} \xi^{\mathrm{int}} / c_v^2$ for a polyatomic gas. The collision number ξ^{int} associated with internal energy relaxation is usually taken to be a simple decreasing function of temperature and the internal heat capacity per unit mass c^{int} is associated with the various internal energy modes like rotation, vibration or electronic. In particular, the internal heat capacity is such that $c^{\mathrm{int}} \geq r_{\mathrm{g}}$ for linear molecules and $c^{\mathrm{int}} \geq \frac{3}{2} r_{\mathrm{g}}$ for nonlinear molecules solely from rotational degrees of freedom. Volume viscosity also arise in dense gases and in liquids so that its absence in monatomic dilute gases is an exception rather than a rule.[6,22]

3.3. *Higher order kinetic entropy estimators*

Following the physical *ansatz* (2.5) and taking into account the simplifications associated with the temperature dependence of transport coefficients (3.11) and with a specific gas constant taken to be unity, we define the $(2k)$th order kinetic entropy corrector $\gamma^{[k]}$ by

$$\gamma^{[k]} = \rho h^{2k} \left(\frac{|\partial^k \rho|^2}{\rho^2} + \frac{|\partial^k v|^2}{T} + c_v \frac{|\partial^k T|^2}{T^2} \right), \tag{3.13}$$

where $h = 1/(T^{\frac{1}{2}-\varkappa}\rho)$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we denote as usual by ∂^α the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and by $|\alpha|$ its order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the square of kth derivatives of a scalar function ϕ, like T, ρ, or v_i, $1 \leq i \leq n$, is defined by

$$|\partial^k \phi|^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha \phi)^2 = \sum_{1 \leq i_1, \ldots, i_k \leq n} (\partial_{i_1} \cdots \partial_{i_k} \phi)^2, \tag{3.14}$$

where $k!/\alpha!$ are the multinomial coefficients.[14,60] Similarly, for a vector function like v we define $|\partial^k v|^2 = \sum_{1 \leq i \leq n} |\partial^k v_i|^2$.

This choice of $\gamma^{[k]}$ yields more convenient higher order entropic estimates. Calculations show that it eliminates various quadratic terms associated with hyperbolic variables, thanks to symmetry properties. This choice can also be associated with symmetrized forms of the system of partial differential equations. Denoting $\mathrm{U} = (\rho, \rho v, \rho(e + \frac{1}{2}|v|^2))^t$ the conservative variable, $\mathrm{V} = -(\partial_{\mathrm{U}} S^{(0)})^t$ the entropic variable, $\mathrm{Z} = (\rho, v, T)^t$ the natural variable, which is also a normal variable,[33,44] and defining the matrix $\overline{A}_0 = (\partial_{\mathrm{Z}} \mathrm{V})^t \partial_{\mathrm{U}} \mathrm{V} (\partial_{\mathrm{Z}} \mathrm{V})$ associated with normal forms of the system of partial differenial equations,[33,44] one can rewrite the higher order entropy correctors in the form $\gamma^{[k]} = h^{2k} \langle \partial^k \mathrm{Z}, \overline{A}_0 \partial^k \mathrm{Z} \rangle$, where h is the weight associated with the dependence of the local mean free path $l = \eta/\rho\sqrt{r_{\mathrm{g}} T}$ on density and temperature. This choice of $\gamma^{[k]}$ can also be associated with a "spatial gradient" Fisher information with for instance $\gamma^{[1]} = h^2 \sum_{\mathrm{I} \in \mathcal{I}} k_{\mathrm{B}} \int_{\mathbb{R}^n} |\partial_x \log f^{(0)}|^2 f^{(0)} \, dc$, where $f^{(0)}$ is the local Maxwellian distribution discussed in Sec. 2.2.

Remark 3.6. We define similarly the pth power of derivatives $|\partial^k\phi|^p$ by

$$|\partial^k\phi|^p = \sum_{|\alpha|=k} \frac{k!}{\alpha!}(\partial^\alpha\phi)^p = \sum_{1\le i_1,\ldots,i_k\le n} (\partial_{i_1}\cdots\partial_{i_k}\phi)^p, \tag{3.15}$$

and these definitions (3.14)–(3.15) are compatible with the classical definition already used in Sec. 2.1 when $p = 2$. These natural definitions also simplify the analytic form of higher order entropies governing equations. In agreement with (3.14) we also set for future use

$$\partial^k\phi\partial^k\psi = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\partial^\alpha\phi\partial^\alpha\psi, \quad \partial^k v\cdot\partial^k\partial_x\rho = \sum_{\substack{|\alpha|=k\\ 1\le i\le n}} \frac{k!}{\alpha!}\partial^\alpha v_i\partial^\alpha\partial_i\rho. \tag{3.16}$$

In order to recast the zeroth order entropy balance equation into a more convenient form we introduce a modified zeroth order entropy $\gamma^{[0]}$. The mathematical fluid entropy $-S^{(0)}$ can be shown to be a strictly convex function of the conservative variables[27,44] $\mathrm{U} = (\rho, \rho v, \rho(e+\frac{1}{2}v\cdot v))^t$. Denoting by $\mathrm{E}^{\mathrm{tot}} = \rho(e+\frac{1}{2}v\cdot v)$ the total energy per unit volume, we define $\gamma^{[0]} = \mathsf{C}_0\psi^{[0]}$ where $\psi^{[0]}$ is the modified zeroth order entropy

$$\psi^{[0]} = -S^{(0)} + S^{(0)}_\infty + (\partial_\rho S^{(0)})_\infty(\rho-\rho_\infty) + (\partial_{\mathrm{E}^{\mathrm{tot}}} S^{(0)})_\infty(\mathrm{E}^{\mathrm{tot}} - \mathrm{E}^{\mathrm{tot}}_\infty),$$

and C_0 is a positive constant that will be taken large enough. The zeroth order term $\gamma^{[0]}$ is easily rewritten in the form

$$\gamma^{[0]}/\mathsf{C}_0 = \left(\rho\log\left(\frac{\rho}{\rho_\infty}\right) - (\rho-\rho_\infty)\right) + \frac{1}{2}\rho\frac{v^2}{T_\infty} + \rho c_v\left(\frac{T-T_\infty}{T_\infty} - \log\left(\frac{T}{T_\infty}\right)\right). \tag{3.17}$$

Thanks to the fact that v and T are parabolic variables, we can expect source terms in the form $|\partial^{k+1}T/T|^2$ and $|\partial^{k+1}v/\sqrt{T}|^2$ to appear in the governing equation for $\gamma^{[k]}$ — up to weight factors. However, since ρ is a hyperbolic variable, there will be no such corresponding source term $|\partial^{k+1}\rho/\rho|^2$ for density. *A priori* estimates for density derivatives and more generally of hyperbolic variables derivatives indeed require to introduce extra entropic corrector terms. These extra corrector terms will yield source terms in the form $|\partial^k\rho/\rho|^2$. These terms are similar to the perturbed quadratic terms introduced by Kawashima[42] in order to obtain hyperbolic variable derivative estimates for linearized equations around equilibrium states and decay estimates.[42] They are used here with renormalized variables, as well as with powers of h as extra weights factors, in order to obtain higher order entropic principles. More specifically, we define the quantity $\gamma^{[k-\frac{1}{2}]}$ by

$$\gamma^{[k-\frac{1}{2}]} = \rho h^{2k-1}\frac{\partial^{k-1}v}{\sqrt{T}}\cdot\frac{\partial^{k-1}\partial_x\rho}{\rho}, \tag{3.18}$$

and we will see that in the $\gamma^{[k-\frac{1}{2}]}$ governing equation there is a source term in the form $|\partial^k\rho/\rho|^2$ — up to weight factors. From a physical point of view, we also note that $\gamma^{[k-\frac{1}{2}]}$ is of the general form (2.4) for $S^{(2k-1)}$. Finally, we define the $(2k)$th order kinetic entropy estimator by

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1\le i\le k} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]}), \quad k \ge 0, \tag{3.19}$$

where a is a parameter that will be chosen small enough. The quantities $\gamma^{[i-\frac{1}{2}]}$, $1 \le i \le k$, are multiplied by the small rescaling factor a in (3.19) so as not to modify the majorizing properties of the correctors $\gamma^{[k]}$, $k \ge 0$.

Similarly, following the physical *ansatz* (2.6), we define the modified $(2k)$th order kinetic entropy corrector $\tilde{\gamma}^{[k]}$ by

$$\tilde{\gamma}^{[k]} = \rho h^{2k}(|\partial^k r|^2 + |\partial^k w|^2 + c_v|\partial^k \tau|^2), \tag{3.20}$$

where $r = \log\rho$, $w = v/\sqrt{T}$, and $\tau = \log T$. We correspondingly define

$$\tilde{\gamma}^{[k-\frac{1}{2}]} = \rho h^{2k-1}\partial^{k-1}w\cdot\partial^{k-1}\partial_x r, \tag{3.21}$$

$\tilde{\gamma}^{[0]} = \gamma^{[0]}$, and introduce the modified $(2k)$th order kinetic entropy estimators

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \sum_{1\le i\le k} (\tilde{\gamma}^{[i]} + a\tilde{\gamma}^{[i-\frac{1}{2}]}), \quad k \ge 0. \tag{3.22}$$

The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$, as well as the estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$, will be shown to have similar properties and both may be used to derive *a priori* estimates. Strictly speaking, we should term $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ "$(2k)$th order kinetic entropy correctors" or "$(2k)$th order kinetic entropy deviation estimators", and $\gamma^{[k-\frac{1}{2}]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$ "$(2k-1)$th order kinetic entropy correctors", and $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "mathematical $(2k)$th order entropies", or "$(2k)$th order kinetic entropy estimators". However, we will often informally term $\gamma^{[k]}$, $\tilde{\gamma}^{[k]}$, $\gamma^{[k-\frac{1}{2}]}$, $\tilde{\gamma}^{[k-\frac{1}{2}]}$, $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "higher order entropies".

Remark 3.7. Entropic correctors can also be defined by using the derivatives of the strain rate tensor $\partial^{k-1}d$ instead of that of velocity $\partial^k v$. We have chosen to work with the derivatives of velocity $\partial^k v$ for the sake of simplicity. It is also possible to define extra entropic correctors in the form $\rho h^{2k-1}\partial^{k-1}(\partial_x\cdot v)\partial^{k-1}\rho/(\sqrt{T}\rho)$ and $\rho h^{2k-1}\partial^{k-1}(\partial_x\cdot w)\partial^{k-1}r$ but their properties are similar to that of $\gamma^{[k-\frac{1}{2}]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$. Entropic estimators can also be defined in the form

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1\le i\le k} \theta^i(\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]}), \quad k \ge 0, \tag{3.23}$$

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \sum_{1\le i\le k} \theta^i(\tilde{\gamma}^{[i]} + a\tilde{\gamma}^{[i-\frac{1}{2}]}), \quad k \ge 0, \tag{3.24}$$

where θ is a fixed parameter smaller than unity, but the corresponding results are similar to the simpler situation $\theta = 1$.

Remark 3.8. As suggested by a referee, it is also possible to define higher order entropic correctors of fractional order $\gamma^{[s]}$, $s > 0$, upon defining fractional derivatives with Fourier transform. The explicit conservation equations for entropy correctors obtained in the next sections have then to be replaced by communators but such generalizations are out of the scope of the present study.

3.4. *Balance equation for $\gamma^{[k]}$ and $\gamma^{[k-\frac{1}{2}]}$*

Our aim is to establish balance equations for $\gamma^{[k]}$ and $\gamma^{[k-\frac{1}{2}]}$. In Sec. 5, we will use these equations to derive *a priori* estimates and to establish that $\Gamma^{[k]}$ satisfies conditional entropic principles.

Proposition 3.9. *Let (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7) *and let $1 \le k \le l$. Then the following balance equation holds in $\mathcal{D}'((0,\bar{t}) \times \mathbb{R}^n)$ and $L^1((0,\bar{t}), W^{l-k-1,1})$*

$$\partial_t \gamma^{[k]} + \partial_x\cdot(v\gamma^{[k]}) + \partial_x\cdot\varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0, \tag{3.25}$$

where $\varphi_\gamma^{[k]}$, $\pi_\gamma^{[k]}$, $\Sigma_\gamma^{[k]}$, $\omega_\gamma^{[k]} \in L^1((0,\bar{t}), W^{l-k,1})$. The term $\pi_\gamma^{[k]}$ is given by

$$\pi_\gamma^{[k]} = 2g^2h^{2(k+1)}\left(\frac{\lambda}{T^\varkappa}\frac{|\partial^{k+1}T|^2}{T^2} + \frac{\eta}{T^\varkappa}\frac{|\partial^{k+1}v|^2}{T} + \frac{\kappa + \frac{n-2}{n}\eta}{T^\varkappa}\frac{|\partial^k(\partial_x\cdot v)|^2}{T}\right), \tag{3.26}$$

where $g = \rho T^{\frac{1}{2}(1-\varkappa)}$ and $h = 1/(\rho T^{\frac{1}{2}-\varkappa})$. The term $\Sigma_\gamma^{[k]}$ is in the form

$$\Sigma_\gamma^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi} T^{\sigma-\varkappa}\partial_T^\sigma \phi \Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{c_v T^\varkappa} g^2 h^{2(k+1)} \frac{|\partial^k\rho|^2}{\rho^2}\frac{\Delta T}{T}, \tag{3.27}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sum extends over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \le \sigma \le k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = gh^{k+1} \prod_{1\le|\alpha|\le k+1} \left(\frac{\partial^\alpha \rho}{\rho}\right)^{\nu_\alpha} \left(\frac{\partial^\alpha v}{\sqrt{T}}\right)^{\nu'_\alpha} \left(\frac{\partial^\alpha T}{T}\right)^{\nu''_\alpha}, \tag{3.28}$$

where v denotes — with a slight abuse of notation — any of its components $v_1, \ldots, v_n$, and ν must be such that $\sum_{1\le|\alpha|\le k+1} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k+1$, $\sum_{|\alpha|=k+1} \nu_\alpha = 0$, $\sum_{|\alpha|=k+1} (\nu'_\alpha + \nu''_\alpha + \mu'_\alpha + \mu''_\alpha) \le 1$. Furthermore the term $\omega_\gamma^{[k]}$ is given by

$$\omega_\gamma^{[k]} = \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}, \tag{3.29}$$

where the summation extends over $\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k$, $\sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = k+1$, and $c_{\nu\mu}$ are constants. Finally the flux $\varphi_\gamma^{[k]} = (\varphi_{\gamma 1}^{[k]}, \ldots, \varphi_{\gamma n}^{[k]})$ is in the form

$$\varphi_{\gamma l}^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi l} T^{\sigma-\varkappa}\partial_T^\sigma \phi h \Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\nu\mu} c_{\nu\mu l} h \Pi_\nu^{(k)}\Pi_\mu^{(k)}.$$

Proof. The proof — given in Appendix A — is lengthy and tedious but presents no serious difficulties. □

In Proposition 3.9, $\varphi_\gamma^{[k]}$ is a flux and $\pi_\gamma^{[k]}$, $\Sigma_\gamma^{[k]}$, $\omega_\gamma^{[k]}$ are source terms. The source term $\pi_\gamma^{[k]}$ only contains the temperature and velocity $(k+1)$th derivatives squared as expected from the hyperbolic–parabolic nature of system of partial differential equations. In the products $\Pi_\nu^{(k+1)}$ appearing in $\Sigma_\gamma^{[k]}$ there is a total number of $k+1$ derivations and there is no derivative of order $k+1$ of density. Moreover, there is at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ so that one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split between two or more derivative factors. The products $\Pi_\mu^{(k+1)}$ appearing in $\omega_\gamma^{[k]}$ are such that $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha+\mu''_\alpha)=0$ and are always split between several derivative factors.

We investigate the $\gamma^{[k-\frac{1}{2}]}$ balance equation for compressible fluids with temperature dependent transport coefficients.

Proposition 3.10. *Let (ρ,v,T) be a smooth solution of the compressible Navier–Stokes equations (3.1)–(3.5) with regularity (3.6), (3.7) and let $1\le k\le l$. Then the following balance equation holds in $\mathcal{D}'((0,\bar t)\times\mathbb{R}^n)$ and $L^1((0,\bar t),W^{l-k-1,1})$*

$$\partial_t\gamma^{[k-\frac{1}{2}]}+\partial_x\cdot(v\gamma^{[k-\frac{1}{2}]})+\partial_x\cdot\varphi_\gamma^{[k-\frac{1}{2}]}+\pi_\gamma^{[k-\frac{1}{2}]}+\Sigma_\gamma^{[k-\frac{1}{2}]}+\omega_\gamma^{[k-\frac{1}{2}]}=0, \tag{3.30}$$

where $\varphi_\gamma^{[k-\frac{1}{2}]}$, $\pi_\gamma^{[k-\frac{1}{2}]}$, $\Sigma_\gamma^{[k-\frac{1}{2}]}$, $\omega_\gamma^{[k-\frac{1}{2}]}\in L^1((0,\bar t),W^{l-k,1})$. The term $\pi_\gamma^{[k-\frac{1}{2}]}$ is given by

$$\pi_\gamma^{[k-\frac{1}{2}]}=g^2h^{2k}\frac{|\partial^k\rho|^2}{\rho^2}, \tag{3.31}$$

where $g=\rho T^{\frac{1}{2}(1-\varkappa)}$ and $h=1/(\rho T^{\frac{1}{2}-\varkappa})$. The term $\Sigma_\gamma^{[k-\frac{1}{2}]}$ is in the form

$$\Sigma_\gamma^{[k-\frac{1}{2}]}=\sum_{\sigma\nu\mu\phi}c_{\sigma\nu\mu\phi}T^{\sigma-\varkappa}\partial_T^\sigma\phi\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}-\frac{\kappa+\frac{2(n-1)}{n}\eta}{T^\varkappa}g^2h^{2k+1}\frac{\partial^k(\partial_x\cdot v)}{\sqrt{T}}\frac{\partial^k\rho}{\rho}, \tag{3.32}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sums are over $\phi\in\{\lambda,\eta,\kappa\}$, $0\le\sigma\le k$, $\nu=(\nu_\alpha,\nu'_\alpha,\nu''_\alpha)_{1\le|\alpha|\le k}$, $\mu=(\mu_\alpha,\mu'_\alpha,\mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha,\nu'_\alpha,\nu''_\alpha,\mu_\alpha,\mu'_\alpha,\mu''_\alpha\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$. The products $\Pi_\nu^{(k)}$ and $\Pi_\mu^{(k+1)}$ are defined as in the governing equation for $\gamma^{[k]}$ and $\sum_{1\le|\alpha|\le k}|\alpha|(\mu_\alpha+\mu'_\alpha+\mu''_\alpha)=k+1$. Furthermore the term $\omega_\gamma^{[k-\frac{1}{2}]}$ is given by

$$\omega_\gamma^{[k-\frac{1}{2}]}=\sum_{\nu\mu}c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k)}+g^2h^{2k}\frac{\partial^kT}{T}\frac{\partial^k\rho}{\rho}-g^2h^{2k}\frac{|\partial^{k-1}(\partial_x\cdot v)|^2}{T}, \tag{3.33}$$

where $c_{\nu\mu}$ are constants and at least one of the two products $\Pi_\nu^{(k)}$ or $\Pi_\mu^{(k)}$ is split between two or more derivative factors. Finally the flux $\varphi_\gamma^{[k-\frac{1}{2}]}=(\varphi_{\gamma 1}^{[k-\frac{1}{2}]},\dots,\varphi_{\gamma n}^{[k-\frac{1}{2}]})$ is in the form

$$\varphi_{\gamma l}^{[k-\frac{1}{2}]}=\sum_{\nu\mu}c_{\nu\mu l}h\Pi_\nu^{(k-1)}\Pi_\mu^{(k)}.$$

Proof. The proof — lengthy and tedious — presents no serious difficulties and is similar to that of Proposition 3.9. □

In Proposition 3.10, $\varphi_\gamma^{[k-\frac{1}{2}]}$ is a flux and $\pi_\gamma^{[k-\frac{1}{2}]}$, $\Sigma_\gamma^{[k-\frac{1}{2}]}$, $\omega_\gamma^{[k-\frac{1}{2}]}$ are source terms. The term $\pi_\gamma^{[k-\frac{1}{2}]}$ will help to complete the missing gradient terms in $\pi_\gamma^{[k-1]}$. The products $\Pi_\mu^{(k+1)}$ in $\Sigma_\gamma^{[k-\frac{1}{2}]}$ do not contain derivatives of order $k+1$ and are thus split between two or more derivative factors.

3.5. *Balance equation for $\tilde{\gamma}^{[k]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$*

We establish balance equations for $\tilde{\gamma}^{[k]}$, and $\tilde{\gamma}^{[k-\frac{1}{2}]}$. In Sec. 5, we will use these equations to derive *a priori* estimates and to establish that $\widetilde{\Gamma}^{[k]}$ satisfies conditional entropic principles.

Proposition 3.11. *Let (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7) *and let $1 \leq k \leq l$. Then the following balance equation holds in $\mathcal{D}'((0,\bar{t}) \times \mathbb{R}^n)$ and $L^1((0,\bar{t}), W^{l-k-1,1})$*

$$\partial_t \tilde{\gamma}^{[k]} + \partial_x \cdot (v \tilde{\gamma}^{[k]}) + \partial_x \cdot \varphi_{\tilde{\gamma}}^{[k]} + \pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]} = 0, \tag{3.34}$$

where $\varphi_{\tilde{\gamma}}^{[k]}$, $\pi_{\tilde{\gamma}}^{[k]}$, $\Sigma_{\tilde{\gamma}}^{[k]}$,
$\varphi_{\tilde{\gamma}}^{[k]}$, $\pi_{\tilde{\gamma}}^{[k]}$, $\Sigma_{\tilde{\gamma}}^{[k]}$, $\omega_{\tilde{\gamma}}^{[k]} \in L^1((0,\bar{t}), W^{l-k,1})$. *The term $\pi_{\tilde{\gamma}}^{[k]}$ is given by*

$$\pi_{\tilde{\gamma}}^{[k]} = 2g^2 h^{2(k+1)} \left(\frac{\lambda}{e^{\varkappa\tau}} |\partial^{k+1}\tau|^2 + \frac{\eta}{e^{\varkappa\tau}} |\partial^{k+1} w|^2 + \frac{\kappa + \frac{n-2}{n}\eta}{e^{\varkappa\tau}} |\partial^k (\partial_x \cdot w)|^2 \right), \tag{3.35}$$

where $g = \rho T^{\frac{1}{2}(1-\varkappa)}$ and $h = 1/(\rho T^{\frac{1}{2}-\varkappa})$. The term $\Sigma_{\tilde{\gamma}}^{[k]}$ is in the form

$$\Sigma_{\tilde{\gamma}}^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi} e^{-\varkappa\tau} \partial_\tau^\sigma \phi \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{e^{\varkappa\tau} c_v} g^2 h^{2(k+1)} |\partial^k r|^2 \Delta\tau, \tag{3.36}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sum extends over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{0\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{0\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = g h^{k+1} \prod_{0\leq|\alpha|\leq k+1} (\partial^\alpha r)^{\nu_\alpha} \ (\partial^\alpha w)^{\nu'_\alpha} \ (\partial^\alpha \tau)^{\nu''_\alpha}, \tag{3.37}$$

where w denotes — with a slight abuse of notation — any of its components $w_1, \ldots, w_n$, and μ and ν must be such that $\sum_{1\leq|\alpha|\leq k+1} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k+1$, $\sum_{|\alpha|=k+1} \nu_\alpha = 0$, $\sum_{|\alpha|=0} (\nu_\alpha + \nu''_\alpha) = 0$, $\sum_{|\alpha|=k+1} (\nu'_\alpha + \nu''_\alpha + \mu'_\alpha + \mu''_\alpha) \leq 1$. Furthermore the term $\omega_{\tilde{\gamma}}^{[k]}$ is given by

$$\begin{aligned}\omega_{\tilde{\gamma}}^{[k]} = &\sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + g^2 h^{2k+1} \partial^k \tau \partial^k (\partial_x \tau) \cdot w + g^2 h^{2k+1} \partial^k r \partial^k (\partial_x \tau) \cdot w \\ &- \frac{1}{c_v} g^2 h^{2k+1} \partial^k w \cdot w \partial^k (\partial_x \cdot w) - \frac{1}{2c_v} g^2 h^{2k+1} \partial^k w \cdot w \partial^k (\partial_x \tau) \cdot w,\end{aligned} \tag{3.38}$$

where we use similar notation for $\Pi_\nu^{(k)}$ *as for* $\Pi_\mu^{(k+1)}$*, the summation extends over* $\sum_{1\leq|\alpha|\leq}|\alpha|(\nu_\alpha+\nu'_\alpha+\nu''_\alpha)=k$, $\sum_{1\leq|\alpha|\leq k}|\alpha|(\mu_\alpha+\mu'_\alpha+\mu''_\alpha)=k+1$, *and* $c_{\nu\mu}$ *are constants. Finally the flux* $\varphi_{\tilde{\gamma}}^{[k]}=(\varphi_{\tilde{\gamma}1}^{[k]},\ldots,\varphi_{\tilde{\gamma}n}^{[k]})$ *is in the form*

$$\varphi_{\tilde{\gamma}l}^{[k]}=\sum_{\sigma\nu\mu\phi}c_{\sigma\nu\mu\phi l}e^{-\varkappa\tau}\partial_\tau^\sigma\phi h\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}+\sum_{\nu\mu}c_{\nu\mu l}h\Pi_\nu^{(k)}\Pi_\mu^{(k)}.$$

Proof. The proof is similar to that of Proposition 3.9 and is omitted. □

In Proposition 3.11, $\varphi_{\tilde{\gamma}}^{[k]}$ is a flux and $\pi_{\tilde{\gamma}}^{[k]}$, $\Sigma_{\tilde{\gamma}}^{[k]}$, $\omega_{\tilde{\gamma}}^{[k]}$ are source terms. The term $\pi_{\tilde{\gamma}}^{[k]}$ only contains the temperature and velocity $(k+1)$th derivatives squared as expected from the hyperbolic–parabolic structure of system of partial differential equations. In the products $\Pi_\nu^{(k+1)}$ appearing in $\Sigma_{\tilde{\gamma}}^{[k]}$ there is a total of $k+1$ derivations and there is no derivative of order $k+1$ of density. Note that powers of the renormalized velocity w may appear in $\Pi_\nu^{(k+1)}$ but not of τ or r. In addition, there is at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ so that one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split between two or more derivative factors. The products $\Pi_\mu^{(k+1)}$ appearing in $\omega_{\tilde{\gamma}}^{[k]}$ are such that $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha+\mu''_\alpha)=0$ and are thus split between two or more derivatives factors.

Proposition 3.12. *Let* (ρ,v,T) *be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7) *and let* $1\leq k\leq l$*. Then the following balance equation holds in* $\mathcal{D}'((0,\bar{t})\times\mathbb{R}^n)$ *and* $L^1((0,\bar{t}),W^{l-k-1,1})$

$$\partial_t\tilde{\gamma}^{[k-\frac12]}+\partial_x\cdot(v\tilde{\gamma}^{[k-\frac12]})+\partial_x\cdot\varphi_{\tilde{\gamma}}^{[k-\frac12]}+\pi_{\tilde{\gamma}}^{[k-\frac12]}+\Sigma_{\tilde{\gamma}}^{[k-\frac12]}+\omega_{\tilde{\gamma}}^{[k-\frac12]}=0,\tag{3.39}$$

where $\varphi_{\tilde{\gamma}}^{[k-\frac12]}$, $\pi_{\tilde{\gamma}}^{[k-\frac12]}$, $\Sigma_{\tilde{\gamma}}^{[k-\frac12]}$, $\omega_{\tilde{\gamma}}^{[k-\frac12]}\in L^1((0,\bar{t}),W^{l-k,1})$*. The term* $\pi_{\tilde{\gamma}}^{[k-\frac12]}$ *is given by*

$$\pi_{\tilde{\gamma}}^{[k-\frac12]}=g^2h^{2k}|\partial^k r|^2,\tag{3.40}$$

where $g=e^{r+\frac12(1-\varkappa)\tau}$ *and* $h=e^{-r-(\frac12-\varkappa)\tau}$*. The term* $\Sigma_{\tilde{\gamma}}^{[k-\frac12]}$ *is in the form*

$$\begin{aligned}\Sigma_{\tilde{\gamma}}^{[k-\frac12]}=&\sum_{\nu\mu}c_{\sigma\nu\mu\phi}e^{-\varkappa\tau}\partial_\tau^\sigma\phi\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}-\frac{\kappa+\frac{2(n-1)}{n}\eta}{e^{\varkappa\tau}}g^2h^{2k+1}\partial^k r\partial^k(\partial_x\cdot w)\\&+\frac12\frac{\lambda}{c_ve^{\varkappa\tau}}g^2h^{2k+1}\partial^{k-1}\partial_x r\cdot w\partial^{k-1}\Delta\tau,\end{aligned}\tag{3.41}$$

where $c_{\sigma\nu\mu\phi}$ *are constants and the sums are over* $\phi\in\{\lambda,\eta,\kappa\}$, $0\leq\sigma\leq k$, $\nu=(\nu_\alpha,\nu'_\alpha,\nu''_\alpha)_{1\leq|\alpha|\leq k}$, $\mu=(\mu_\alpha,\mu'_\alpha,\mu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha,\nu'_\alpha,\nu''_\alpha,\mu_\alpha,\mu'_\alpha,\mu''_\alpha\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$*. The products* $\Pi_\nu^{(k)}$ *and* $\Pi_\mu^{(k+1)}$ *are defined as in the governing equation for* $\tilde{\gamma}^{[k]}$ *and*

$\sum_{1\leq|\alpha|\leq k}|\alpha|(\mu_\alpha+\mu'_\alpha+\mu''_\alpha)=k+1$. *Furthermore the term* $\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ *is given by*

$$\begin{aligned}\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]} &= \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k)} + g^2h^{2k}\partial^k\tau\partial^k r - g^2h^{2k}|\partial^{k-1}(\partial_x\cdot w)|^2\\ &\quad - \frac{1}{2}g^2h^{2k}\partial^{k-1}(\partial_x\cdot w)\partial^{k-1}(\partial_x\tau)\cdot w - \frac{1}{2c_v}g^2h^{2k}\partial^{k-1}(\partial_x\cdot w)\partial^{k-1}(\partial_x r)\cdot w\\ &\quad - \frac{1}{4c_v}g^2h^{2k}\partial^{k-1}(\partial_x\tau)\cdot w\partial^{k-1}(\partial_x r)\cdot w, \qquad (3.42)\end{aligned}$$

where $c_{\nu\mu}$ *are constants and at least one of the products* $\Pi_\nu^{(k)}$ *or* $\Pi_\mu^{(k)}$ *is split between derivatives factors. Finally the flux* $\varphi_{\tilde{\gamma}}^{[k-\frac{1}{2}]} = (\varphi_{\tilde{\gamma}1}^{[k-\frac{1}{2}]},\ldots,\varphi_{\tilde{\gamma}n}^{[k-\frac{1}{2}]})$ *is of the form*

$$\varphi_{\tilde{\gamma}l}^{[k-\frac{1}{2}]} = \sum_{\nu\mu} c_{\nu\mu l} h\Pi_\nu^{(k-1)}\Pi_\mu^{(k)}.$$

In Proposition 3.12, $\varphi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ is a flux and $\pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$, $\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$, $\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ are source terms. Note that $\pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ will help to complete the missing gradient terms in $\pi_{\tilde{\gamma}}^{[k-1]}$. In addition, in $\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$, there is no derivative of order $k+1$ in $\Pi_\mu^{(k+1)}$.

3.6. *Higher order entropies for zero Mach number flows*

Asymptotic expansions of higher order entropies with respect to small Mach and Knudsen numbers have been investigated.[31] These asymptotic studies have been performed by using rescaled variables and rescaled equations in terms of the Mach and Knudsen numbers as well as by using molecular coordinates.[31] Kinetic entropy estimators have been shown to be related to the Sobolev norm of the variable $(\log(\rho/\rho_\infty), v/\sqrt{T}, \log(T/T_\infty))$ in molecular coordinates.

In this asymptotic framework,[31] upon reordering higher order entropies in terms of the Mach number, it is easily checked that the velocity and the gradient of the density are then of the same order. More specifically, upon reordering the higher order correctors in terms of the Mach number, the following variant of the $(2k)$th order kinetic entropy corrector $\gamma^{[k]}$ is naturally obtained

$$\widehat{\gamma}^{[k]} = \rho h^{2(k-1)}\left(h^2\frac{|\partial^k\rho|^2}{\rho^2} + \frac{|\partial^{(k-1)}v|^2}{T} + c_v h^2\frac{|\partial^k T|^2}{T^2}\right). \qquad (3.43)$$

These variants are especially adapted to the zero Mach number equations where $\rho T=$ Cte and where the energy conservation equation is a pure thermal balance[27] which can first be used to estimate the temperature. Then, at the next steps, $\partial^{k-1}v$ and $\partial^k T$ and $\partial^k\rho$ have to be estimated simultaneously upon using $\widehat{\gamma}^{[l]}$, $0\leq l\leq k$.

On the other hand, as pointed out by a referee, in order to obtain estimates for data with low regularity, Bresch and Desjardin[8] and Danchin[16] have used functional spaces such that the gradient of the density and the velocity have the same regularity. An interesting extension of this work would thus be to consider the variant entropy estimators $\widehat{\gamma}^{[0]} + \cdots + \widehat{\gamma}^{[k]}$, $1 \le k \le l$, either for the zero Mach number equations or else for the fully compressible equations.

3.7. *Transport coefficients in a dense gas*

In this section, for self completeness, we address some of the model modifications required for dense gases. Indeed, the behavior of compressible fluid models at low/high densities and at low/high temperatures has been a key ingredient in recent advances concerning the existence of global weak solutions[8,9,23,24,49] as well as classical solutions.[62] A first fundamental point is the state law which may deviate from the ideal gas law, as for instance Van der Waal's equation of state. A second ingredient is the dependence of transport coefficients in terms of density. In the remaining part of this section we discuss some results obtained from the kinetic theory of dense gases.

The status of the kinetic theory of dense gases is not as well developed as that of dilute gases. A first attempt towards a kinetic theory of dense gases is that of Enskog for hard spheres. The advantages of the rigid sphere model is that collisions are instantaneous so that the probability of simultaneous multiple encounters is negligible. Enskog corrections involve the mechanism of *collisional transfer* which is the principal transport mechanism in dense gases — since the particles are almost packed together — so that transport by molecular flow becomes very difficult.[12,22] The transport coefficients $\lambda(T,\rho)$, $\eta(T,\rho)$, and $\kappa(T,\rho)$, obtained from Enskog theory of dense gases, are in the form

$$\lambda = \frac{\lambda_0}{\mathsf{g}} + \lambda_1 \mathsf{b}\rho + \lambda_2 \mathsf{g}(\mathsf{b}\rho)^2, \quad \eta = \frac{\eta_0}{\mathsf{g}} + \eta_1 \mathsf{b} + \eta_2 \mathsf{g}(\mathsf{b}\rho)^2, \quad \kappa = \kappa_2 \mathsf{g}(\mathsf{b}\rho)^2,$$

where λ_0, λ_1, λ_2, η_0, η_1, η_2 and κ_2 are only functions of temperature. In these relations, b denotes the covolume, which can be taken to be constant, and g denotes a function of the state of the gas which models the increase of probability for collisions due to the volume occupied by the gas. This probability factor g can be modeled[12,22] as a series or as a rational fraction in the density $\mathsf{g}(\rho)$. More generally, it is also possible to relate the quantities g and b to the state law.[12,22] The probability factor $\mathsf{g}(\rho)$ must be such that $\mathsf{g}(0) = 1$ in such a way that λ_0 and η_0 correspond to the coefficients for a dilute gas. It is worthwhile to note than even for a monatomic gas of hard spheres there is a nonzero volume viscosity for dense states.

More general theories of dense gases are based on multiple velocity-distribution functions and on the BBGKY-hierarchy of equations.[12,22] Formal expressions of the transport coefficients have been obtained upon assuming that the two-particle distribution is a time-independent functional of the usual one-particle distribution function — Bogoliubov's functional assumption. The transport coefficients are typically expressed in the form[22]

$$\varphi = \varphi_0 + \varphi_1 \rho + \varphi_2 \rho^2 + \widetilde{\varphi}_2 \rho^2 \log \rho + \cdots,$$

where φ denotes either λ, η, or κ, and the functions φ_0, φ_1, φ_2, and $\widetilde{\varphi}_2$ only depend on temperature. In particular, the expansions contain terms in the form $\rho^2 \log \rho$ associated with recollisions.

Even though the results obtained for dense gases are far from being completed, the models proposed so far shed light on the dependence of transport coefficients as suggested by theoretical physics. In particular, the coefficients usually share a common functional dependence in terms of temperature and density and there is a coefficient of volume viscosity. A very interesting application of the density dependence of transport coefficients is that of Bresch and Desjardins[8,9] as discussed in Sec. 5.4.

4. Weighted Inequalities

We investigate weighted inequalities in Sobolev and Lebesgue spaces.[13,26,29,35,36,53] These inequalities are required for renormalized variables with powers of temperature and density as weights as well as for temperature dependent thermal conductivity and viscosities.

4.1. *Differential identities*

Let α_i, $1 \leq i \leq n$, be nonnegative integers and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be the corresponding multi-index. We denote by ∂^α the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and by $|\alpha|$ its order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The derivative of superpositions has been investigated in particular by Vol'pert and Hudjaev[63] and the following proposition is established by induction on $|\alpha|$.

Lemma 4.1. *Let $l \geq 1$, f be a smooth scalar function of $\mathsf{u} \in \mathbb{R}^l$, $\mathsf{u}_1, \dots, \mathsf{u}_l$ be smooth scalar functions of $x \in \mathbb{R}^n$, and let α be a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \geq 1$. The partial derivatives of the superposition $f \circ \mathsf{u} = f(\mathsf{u}_1, \dots \mathsf{u}_l)$ can be written in the form*

$$\partial^\alpha (f \circ \mathsf{u}) = \sum_{\sigma\mu} c_{\sigma\mu} \partial^\sigma f \prod_{\substack{1 \leq |\beta| \leq |\alpha| \\ 1 \leq j \leq l}} (\partial^\beta \mathsf{u}_j)^{\mu_{\beta j}}, \tag{4.1}$$

where $c_{\sigma\mu}$ are non-negative integer and the sum is over $\sigma \in \mathbb{N}^l$, $1 \leq |\sigma| \leq |\alpha|$, $\mu = (\mu_{\beta j})_{1 \leq |\beta| \leq |\alpha|, 1 \leq j \leq l}$ with $\mu_{\beta j} \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $j \in \mathbb{N}$, such that

$$\sum_{1 \leq |\beta| \leq |\alpha|} \mu_{\beta j} = \sigma_j, \quad \sum_{\substack{1 \leq |\beta| \leq |\alpha| \\ 1 \leq j \leq l}} \beta \mu_{\beta j} = \alpha, \tag{4.2}$$

so that we have in particular $\sum_{\beta j} |\beta| \mu_{\beta j} = |\alpha|$.

When $l = 1$, that is when u is scalar, the identity (4.1) is sometimes called Faá di Bruno's formula although it seems to have first been published by Tiburce Abadie.[40] The rescaled unknowns $r = \log \rho$, $w = v/\sqrt{T}$, and $\tau = \log T$, naturally appear in higher order entropy estimates. We will need the following differential identities[29] easily established by induction on $|\alpha|$ and the next lemma will be used for temperature as well as for density.

Lemma 4.2. *Let T be smooth and positive and α be a multi-index. Then we have*

$$\frac{\partial^\alpha T}{T} = \sum_\mu c_\mu \prod_{1\leq|\beta|\leq|\alpha|} (\partial^\beta \tau)^{\mu_\beta} = \partial^\alpha \tau + \sum_\mu c_\mu \prod_{1\leq|\beta|\leq|\alpha|-1} (\partial^\beta \tau)^{\mu_\beta}, \tag{4.3}$$

where $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and c_μ are non-negative integer coefficients. The sum is extended over the μ such that

$$\sum_{1\leq|\beta|\leq|\alpha|} \beta\mu_\beta = \alpha,$$

so that we have in particular $\sum_{1\leq|\beta|\leq|\alpha|} |\beta|\mu_\beta = |\alpha|$, and the only term with $|\beta| = |\alpha|$ corresponds to $\partial^\alpha\tau$. Conversely, we have

$$\partial^\alpha \tau = \sum_\mu c'_\mu \prod_{1\leq|\beta|\leq|\alpha|} \left(\frac{\partial^\beta T}{T}\right)^{\mu_\beta} = \frac{\partial^\alpha T}{T} + \sum_\mu c'_\mu \prod_{1\leq|\beta|\leq|\alpha|-1} \left(\frac{\partial^\beta T}{T}\right)^{\mu_\beta}, \tag{4.4}$$

where c'_μ are integer coefficients and the sum is extended over the same set of μ.

Lemma 4.3. *Let T and v be smooth, T be positive, i with $1 \leq i \leq n$, and α be a multi-index. Then we have*

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} (\partial^\beta \tau)^{\mu_\beta} \partial^{\tilde{\alpha}} w_i, \tag{4.5}$$

where $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$, $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $\tilde{\alpha} \in \mathbb{N}^n$, $c_{\mu\tilde{\alpha}}$ are non-negative integer coefficients, and the sum is extended over the μ and $\tilde{\alpha}$, such that

$$0 \leq \tilde{\alpha} \leq \alpha, \qquad \sum_{1\leq|\beta|\leq|\alpha|} \beta\mu_\beta + \tilde{\alpha} = \alpha.$$

More precisely, isolating the only term $\partial^\alpha w_i$ corresponding to $\tilde{\alpha} = \alpha$ and all the terms corresponding to $\tilde{\alpha} = (0, \dots, 0)$, we have

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \partial^\alpha w_i + \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} (\partial^\beta \tau)^{\mu_\beta} \partial^{\tilde{\alpha}} w_i + \sum_\mu c_{\mu 0} \prod_{1\leq|\beta|\leq|\alpha|} (\partial^\beta \tau)^{\mu_\beta} w_i, \tag{4.6}$$

where the $\tilde{\alpha}$ in the middle sum are such that $1 \leq |\tilde{\alpha}| < |\alpha|$. Conversely, we have

$$\partial^\alpha w_i = \sum_{\mu\tilde{\alpha}} c'_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} \left(\frac{\partial^\beta T}{T}\right)^{\mu_\beta} \frac{\partial^{\tilde{\alpha}} v_i}{\sqrt{T}}, \tag{4.7}$$

and more precisely

$$\partial^\alpha w_i = \frac{\partial^\alpha v_i}{\sqrt{T}} + \sum_{\mu\tilde{\alpha}} c'_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} \left(\frac{\partial^\beta T}{T}\right)^{\mu_\beta} \frac{\partial^{\tilde{\alpha}} v_i}{\sqrt{T}} + \sum_\mu c'_{\mu 0} \prod_{1\leq|\beta|\leq|\alpha|} \left(\frac{\partial^\beta T}{T}\right)^{\mu_\beta} \frac{v_i}{\sqrt{T}}, \tag{4.8}$$

where $c'_{\mu\tilde{\alpha}}$ are integer coefficients and the sums are extended over the same sets.

4.2. *Weighted operators*

A natural condition associated with weights[13,26,53] has been shown to be the Muckenhoupt property A_p, where $1 \le p \le \infty$.

Definition 4.4. Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ be positive and let $1 < p < \infty$. The function g satisfies the Muckenhoupt condition A_p if

$$[g]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q g\, dx \right) \left(\frac{1}{|Q|} \int_Q g^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q.

For detailed studies about the Muckenhoupt property we refer to the book of Garcia-Cuerva and Rubio de Francia.[26] We have in particular $A_p \cap A_q = A_{\min(p,q)}$ and the weights of A_p have their logarithms in BMO.[26,53] A locally summable function f belongs to the space $\text{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - \bar{f}_Q|\, dx < \infty,$$

where the supremum is taken over all cubes Q and where $\bar{f}_Q = {}^{1}/_{|Q|} \int_Q f(x)\, dx$ denotes the average of f over Q.[52] The function space BMO has been introduced by John and Nirenberg[39] and naturally arises when estimating the norms of the weighted operators $T^\theta R_i T^{-\theta}$ where $R_i = (-\Delta)^{-1/2}\partial_i$, $1 \le i \le n$, are Riesz transforms, or when using the Coifman and Meyer inequalities.[54,55] The space BMO and its dual $\mathcal{H}^1$ have already been used in the context of the Navier–Stokes equations.[46,48,49]

Theorem 4.5. *There exist constants* $b(n)$ *and* $B(n)$ *such that for any* $\theta \in \mathbb{R}$, *any* $\mathsf{u} \in \text{BMO}$, *and any* $1 < p < \infty$, *the condition*

$$|\theta| \|\mathsf{u}\|_{\text{BMO}} < \frac{1}{2} b(n) \min(1, p-1),$$

implies that $\exp(\theta u) \in A_p$ *and*

$$[\exp(\theta u)]_{A_p} \le (1 + B(n))^p.$$

Moreover, the constants $b(n)$ *and* $B(n)$ *only depend on* n *and are thus invariant by a change of scale in the coordinate system.*

Proof. These estimates are proved in Ref. 29 and the scale invariance of $b(n)$ and $B(n)$ is straightforward since both the BMO seminorm and the A_p condition number $[g]_{A_p}$ are scale invariant. □

We now investigate the continuity of Calderón–Zygmund operators in weighted Lebesgue spaces. In the following theorem the quantitites c_0, c_1, c_2 are the constants naturally associated with the norm of a Calderón–Zygmund operator $\mathcal{G}$.[53]

Theorem 4.6. *Let $\mathcal{G}$ be a Calderón–Zygmund operator, let $1 < p < \infty$, and let g^p be a weight in A_p. Then the operator $\mathcal{G}$ is bounded in $L^p(g^p dx)$, or equivalently, the operator $g\mathcal{G}g^{-1}$ is bounded in L^p, with norm lower than $\mathcal{C}(c_0, c_1, c_2, n, p, [g^p]_{A_p})$, where c_0, c_1, c_2 are the constants naturally associated with the norm of $\mathcal{G}$.*

Proof. We refer to the books of Garcia–Cuerva and Rubio de Francia[26] and of Yves Meyer.[53] □

4.3. *Multilinear estimates*

We investigate weighted multilinear estimates for derivatives with weights in A_p classes[13,26,29,35,36,53] and we denote by $\mathcal{C}_0^0(\mathbb{R}^n)$ the set of continuous function that vanish at infinity. The following multilinear estimates have been obtained in previous work[29] by using the Wiener algebra $A(\mathbb{R}^n)$ instead of the space $\mathcal{C}_0^0(\mathbb{R}^n)$ but the proofs are similar thanks to the density of $\mathcal{D}(\mathbb{R}^n)$ in $W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$. The proof of this theorem essentially relies on the Coifman–Meyer theory and on Theorem 4.6.

Theorem 4.7. *Let $k \geq 1$, $l \geq 1$ be integers, and α^j, $1 \leq j \leq l$, be multi-indices such that $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $k = \sum_{1\leq j\leq l} |\alpha^j|$. Let $1 < p < \infty$, $g^p \in A_p$ and $\mathsf{u}_1, \ldots, \mathsf{u}_l$, be such that there exist constants $\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, and such that $g\partial^k u_j \in L^p$, $1 \leq j \leq l$. There exists a constant $c = c(k, n, p, [g^p]_{A_p})$ only depending on $(k, n, p, [g^p]_{A_p})$, such that*

$$\Big\| g \prod_{1\leq j\leq l} \partial^{\alpha^j} \mathsf{u}_j \Big\|_{L^p} \leq c \sum_{1\leq i\leq l} \Bigg(\prod_{\substack{1\leq j\leq l\\ j\neq i}} \|\mathsf{u}_j\|_{\mathrm{BMO}} \Bigg) \|g\partial^k \mathsf{u}_i\|_{L^p}, \tag{4.9}$$

and thus

$$\Big\| g \prod_{1\leq j\leq l} \partial^{\alpha^j} \mathsf{u}_j \Big\|_{L^p} \leq c\|\mathsf{u}\|_{\mathrm{BMO}}^{l-1} \|g\partial^k \mathsf{u}\|_{L^p}, \tag{4.10}$$

where

$$\|\mathsf{u}\|_{\mathrm{BMO}} = \sum_{1\leq j\leq l} \|\mathsf{u}_j\|_{\mathrm{BMO}}, \quad \|g\partial^m \mathsf{u}\|_{L^p}^p = \sum_{1\leq j\leq l} \|g\partial^m \mathsf{u}_j\|_{L^p}^p.$$

We now investigate multilinear estimates where a weighted L^∞ norm of the gradient is used to decrease the total number of derivations k in the upper bound. We denote by $\mathcal{C}_0^1(\mathbb{R}^n)$ the set of continuously differentiable functions that vanish at infinity with their gradients.

Theorem 4.8. *Let $k \geq 2$, $l \geq 2$ be integers, and α^j, $1 \leq j \leq l$, be multi-indices such that $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $k = \sum_{1\leq j\leq l} |\alpha^j|$. Let $1 < p < \infty$, g be positive, $g \in L^1_{\mathrm{loc}}$ with $\log g \in \mathrm{BMO}$, and $\mathsf{u}_1, \ldots, \mathsf{u}_l$, be such that there exist constants $\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{k-1,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$. Let h be the weight $h = \exp(\theta_1 u_1 + \cdots + \theta_l u_l)$,* *where* $|\theta_j| \le \bar{\theta}$ *and* $\bar{\theta} > 0$. *There exist constants* $\delta = \delta(k,n,p,\bar{\theta})$ *and* $c = c(k,n,p,\bar{\theta})$, *only depending on* $(k,n,p,\bar{\theta})$, *such that if* $\|\log g\|_{\mathrm{BMO}} + \sum_{1\le j\le l} \|u_j\|_{\mathrm{BMO}} < \delta$, *then, whenever* $gh^{k-1}\partial^{k-1}u_j \in L^p$ *and* $gh^{k-2}\partial^{k-2}u_j \in L^p$, $1 \le j \le l$, *the following estimates hold*

$$\begin{aligned}\Big\| gh^k \prod_{1\le j\le l} \partial^{\alpha^j} \mathsf{u}_j \Big\|_{L^p} &\le c\|\mathsf{u}\|_{\mathrm{BMO}}^{l-2} \|h\partial_x \mathsf{u}\|_{L^\infty} \|gh^{k-1}\partial^{k-1}\mathsf{u}\|_{L^p} \\ &\quad + c\ \mathbf{1}_{k>3} \|\mathsf{u}\|_{\mathrm{BMO}}^{(l-3)^+} \|h\partial_x \mathsf{u}\|_{L^\infty}^2 \|gh^{k-2}\partial^{k-2}\mathsf{u}\|_{L^p},\end{aligned} \tag{4.11}$$

where

$$\|h\partial_x \mathsf{u}\|_{L^\infty} = \sum_{1\le j\le l} \|h\partial_x \mathsf{u}_j\|_{L^\infty}, \quad \|gh^m\partial^m \mathsf{u}\|_{L^p}^p = \sum_{1\le j\le l} \|gh^m \partial^m \mathsf{u}_j\|_{L^p}^p,$$

and where $\mathbf{1}_{k>3} = 1$ *if* $k > 3$ *and* $\mathbf{1}_{k>3} = 0$ *if* $k \le 3$ *so that in the special situation* $2 \le k \le 3$, *the second term on the right-hand side of* (4.11) *is absent.*

Proof. If there exists one multi-index α^{j_0} such that $|\alpha^{j_0}| = 1$ we can directly write that

$$\Big\| gh^k \prod_{1\le j\le l} \partial^{\alpha^j} \mathsf{u}_j \Big\|_{L^p} \le \|h\partial_x \mathsf{u}_{j_0}\|_{L^\infty} \Big\| gh^{k-1} \prod_{\substack{1\le j\le l \\ j\ne j_0}} \partial^{\alpha^j} \mathsf{u}_j \Big\|_{L^p}, \tag{4.12}$$

and use the multilinear estimates of Theorem 4.7. The weight gh^{k-1} is in the A_p class and $[g^p h^{p(k-1)}]_{A_p}$ is bounded by a constant only depending on n and p from Theorem 4.5 for δ small enough since $\|\log(gh^{k-1})\|_{\mathrm{BMO}} \le (1 + k\bar{\theta})\delta$ provided we select $\delta \le \frac{1}{2} b(n) \min(1, p-1)/(1+k\bar{\theta})$. This covers in particular the situation where $2 \le k \le 3$ since it is assumed that $l \ge 2$ so that there is at least one first-order derivative factor $\partial^{\alpha^{j_0}} \mathsf{u}_{j_0}$ with $|\alpha^{j_0}| = 1$ in this case.

Keeping in mind that $l \ge 2$, we can now assume that $|\alpha^1| \ge 2$ and $|\alpha^2| \ge 2$, so that $k \ge 4$, and write $\alpha^1 = \tilde{\alpha}^1 + e_{i_1}$, $\alpha^2 = \tilde{\alpha}^2 + e_{i_2}$, where $|\tilde{\alpha}^1| = k-1$, $|\tilde{\alpha}^2| = k-1$, and $i_1, i_2 \in \{1, \dots, n\}$. We have denoted by e_i, $1 \le i \le n$, the canonical basis of $\mathbb{R}^n$ with $e_i = (\delta_{i1}, \dots, \delta_{in})$, where δ_{ij} is the Kronecker symbol, so that $\partial^{e_i} = \partial_i$. We introduce the auxiliary functions $\mathsf{v}_1 = h\partial_{i_1}\mathsf{u}_1$ and $\mathsf{v}_2 = h\partial_{i_2}\mathsf{u}_2$ and write that

$$gh^k \prod_{1\le j\le l} \partial^{\alpha^j} \mathsf{u}_j = gh^k \partial^{\tilde{\alpha}_1} \Big(\frac{\mathsf{v}_1}{h}\Big) \partial^{\tilde{\alpha}_2} \Big(\frac{\mathsf{v}_2}{h}\Big) \prod_{3\le j\le l} \partial^{\alpha^j} \mathsf{u}_j.$$

Next we expand the derivatives by using Lemma 4.1

$$\partial^{\tilde{\alpha}_1} \Big(\frac{\mathsf{v}_1}{h}\Big) = \frac{1}{h} \sum_{\tilde{\beta}_1 \mu} c_{\tilde{\beta}_1 \mu} \partial^{\tilde{\beta}_1} \mathsf{v}_1 \prod_{\substack{1\le|\beta|\le|\alpha| \\ 1\le j\le l}} (\partial^\beta \mathsf{u}_j)^{\mu_{\beta j}}, \tag{4.13}$$

where $c_{\tilde\beta_1\mu}$ are non-negative integer coefficients, and the sum is over $0 \le \tilde\beta_1 \le \tilde\alpha_1$ and $\mu = (\mu_{\beta j})_{1\le|\beta|\le|\tilde\alpha_1|,1\le j\le l}$ with $\mu_{\beta j} \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $1 \le j \le l$, and $\sum_{\beta j} \beta\mu_{\beta j} = \tilde\alpha_1 - \tilde\beta_1$. We can thus write that

$$gh^k \prod_{1\le j\le l} \partial^{\alpha^j} \mathsf{u}_j = \sum_{\tilde\beta_1\tilde\beta_2\hat\alpha} c_{\tilde\beta_1\tilde\beta_2\hat\alpha} gh^{k-2}\partial^{\tilde\beta_1}\mathsf{v}_1\partial^{\tilde\beta_2}\mathsf{v}_2 \prod_{3\le j\le l} \partial^{\alpha^j}\mathsf{u}_j \prod_{1\le j\le \hat l} \partial^{\hat\alpha^j}\hat{\mathsf{u}}_j, \tag{4.14}$$

where the derivative factors arising from the derivation of $1/h$ in (4.13) are rewritten in the form $\prod_{1\le j\le\hat l} \partial^{\hat\alpha^j}\hat{\mathsf{u}}_j$, where $(\hat{\mathsf{u}}_1,\dots,\hat{\mathsf{u}}_{\hat l})$ are proper replicates of $\mathsf{u}_1,\dots,\mathsf{u}_l$. We can then use the inequality (4.9) of Theorem 4.7 to estimate the L^p norm of each term in the sum (4.14). Inequality (4.9) is used with the weight gh^{k-2} and with the variables $(\mathsf{v}_1,\mathsf{v}_2,\mathsf{u}_3,\dots,\mathsf{u}_j,\hat{\mathsf{u}}_1,\dots,\hat{\mathsf{u}}_{\hat l})$. The weight gh^{k-2} is in the A_p class for δ small enough and $[g^p h^{p(k-2)}]_{A_p}$ is bounded by a constant only depending on n and p from Theorem 4.5 provided that $\delta \le \frac12 b(n)\min(1,p-1)/(1+k\bar\theta)$. We can thus estimate the L^p norm of $gh^k\prod_{1\le j\le l}\partial^{\alpha^j}\mathsf{u}_j$, up to multiplicative constants depending on $(k,n,p,\bar\theta)$, in terms of

$$\|\mathsf{v}_2\|_{\mathrm{BMO}} \prod_{3\le j\le l} \|\mathsf{u}_j\|_{\mathrm{BMO}} \prod_{1\le j\le\hat l} \|\hat{\mathsf{u}}_j\|_{\mathrm{BMO}} \big\|gh^{k-2}\partial^{k-2}\mathsf{v}_1\big\|_{L^p},$$

$$\|\mathsf{v}_1\|_{\mathrm{BMO}} \prod_{3\le j\le l} \|\mathsf{u}_j\|_{\mathrm{BMO}} \prod_{1\le j\le\hat l} \|\hat{\mathsf{u}}_j\|_{\mathrm{BMO}} \big\|gh^{k-2}\partial^{k-2}\mathsf{v}_2\big\|_{L^p},$$

$$\|\mathsf{v}_1\|_{\mathrm{BMO}}\|\mathsf{v}_2\|_{\mathrm{BMO}} \prod_{\substack{3\le j\le l\\ j\ne i}} \|\mathsf{u}_j\|_{\mathrm{BMO}} \prod_{1\le j\le\hat l} \|\hat{\mathsf{u}}_j\|_{\mathrm{BMO}} \big\|gh^{k-2}\partial^{k-2}\mathsf{u}_i\big\|_{L^p}, \quad 3\le i\le l,$$

and

$$\|\mathsf{v}_1\|_{\mathrm{BMO}}\|\mathsf{v}_2\|_{\mathrm{BMO}} \prod_{3\le j\le l} \|\mathsf{u}_j\|_{\mathrm{BMO}} \prod_{\substack{1\le j\le\hat l\\ j\ne i}} \|\hat{\mathsf{u}}_j\|_{\mathrm{BMO}} \big\|gh^{k-2}\partial^{k-2}\hat{\mathsf{u}}_i\big\|_{L^p}, \quad 1\le i\le \hat l.$$

Expanding then the derivatives $\partial^{k-2}\mathsf{v}_j = \partial^{k-2}(h\partial_{i_j}\mathsf{u}_j)$, $j=1,2$, it is easily checked that $\|gh^{k-2}\partial^{k-2}\mathsf{v}_j\|_{L^p}$ is majorized by a multiplicative constant multiplied by $\sum_{1\le i\le l}\|gh^{k-1}\partial^{k-1}\mathsf{u}_i\|_{L^p}$ and the proof is complete since one may choose δ such that $0<\delta\le 1$. □

Remark 4.9. The space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{k,2}(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n)$ — for the norm $\|\cdot\|_{W^{k,2}}+\|\cdot\|_{\mathrm{BMO}}$ of course — if and only if $k\ge n/2$. Indeed, for $k<n/2$, $\mathcal{D}(\mathbb{R}^n)$ is not even dense in $W^{k,2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and counterexemples are classically found in the form of series of needles.[29] On the other hand, for $k=n/2$, we have $W^{k,2}(\mathbb{R}^n)\cap\mathrm{BMO}(\mathbb{R}^n)=W^{k,2}(\mathbb{R}^n)$, whereas for $k>n/2$, $W^{k,2}(\mathbb{R}^n)$ is included in $\mathcal{C}^0_0(\mathbb{R}^n)$. We have introduced the natural simplifying assumption $\mathsf{u}_j-\mathsf{u}_{j,\infty}\in W^{k,2}(\mathbb{R}^n)\cap\mathcal{C}^0_0(\mathbb{R}^n)$ since it will be sufficient for our applications, since $\mathcal{C}^0_0(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)\subset\mathrm{BMO}(\mathbb{R}^n)$, and since $\mathcal{D}(\mathbb{R}^n)$ is always dense in $W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, even for $k < n/2$, as easily established by truncation and smoothing. Similarly, when using the gradient norms $\|h\partial_x \mathsf{u}_j\|_{L^\infty}$, $1 \le j \le l$, we have introduced the natural simplifying assumption that $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ since it will be sufficient for our applications.

4.4. *Weighted products of derivatives*

We first investigate products of derivatives of the rescaled unknowns τ and w with powers of temperature and density as natural weights.[29] Since in our applications w and τ will be parabolic variables, the total number of derivations k is left unchanged in the estimates.

Theorem 4.10. *Let $k \ge 1$ be an integer, $\bar\theta > 0$ be positive, $1 < p < \infty$, τ be such that $\tau - \tau_\infty \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ for some constant τ_∞ and let $r \in \mathrm{BMO}$. There exist positive constants $\delta(n,p,\bar\theta)$ and $c(k,n,p)$, only depending on $(n,p,\bar\theta)$ and (k,n,p), respectively, such that if $\|r\|_{\mathrm{BMO}} + \|\tau\|_{\mathrm{BMO}} < \delta$, then for any a, b with $|a| + |b| \le \bar\theta$, any integer $l \ge 1$, and any multi-indices α^j, $1 \le j \le l$, with $|\alpha^j| \ge 1$, $1 \le j \le l$, and $\sum_{1\le j\le l} |\alpha^j| = k$, whenever $e^{a\tau+br}\partial^k\tau \in L^p(\mathbb{R}^n)$, the following inequality holds*

$$\Big\| e^{a\tau+br} \prod_{1\le j\le l} \partial^{\alpha^j}\tau \Big\|_{L^p} \le c\|\tau\|_{\mathrm{BMO}}^{l-1} \|e^{a\tau+br}\partial^k\tau\|_{L^p}. \tag{4.15}$$

Further assuming that $w \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $e^{a\tau+br}\partial^k w \in L^p(\mathbb{R}^n)$, and $0 \le \bar l \le l$, then

$$\begin{aligned}\Big\| e^{a\tau+br} \prod_{1\le j\le \bar l} \partial^{\alpha^j} w \prod_{\bar l+1\le j\le l} \partial^{\alpha^j}\tau \Big\|_{L^p} &\le c\left(\|w\|_{\mathrm{BMO}} + \|\tau\|_{\mathrm{BMO}}\right)^{l-1} \\ &\quad \times \left(\|e^{a\tau+br}\partial^k w\|_{L^p} + \|e^{a\tau+br}\partial^k\tau\|_{L^p}\right),\end{aligned} \tag{4.16}$$

where we have naturally defined $\|e^{a\tau+br}\partial^k w\|_{L^p}^p = \sum_{1\le i\le n} \|e^{a\tau+br}\partial^k w_i\|_{L^p}^p$ and on the left-hand member of (4.16), *with a slight abuse of notation, we have denoted by w any of its components $w_1, \dots, w_n$.*

We now investigate products of derivatives of the rescaled unknowns r, τ and w. Since in our applications r will be a hyperbolic variable, the total number of derivations appearing in the estimates needs to be decreased by using a weighted L^∞ norm of the gradients.

Theorem 4.11. *Let $k \ge 2$ be an integer, $\bar\theta > 0$ be positive, $1 < p < \infty$, τ, r, w be such that $\tau - \tau_\infty, r - r_\infty, w \in W^{k-1,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ for some constants τ_∞ and r_∞. Let a, b, $\bar a$ and $\bar b$ be constants with $|a| + |b| \le \bar\theta$, $|\bar a| + |\bar b| \le \bar\theta$, and let $g = \exp(a\tau + br)$ and $h = \exp(\bar a\tau + \bar b r)$. Let $l \ge 2$, let α^j, $1 \le j \le l$, be multi-indices with $|\alpha^j| \ge 1$, $1 \le j \le l$, and $\sum_{1\le j\le l} |\alpha^j| = k$. There exist positive constants $\delta(k,n,p,\bar\theta)$*

and $c(k,n,p,\bar\theta)$, only depending on $(k,n,p,\bar\theta)$, such that if $\|r\|_{\mathrm{BMO}} + \|\tau\|_{\mathrm{BMO}} < \delta$, then whenever $gh^{k-1}\partial^{k-1}r$, $gh^{k-1}\partial^{k-1}w$, $gh^{k-1}\partial^{k-1}\tau$, $gh^{k-2}\partial^{k-2}r$, $gh^{k-2}\partial^{k-2}w$, $gh^{k-2}\partial^{k-2}\tau \in L^p(\mathbb{R}^n)$, and $1 \le \bar l \le \tilde l \le l$, we have the estimates

$$\begin{aligned}
\Big\| gh^k \prod_{1\le j\le \bar l} \partial^{\alpha^j} r \prod_{\bar l+1\le j\le \tilde l} \partial^{\alpha^j} w \prod_{\tilde l+1\le j\le l} \partial^{\alpha^j} \tau \Big\|_{L^p} \\
\le c\|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}}^{l-2} \|h\partial_x \widetilde{\mathrm{z}}\|_{L^\infty} \|gh^{k-1}\partial^{k-1}\widetilde{\mathrm{z}}\|_{L^p} \\
+ c\mathbf{1}_{k>3} \|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}}^{(l-3)^+} \|h\partial_x \widetilde{\mathrm{z}}\|_{L^\infty}^2 \|gh^{k-2}\partial^{k-2}\widetilde{\mathrm{z}}\|_{L^p},
\end{aligned} \tag{4.17}$$

where we have denoted $\widetilde{\mathrm{z}} = (r,w,\tau)$ and

$$\|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}} = \|r\|_{\mathrm{BMO}} + \|w\|_{\mathrm{BMO}} + \|\tau\|_{\mathrm{BMO}}, \tag{4.18}$$

$$\|h\partial_x \widetilde{\mathrm{z}}\|_{L^\infty} = \|h\partial_x r\|_{L^\infty} + \|h\partial_x w\|_{L^\infty} + \|h\partial_x \tau\|_{L^\infty}, \tag{4.19}$$

$$\|gh^m\partial^m \widetilde{\mathrm{z}}\|_{L^p}^p = \|gh^m\partial^m r\|_{L^p}^p + \|gh^m\partial^m w\|_{L^p}^p + \|gh^m\partial^m \tau\|_{L^p}^p \tag{4.20}$$

for any $m \in \mathbb{N}^*$ and on the left-hand member of (4.17), *with a slight abuse of notation, we have denoted by w any of its components $w_1,\dots,w_n$. In particular, in the situation where $2 \le k \le 3$, the second term on the right-hand side of in* (4.17) *is absent.*

Proof. Theorems 4.10 and 4.11 are direct consequences of the multilinear estimates of Theorems 4.7 and 4.8. □

4.5. *Weighted products of renormalized derivatives*

We now estimate products of derivatives of density, temperature and velocity components rescaled by the proper renormalizing factors.

Theorem 4.12. *Let $k \ge 1$ be an integer, $\bar\theta > 0$ be positive, $1 < p < \infty$, T be such that $T \ge T_{\min} > 0$ and $T - T_\infty \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ for some positive constant T_∞ and ρ be positive such that $r = \log\rho \in$* BMO. *There exist positive constants $\delta(n,p,\bar\theta)$ and $c(k,n,p)$, only depending on $(n,p,\bar\theta)$ and (k,n,p), respectively, such that if $\|\log\rho\|_{\mathrm{BMO}} + \|\log T\|_{\mathrm{BMO}} < \delta$, then for any real a and b such that $|a|+|b| \le \bar\theta$, any integer $l \ge 1$, and any multi-indices α^j, $1 \le j \le l$, with $|\alpha^j| \ge 1$, $1 \le j \le l$, and $\sum_{1\le j\le l} |\alpha^j| = k$, whenever $T^a\rho^b(\partial^k T)/T \in L^p(\mathbb{R}^n)$, we have the estimates*

$$\Big\| T^a\rho^b \prod_{1\le j\le l} \frac{\partial^{\alpha^j} T}{T} \Big\|_{L^p} \le c\|\log T\|_{\mathrm{BMO}}^{l-1} \Big\| T^a\rho^b \frac{\partial^k T}{T} \Big\|_{L^p}. \tag{4.21}$$

Assuming $v \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $\|\log\rho\|_{\mathrm{BMO}} + \|v/\sqrt{T}\|_{L^\infty} + \|\log T\|_{\mathrm{BMO}} < \delta$, *whenever* $T^a\rho^b(\partial^k v)/\sqrt{T} \in L^p(\mathbb{R}^n)$, *we have for* $0 \le \bar{l} \le l$

$$\left\| T^a\rho^b \prod_{1\le j\le \bar{l}} \frac{\partial^{\alpha^j} v}{\sqrt{T}} \prod_{\bar{l}+1\le j\le l} \frac{\partial^{\alpha^j} T}{T} \right\|_{L^p} \le c\left(\left\| \frac{v}{\sqrt{T}} \right\|_{L^\infty} + \|\log T\|_{\mathrm{BMO}} \right)^{l-1} \times \left(\left\| T^a\rho^b \frac{\partial^k v}{\sqrt{T}} \right\|_{L^p} + \left\| T^a\rho^b \frac{\partial^k T}{T} \right\|_{L^p} \right), \tag{4.22}$$

where, on the left-hand member, with a slight abuse of notation, we have denoted by v *any of its components* $v_1, \ldots, v_n$.

Theorem 4.13. *Let* $k \ge 2$ *be an integer,* $\bar{\theta} > 0$ *be positive,* $1 < p < \infty$, ρ, v, T, *be such that* $\rho \ge \rho_{\min}$, $T \ge T_{\min}$, *and* $\rho - \rho_\infty, v, T - T_\infty \in W^{k-1,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ *for positive constants* ρ_∞, $\rho_{\min}$, T_∞ *and* $T_{\min}$. *Let* a, b $\bar{a}$, *and* $\bar{b}$ *be constants with* $|a| + |b| \le \bar{\theta}$, $|\bar{a}| + |\bar{b}| \le \bar{\theta}$, *and let* $g = T^a\rho^b$, $h = T^{\bar{a}}\rho^{\bar{b}}$. *Let* $l \ge 2$, α^j, $1 \le j \le l$, *be multi-indices with* $|\alpha^j| \ge 1$, $1 \le j \le l$, *and* $\sum_{1\le j\le l} |\alpha^j| = k$. *There exist positive constants* $\delta(k, n, p, \bar{\theta})$ *and* $c(k, n, p, \bar{\theta})$, *only depending on* $(k, n, p, \bar{\theta})$, *such that if* $\|\log\rho\|_{\mathrm{BMO}} + \|v/\sqrt{T}\|_{L^\infty} + \|\log T\|_{\mathrm{BMO}} < \delta(k, n, p, \bar{\theta})$, *then whenever* $gh^{k-1}(\partial^{k-1}\rho)/\rho$, $gh^{k-1}(\partial^{k-1}v)/\sqrt{T}$, $gh^{k-1}(\partial^{k-1}T)/T$, $gh^{k-2}(\partial^{k-2}\rho)/\rho$, $gh^{k-2}(\partial^{k-2}v)/\sqrt{T}$, $gh^{k-2}(\partial^{k-2}T)/T \in L^p(\mathbb{R}^n)$, *we have for* $0 \le \bar{l} \le \tilde{l} \le l$

$$\begin{aligned} &\left\| gh^k \prod_{1\le j\le \bar{l}} \frac{\partial^{\alpha^j}\rho}{\rho} \prod_{\bar{l}+1\le j\le \tilde{l}} \frac{\partial^{\alpha^j} v}{\sqrt{T}} \prod_{\tilde{l}+1\le j\le l} \frac{\partial^{\alpha^j} T}{T} \right\|_{L^p} \\ &\quad \le c\|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}}^{\prime l-2} \|h\partial_x\widetilde{\mathrm{z}}\|'_{L^\infty} \|gh^{k-1}\partial^{k-1}\widetilde{\mathrm{z}}\|'_{L^p} \\ &\quad\quad + c\mathbf{1}_{k>3}\|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}}^{\prime(l-3)^+} \|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty}^{\prime 2} \|gh^{k-2}\partial^{k-2}\widetilde{\mathrm{z}}\|'_{L^p}, \end{aligned} \tag{4.23}$$

where, on the left-hand member, with a slight abuse of notation, we have denoted by v *any of its components* $v_1, \ldots, v_n$, *and where* $\widetilde{\mathrm{z}} = (r, w, \tau)$ *and*

$$\|\widetilde{\mathrm{z}}\|'_{\mathrm{BMO}} = \|\log\rho\|_{\mathrm{BMO}} + \|v/\sqrt{T}\|_{L^\infty} + \|\log T\|_{\mathrm{BMO}}, \tag{4.24}$$

$$\|h\partial_x\widetilde{\mathrm{z}}\|'_{L^\infty} = \left\| h\frac{\partial_x\rho}{\rho} \right\|_{L^\infty} + \left\| h\frac{\partial_x v}{\sqrt{T}} \right\|_{L^\infty} + \left\| h\frac{\partial_x T}{T} \right\|_{L^\infty}, \tag{4.25}$$

$$\|gh^m\partial^m\widetilde{\mathrm{z}}\|_{L^p}^{\prime p} = \left\| gh^m\frac{\partial^m\rho}{\rho} \right\|_{L^p}^p + \left\| gh^m\frac{\partial^m v}{\sqrt{T}} \right\|_{L^p}^p + \left\| gh^m\frac{\partial^m T}{T} \right\|_{L^p}^p, \tag{4.26}$$

for any $m \in \mathbb{N}^*$. *In particular, in the situation where* $2 \le k \le 3$, *the second term on the right-hand side of* (4.23) *is absent. Note that there is a* L^∞ *norm for the rescaled velocity* w *in* $\|\widetilde{\mathrm{z}}\|'_{\mathrm{BMO}}$.

Proof. The proof of Theorems 4.12 and 4.13 essentially relies on Theorems 4.10 and 4.11 and on the differential identities established in Lemmas 4.2 and 4.3. Considering temperature as a typical example, the differential identities and Theorem 4.10 yield estimates in the form

$$\left\| T^a \rho^b \prod_{1\leq j\leq l} \frac{\partial^{\alpha^j} T}{T} \right\|_{L^p} \leq c \|\log T\|_{\mathrm{BMO}}^{l-1} \|T^a \rho^b \partial^k \tau\|_{L^p}, \tag{4.27}$$

and similarly that

$$\left\| T^a \rho^b \left(\frac{\partial^k T}{T} - \partial^k \tau \right) \right\|_{L^p} \leq c \|\log T\|_{\mathrm{BMO}} \|T^a \rho^b \partial^k \tau\|_{L^p},$$

where $c = c(k,n,p)$. Therefore for $c(k,n,p)\|\log T\|_{\mathrm{BMO}} < 1/2$ we have

$$\frac{1}{2} \|T^a \rho^b \partial^k \tau\|_{L^p} \leq \left\| T^a \rho^b \frac{\partial^k T}{T} \right\|_{L^p} \leq \frac{3}{2} \|T^a \rho^b \partial^k \tau\|_{L^p}, \tag{4.28}$$

and reinserting (4.28) in (4.27) completes the proof of (4.21). The same procedure can be applied to get estimates of $\|T^a \rho^b \partial^k \rho/\rho\|_{L^p}$ and $\|T^a \rho^b \partial^k v/\sqrt{T}\|_{L^p}$ and then to obtain (4.22) and (4.23). $\square$

Remark 4.14. Assuming that $T - T_\infty \in W^{2,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$, $T \geq T_{\min} > 0$ and $\|\log T\|_{\mathrm{BMO}}$ is small enough, we obtain from Theorem 4.13 that

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^6}{T^{5+a}}\, dx \leq c \|\log T\|_{\mathrm{BMO}}^2 \left\| \frac{\partial_x T}{T} \right\|_{L^\infty}^2 \int_{\mathbb{R}^n} \frac{|\partial^2 T|^2}{T^{1+a}}\, dx. \tag{4.29}$$

In contrast, when $T - T_\infty \in W^{3,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $T \geq T_{\min} > 0$, and $\|\log T\|_{\mathrm{BMO}}$ is small enough, we obtain from Theorem 4.12 that

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^6}{T^{5+a}}\, dx \leq c \|\log T\|_{\mathrm{BMO}}^4 \int_{\mathbb{R}^n} \frac{|\partial^3 T|^2}{T^{1+a}}\, dx. \tag{4.30}$$

5. Higher Order Entropy Estimates

In this section we investigate higher order entropy estimates for compressible flows spanning the whole space. We establish entropic inequalities when the quantities $\|\log\rho\|_{\mathrm{BMO}}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{\mathrm{BMO}}$, $\|h\partial_x \rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$ are small enough, where $h = 1/(\rho T^{\frac{1}{2}-\varkappa})$ is a weight associated with the dependence of the local mean free path $l = \eta/\rho\sqrt{r_{\mathrm{g}} T}$ on density and temperature. In the following, all constants associated with *a priori* estimates and entropic inequalities may depend on the system parameters $\underline{\mathfrak{a}}$, $\overline{\mathfrak{a}}$, $\overline{\mathfrak{a}}_\sigma$, $\sigma \geq 1$, $\varkappa$, and c_v. However, these dependencies are made implicit in order to avoid notational complexities and only the dependence on k and n is made explicit.

5.1. *Preliminaries*

The balance equations of higher order correctors can be integrated over $\mathbb{R}^n$ and $[0,t]$ where $0 \leq t \leq \bar{t}$ thanks to the regularity properties of the solution. Considering the $\gamma^{[k]}$ balance equation (3.25) as a typical example, we have the following result.

Lemma 5.1. *Let (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6)–(3.7) *and let $1 \leq k \leq l$. Then the following equation holds in $\mathcal{D}'(0,\bar{t})$ and $L^1(0,\bar{t})$*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]} dx + \int_{\mathbb{R}^n} (\pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]})\, dx = 0, \tag{5.1}$$

and the following equation holds in $C^0[0,\bar{t}\,]$

$$\int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \int_0^t \int_{\mathbb{R}^n} (\pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]})\, dx = \int_{\mathbb{R}^n} \gamma_0^{[k]}\, dx, \tag{5.2}$$

where $\gamma_0^{[k]}$ denotes the functional $\gamma^{[k]}$ evaluated at initial conditions.

Proof. This lemma results from standard manipulations using distributional derivatives and test functions in the form of tensor products $\varphi(t)\psi(x)$. □

As a consequence of Lemma 5.1, integrating the balance equation (3.25) for $\gamma^{[k]}$ with $1 \leq k \leq l$, we deduce that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx \leq \int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx + \int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx, \tag{5.3}$$

so that we have to investigate the integrals $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}| dx$ and $\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}| dx$. Similarly, we obtain by integrating the balance equation (3.30) for $\gamma^{[k-\frac{1}{2}]}$ that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k-\frac{1}{2}]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[k-\frac{1}{2}]}\, dx \leq \int_{\mathbb{R}^n} |\Sigma_\gamma^{[k-\frac{1}{2}]}|\, dx + \int_{\mathbb{R}^n} |\omega_\gamma^{[k-\frac{1}{2}]}|\, dx, \tag{5.4}$$

and we have to investigate the integrals $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k-\frac{1}{2}]}| dx$ and $\int_{\mathbb{R}^n} |\omega_\gamma^{[k-\frac{1}{2}]}| dx$. We will simultaneously estimate the analogous integrals $\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k]}| dx$ and $\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}| dx$ associated with the balance equation of $\tilde{\gamma}^{[k]}$ as well as the integrals $\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]}| dx$ and $\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]}| dx$ associated with the balance equations for $\tilde{\gamma}^{[k-\frac{1}{2}]}$.

It will be convenient to denote by χ_γ the quantity

$$\begin{aligned}\chi_\gamma = {}& \|\log \rho\|_{\mathrm{BMO}} + \left\| \frac{v}{\sqrt{T}} \right\|_{L^\infty} + \|\log T\|_{\mathrm{BMO}} \\ & + \left\| h \frac{\partial_x \rho}{\rho} \right\|_{L^\infty} + \left\| h \frac{\partial_x v}{\sqrt{T}} \right\|_{L^\infty} + \left\| h \frac{\partial_x T}{T} \right\|_{L^\infty} + \left\| h^2 \frac{\partial_x^2 T}{T} \right\|_{L^\infty},\end{aligned} \tag{5.5}$$

and similarly by $\chi_{\tilde\gamma}$ the quantity

$$\begin{aligned}\chi_{\tilde\gamma} = \|r\|_{\mathrm{BMO}} + \|w\|_{L^\infty} + \|\tau\|_{\mathrm{BMO}} \\ + \|h\partial_x r\|_{L^\infty} + \|h\partial_x w\|_{L^\infty} + \|h\partial_x \tau\|_{L^\infty} + \|h^2\partial_x^2\tau\|_{L^\infty}.\end{aligned} \tag{5.6}$$

It can easily be established that $\chi_\gamma \le \chi_{\tilde\gamma}(1+\chi_{\tilde\gamma})$ and $\chi_{\tilde\gamma} \le \chi_\gamma(1+\chi_\gamma)$ so that $\chi_\gamma \le 1$ implies that $\frac{1}{3}\chi_\gamma \le \chi_{\tilde\gamma} \le 2\chi_\gamma$, and $\chi_{\tilde\gamma} \le 1$ implies that $\frac{1}{3}\chi_{\tilde\gamma} \le \chi_\gamma \le 2\chi_{\tilde\gamma}$, and assuming that either χ_γ or $\chi_{\tilde\gamma}$ is small is equivalent. We will establish that entropic inequalities hold for $\Gamma^{[k]}$ and $\widetilde\Gamma^{[k]}$ when χ_γ or $\chi_{\tilde\gamma}$ are small enough. These quantities χ_γ and $\chi_{\tilde\gamma}$ are invariant under the change of scales (3.8) described in Remark 3.1. They can also be interpreted as involving the natural variables $\log\rho$, $v/\sqrt{r_{\mathrm{g}}T}$, and $\log T$, appearing in Maxwellian distributions[11] and the natural scale h associated with the local mean free path $\eta/\rho\sqrt{r_{\mathrm{g}}T}$. Since we have formally $v/\sqrt{r_{\mathrm{g}}T} = \mathcal{O}(\mathrm{Ma})$, $\log(T/T_\infty) = \mathcal{O}(\mathrm{Ma})$, and $\log(\rho/\rho_\infty) = \mathcal{O}(\mathrm{Ma})$, where Ma denotes the Mach number, the constraint that χ_γ or $\chi_{\tilde\gamma}$ remain small may be interpreted as a small Mach number constraint, which is consistent with Enskog expansion.[34]

5.2. *A priori estimates*

We first investigate the integrals $\int_{\mathbb{R}^n} |\Sigma_\xi^{[k]}|dx$ and $\int_{\mathbb{R}^n} |\omega_\xi^{[k]}|dx$, where ξ denotes any of the symbols γ or $\tilde\gamma$, by using the weighted inequalities established in Sec. 4.

Proposition 5.2. *Let* (ρ, v, T) *be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7), *let* $1 \le k \le l$, *and let* ξ *denote any of the symbols* γ *or* $\tilde\gamma$. *There exist positive constants* $\delta(k,n)$ *and* $c_k = c(k,n)$ *such that for* $\chi_\xi < \delta$ *we have*

$$\int_{\mathbb{R}^n} |\Sigma_\xi^{[k]}|\,dx \le c_k\chi_\xi \int_{\mathbb{R}^n} \big(\pi_\xi^{[k]} + \pi_\xi^{[k-\frac{1}{2}]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k>2}(\pi_\xi^{[k-\frac{3}{2}]} + \pi_\xi^{[k-2]})\big)\,dx. \tag{5.7}$$

$$\int_{\mathbb{R}^n} |\omega_\xi^{[k]}|\,dx \le c_k\chi_\xi \int_{\mathbb{R}^n} \big(\pi_\xi^{[k]} + \pi_\xi^{[k-\frac{1}{2}]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k>2}(\pi_\xi^{[k-\frac{3}{2}]} + \pi_\xi^{[k-2]})\big)\,dx. \tag{5.8}$$

Proof. We only give the proof for $\xi = \tilde\gamma$ since the proof for $\xi = \gamma$ is similar. We have from (3.36)

$$\Sigma_{\tilde\gamma}^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi} e^{-\varkappa\tau}\partial_\tau^\sigma\phi \Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{e^{\varkappa\tau}c_v} g^2 h^{2(k+1)}|\partial^k r|^2\Delta\tau,$$

and the integral associated with the last term is directly majorized by

$$\int_{\mathbb{R}^n} \frac{\lambda}{e^{\varkappa\tau}} g^2 h^{2(k+1)}|\partial^k r|^2|\Delta\tau|\,dx \le c\|h^2\partial^2\tau\|_{L^\infty} \int_{\mathbb{R}^n} \pi_{\tilde\gamma}^{[k-\frac{1}{2}]}\,dx,$$

where c is a constant since $\lambda e^{-\varkappa\tau}$ is bounded. Considering then the terms of $\Sigma_{\tilde\gamma}^{[k]}$ appearing in the sum we observe that the quantities $e^{-\varkappa\tau}\partial_\tau^\sigma\phi$ are bounded since

$\partial_\tau^\sigma \phi = \sum_{1\le m\le\sigma} c_{\sigma m} T^m \partial_T^m \phi$, where $c_{\sigma m}$ are constants and where $\phi \in \{\kappa, \eta, \lambda\}$, so that we only have to estimate the L^2 norms of the products $\Pi_\nu^{(k+1)}$.

First note that any factor w in $\Pi_\nu^{(k+1)}$ is independently estimated by $\|w\|_{L^\infty}$. When $\Pi_\nu^{(k+1)}$ only contains derivatives of w and τ — in particular if there is a derivative of order $k+1$ — we obtain from Theorem 4.10 applied to (w,τ) with k replaced by $k+1$, that when $\chi_{\tilde\gamma}$ is small enough

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \le c(\|\tau\|_{\mathrm{BMO}} + \|w\|_{L^\infty})^{N_\nu - 1} \left\{ \int_{\mathbb{R}^n} \pi_{\tilde\gamma}^{[k]}\, dx \right\}^{\frac{1}{2}}, \tag{5.9}$$

where $N_\nu = \sum_{1\le|\alpha|\le k+1} (\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = \sum_{1\le|\alpha|\le k+1} (\nu'_\alpha + \nu''_\alpha)$. However, if the product $\Pi_\nu^{(k+1)}$ is split — in particular if there is a derivative of density — we obtain from Theorem 4.11 applied to (r,w,τ) with k replaced by $k+1$, that when $\chi_{\tilde\gamma}$ is small enough

$$\begin{aligned}\|\Pi_\nu^{(k+1)}\|_{L^2} &\le c\|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}}^{N_\nu-2} \|h\partial_x \widetilde{\mathrm{z}}\|_{L^\infty} \|g h^k \partial^k \widetilde{\mathrm{z}}\|_{L^2}\\ &\quad + c\mathbf{1}_{k>2} \|\widetilde{\mathrm{z}}\|_{\mathrm{BMO}}^{(N_\nu-3)^+} \|h\partial_x \widetilde{\mathrm{z}}\|_{L^\infty}^2 \|g h^{k-1} \partial^{k-1} \widetilde{\mathrm{z}}\|_{L^2},\end{aligned}$$

keeping the notation of Theorems 4.11 for $\|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty}$ and $\|gh^m\partial^m\widetilde{\mathrm{z}}\|_{L^2}$. Therefore, we obtain that

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \le c\chi_{\tilde\gamma}^{N_\nu-1} \left\{ \int_{\mathbb{R}^n} (\pi_{\tilde\gamma}^{[k-\frac{1}{2}]} + \pi_{\tilde\gamma}^{[k-1]} + \mathbf{1}_{k>2} (\pi_{\tilde\gamma}^{[k-\frac{3}{2}]} + \pi_{\tilde\gamma}^{[k-2]}))\, dx \right\}^{\frac{1}{2}}, \tag{5.10}$$

where $c = c(k,n)$ thanks to $\chi_{\tilde\gamma} \le 1$ and

$$\|gh^i\partial^i\widetilde{\mathrm{z}}\|_{L^2}^2 \le b \int_{\mathbb{R}^n} (\pi_{\tilde\gamma}^{[i-\frac{1}{2}]} + \pi_{\tilde\gamma}^{[i-1]})\, dx, \quad 1 \le i \le k,$$

where b is independent of i and n. Since one of the two products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split, we can combine the inequalities (5.9) and (5.10) in the form

$$\|\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}\|_{L^1} \le c\chi_{\tilde\gamma} \int_{\mathbb{R}^n} (\pi_{\tilde\gamma}^{[k]} + \pi_{\tilde\gamma}^{[k-1]} + \pi_{\tilde\gamma}^{[k-\frac{1}{2}]} + \mathbf{1}_{k>2}(\pi_{\tilde\gamma}^{[k-\frac{3}{2}]} + \pi_{\tilde\gamma}^{[k-2]}))\, dx,$$

where c depends on k and n. On the other hand, in the expression of $\omega_{\tilde\gamma}^{[k]}$, the products $\Pi_\mu^{(k+1)}$ are always split between several derivative factors, so that the inequality (5.8) is established in a similar way. The proof in the situation $\xi = \gamma$ is similar with Theorems 4.10 and 4.11 replaced by Theorems 4.12 and 4.13. $\square$

Proposition 5.3. *Let* (ρ, v, T) *be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7), *let* $1 \le k \le l$, *and let* ξ *denote any of the symbols* γ *or* $\tilde\gamma$. *There exist positive constants* $\delta(k,n)$ *and*

$c_k = c(k,n)$ such that for $\chi_\xi < \delta$ we have

$$\int_{\mathbb{R}^n} |\Sigma_\xi^{[k-\frac12]}|\,dx \le c_k\chi_\xi \int_{\mathbb{R}^n} (\pi_\xi^{[k]} + \pi_\xi^{[k-\frac12]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k>2}(\pi_\xi^{[k-\frac32]} + \pi_\xi^{[k-2]}))\,dx + \mathsf{c}_0 \left\{\int_{\mathbb{R}^n} \pi_\xi^{[k]}\,dx\right\}^{\frac12} \left\{\int_{\mathbb{R}^n} \pi_\xi^{[k-\frac12]}\,dx\right\}^{\frac12}, \tag{5.11}$$

$$\int_{\mathbb{R}^n} |\omega_\xi^{[k-\frac12]}|\,dx \le c_k\chi_\xi \int_{\mathbb{R}^n} (\pi_\xi^{[k]} + \pi_\xi^{[k-\frac12]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k>2}(\pi_\xi^{[k-\frac32]} + \pi_\xi^{[k-2]}))\,dx + \mathsf{c}_0' \int_{\mathbb{R}^n} \pi_\xi^{[k-1]}\,dx + \mathsf{c}_0' \left\{\int_{\mathbb{R}^n} \pi_\xi^{[k-\frac12]}\,dx\right\}^{\frac12} \left\{\int_{\mathbb{R}^n} \pi_\xi^{[k-1]}\,dx\right\}^{\frac12}, \tag{5.12}$$

where c_0 and c_0' are constants independent of k and n.

Proof. Considering first the case $\xi = \gamma$ and the expression (3.41) for $\Sigma_\gamma^{[k-\frac12]}$, all terms in the sum are estimated as in the proof of Proposition 5.2. More specifically, the L^2 norm of $\Pi_\nu^{(k)}$ is estimated with Theorem 4.12 applied to ρ, v, and T, whereas the L^2 norm of the split product $\Pi_\mu^{(k+1)}$ is estimated with Theorem 4.13 applied to ρ, v, ad T with k replaced by $k+1$. Furthermore, the remaining extra terms are directly estimated in terms of $\pi_\gamma^{[k]}$, $\pi_\gamma^{[k-\frac12]}$ and $\pi_\gamma^{[k-1]}$. The same argument is valid for $\omega_\gamma^{[k-\frac12]}$ using the expression (3.33) as well as in the case $\xi = \tilde\gamma$ using (3.42) and (3.42). $\square$

5.3. *Zeroth order entropic inequalities*

We now recast the classical zeroth order entropic inequality into a convenient form that will be used to investigate entropic principles associated with $\Gamma^{[k]}$.

Proposition 5.4. *Let $\gamma^{[0]}$ be given by (3.17). Then $\gamma^{[0]} \ge 0$ and the following balance equation holds*

$$\partial_t\gamma^{[0]}/\mathsf{C}_0 + \partial_x\cdot\left(\rho v(s_\infty - s) + \rho v c_p\frac{T-T_\infty}{T_\infty} + \rho v\frac{|v|^2}{2T_\infty}\right) + \partial_x\cdot\left(\frac{q}{T_\infty} - \frac{q}{T} + \frac{\Pi\cdot v}{T}\right) + \left(\frac{\lambda|\partial_x T|^2}{T^2} + \frac{\eta|d|^2}{2T} + \frac{\kappa(\partial_x\cdot v)^2}{T}\right)dx = 0. \tag{5.13}$$

Moreover, there exist positive constants B_0 and $\delta_0 > 0$ such that for $\mathsf{C}_0 \ge \mathsf{B}_0$ and $\chi_\gamma < \delta_0$ we have

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\,dx + \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\,dx \le 0, \tag{5.14}$$

where we define from (3.26)

$$\pi_\gamma^{[0]} = 2g^2h^2\left(\frac{\lambda}{T^\varkappa}\frac{|\partial_x T|^2}{T^2} + \frac{\eta}{T^\varkappa}\frac{|\partial_x v|^2}{T} + \frac{\kappa + \frac{n-2}{n}\eta}{T^\varkappa}\frac{(\partial_x\cdot v)^2}{T}\right).$$

Proof. It is easily established that both the temperature and density parts of $\gamma^{[0]}$ are non-negative so that $\gamma^{[0]} \geq 0$. Multiplying the total mass equation by $(\partial_\rho S^{(0)})_\infty = s_\infty - e_\infty/T_\infty - r_{\mathrm{g}}$, the total energy equation by $(\partial_{\mathrm{E^{tot}}} S^{(0)})_\infty = 1/T_\infty$, and subtracting to this linear combination the fluid entropy governing equation yields (5.13). Integrating this balance equation (5.13), keeping in mind the regularity assumptions such that fluxes and sources are in $L^1((0,\bar{t}), L^1(\mathbb{R}^n))$, we obtain that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \mathsf{C}_0 \int_{\mathbb{R}^n} \left(\frac{\lambda |\partial_x T|^2}{T^2} + \frac{\eta |d|^2}{2T} + \frac{\kappa (\partial_x \cdot v)^2}{T} \right) dx = 0.$$

From the properties of the transport coefficients we obtain

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \mathsf{C}_0 (\underline{\mathfrak{a}}/2) \int_{\mathbb{R}^n} T^{\varkappa} \left(\frac{|\partial_x T|^2}{T^2} + \frac{|\partial_x v + (\partial_x v)^t|^2}{T} \right) dx \leq 0.$$

On the other hand, for any $v \in W^{1,2}$ and any index pair (i,j) we have[61]

$$2\partial_j v_i = (\partial_j v_i + \partial_i v_j) - \sum_{1 \leq l \leq n} R_l R_j (\partial_l v_i + \partial_i v_l) + \sum_{1 \leq l \leq n} R_l R_i (\partial_l v_j + \partial_j v_l), \quad (5.15)$$

where $R_i = (-\Delta)^{-1/2} \partial_i$ are the Riesz transforms, $1 \leq i \leq n$, and from the continuity of Calderòn–Zygmund operators in weighted Legesgue spaces established in Theorem 4.6 we deduce that there exists a constant $\bar{c}(n, \varkappa)$ such that

$$\int_{\mathbb{R}^n} \frac{|\partial_x v|^2}{T^{1-\varkappa}} dx \leq \bar{c} \int_{\mathbb{R}^n} \frac{|\partial_x v + (\partial_x v)^t|^2}{T^{1-\varkappa}}\, dx,$$

for $\|\log T\|_{\mathrm{BMO}} < \delta(n, \varkappa)$ small enough. By combining these estimates and by using that $T^{\varkappa} = g^2 h^2$ we obtain

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \mathsf{C}_0 (\underline{\mathfrak{a}}/2\overline{\mathfrak{a}}) \frac{1}{1+\bar{c}} \frac{1}{1+4n} \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx \leq 0,$$

and selecting $\mathsf{C}_0 \geq 2(1+\bar{c})(1+4n)\overline{\mathfrak{a}}/\underline{\mathfrak{a}}$ completes the proof. □

We also recast the classical zeroth order entropic inequality into a convenient form that will be needed to investigate entropic principles associated with $\widetilde{\Gamma}^{[k]}$.

Proposition 5.5. *Let* $\tilde{\gamma}^{[0]} = \gamma^{[0]}$ *by given be* (3.17). *Then* $\tilde{\gamma}^{[0]} \geq 0$ *and the balance equation* (5.13) *holds. Moreover, there exist positive constants* B_0 *and* $\delta_0 > 0$ *such that for* $\mathsf{C}_0 \geq \mathsf{B}_0$ *and* $\chi_{\tilde{\gamma}} < \delta_0$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx \leq 0, \quad (5.16)$$

where we define from (3.35)

$$\pi_{\tilde{\gamma}}^{[0]} = 2g^2 h^2 \left(\frac{\lambda}{e^{\varkappa\tau}} |\partial_x \tau|^2 + \frac{\eta}{e^{\varkappa\tau}} |\partial_x w|^2 + \frac{\kappa + \frac{n-2}{n}\eta}{e^{\varkappa\tau}} (\partial_x \cdot w)^2 \right).$$

Proof. This is a consequence of the proof of Proposition 5.4 and of the differential relations

$$\frac{\partial_i v}{\sqrt{T}} = \partial_i w + \frac{1}{2} w \partial_i \tau, \quad 1 \le i \le n,$$

which yield that $\int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx$ is minorized by $(1 - c\chi_{\tilde{\gamma}}) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx$. □

5.4. *Higher order entropic inequalities*

Our goal in this section is to obtain entropic inequalities for the $(2k)$th order kinetic entropy estimators

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1 \le i \le k} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]}) \quad k \ge 0 \tag{5.17}$$

and

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \sum_{1 \le i \le k} (\tilde{\gamma}^{[i]} + a\tilde{\gamma}^{[i-\frac{1}{2}]}) \quad k \ge 0. \tag{5.18}$$

The quantities $\gamma^{[i-\frac{1}{2}]}$ and $\tilde{\gamma}^{[i-\frac{1}{2}]}$, $1 \le i \le k$, are multiplied by a small rescaling factor a in (5.17) and (5.18) so as not to modify the majorizing properties of the correctors $\gamma^{[i]}$ and $\tilde{\gamma}^{[i]}$, $i \ge 0$.

Lemma 5.6. *Let (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7), *assume that $T \ge T_{\min}$. There exists* $\mathsf{B}_0(T_{\min}/T_\infty)$ *such that for* $\mathsf{C}_0 \ge \mathsf{B}_0$, $0 < a \le 1$, *and* $0 \le k \le l$

$$\frac{1}{2}(\gamma^{[0]} + \cdots + \gamma^{[k]}) \le \Gamma^{[k]} \le \frac{3}{2}(\gamma^{[0]} + \cdots + \gamma^{[k]}), \quad 0 \le k \le l, \tag{5.19}$$

$$\frac{1}{2}(\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}) \le \widetilde{\Gamma}^{[k]} \le \frac{3}{2}(\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}), \quad 0 \le k \le l. \tag{5.20}$$

Moreover, assuming that $T \ge T_{\min}$ and $\rho \le \rho_{\max}$, there exists $\mathsf{B}_0(T_{\min}/T_\infty, \rho_\infty/\rho_{\max})$ *such that for* $\mathsf{C}_0 \ge \mathsf{B}_0$,

$$\rho(|r - r_\infty|^2 + |w|^2 + c_v|\tau - \tau_\infty|^2) \le \gamma^{[0]}. \tag{5.21}$$

Proof. Using the Cauchy–Schwarz inequality, it is straightforward to check that for any $1 \le i \le k \le l$

$$\begin{aligned} |\gamma^{[i-\frac{1}{2}]}| &\le \left\{ \rho h^{2(i-1)} \left| \frac{\partial^{i-1} v}{\sqrt{T}} \right|^2 \right\}^{\frac{1}{2}} \left\{ \rho h^{2i} \left| \frac{\partial^i \rho}{\rho} \right|^2 \right\}^{\frac{1}{2}} \\ &\le \frac{1}{2} \left(\rho h^{2(i-1)} \left| \frac{\partial^{i-1} v}{\sqrt{T}} \right|^2 + \rho h^{2i} \left| \frac{\partial^i \rho}{\rho} \right|^2 \right). \end{aligned}$$

Therefore, half of the density part of $\gamma^{[i]}$ and of the velocity part of $\gamma^{[i-1]}$ compensate for $|\gamma^{[i-\frac{1}{2}]}|$ provided we ensure that $\gamma^{[0]} \ge \rho |v/\sqrt{T}|^2$ but this is a consequence of $\mathsf{C}_0 \ge 2T_\infty/T_{\min}$. The same method also applies for the modified estimators

$\tilde{\gamma}^{[i-\frac{1}{2}]}$, $1 \le i \le k$, and this yields inequalities (5.19) and (5.20) upon summing over $1 \le i \le k$. Inequality (5.21) is a consequence of

$$\frac{T_{\min}}{2T_\infty}|w|^2 \le \frac{|v|^2}{2T_\infty},$$

$$\frac{T_{\min}}{2T_\infty}|\tau - \tau_\infty|^2 \le \exp(\tau - \tau_\infty) - 1 - (\tau - \tau_\infty),$$

valid for $\tau_{\min} \le \tau$, where $\tau_{\min} = \log T_{\min}$, $\tau_\infty = \log T_\infty$ and $T_{\min} \le T_\infty$, and of

$$\frac{\rho_\infty}{2\rho_{\max}}|r - r_\infty|^2 \le \exp(r_\infty - r) - 1 - (r_\infty - r),$$

valid for $r \le r_{\max}$, where $r_{\max} = \log \rho_{\max}$, $r_\infty = \log \rho_\infty$ and $r_\infty \le r_{\max}$ letting $\mathsf{B}_0 = \max(1, \frac{2T_\infty}{T_{\min}}, \frac{\rho_{\max}}{2\rho_\infty})$ and $\mathsf{C}_0 \ge \mathsf{B}_0$. □

In the following, we assume that C_0 has been chosen large enough such that the inequalities of Propositions 5.4–5.5, and 5.6 hold.

Theorem 5.7. *Let (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6)–(3.7) *and let $1 \le k \le l$. There exist positive constants $\bar{a} \le 1$ and $\delta_{\mathrm{N}}(k,n)$ such that for $a \le \bar{a}$ and $\chi_\gamma < \delta_{\mathrm{N}} a$ we have*

$$\partial_t \int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + \frac{1}{5}\int_{\mathbb{R}^n} \left(\pi_\gamma^{[0]} + \sum_{1\le i\le k} (\pi_\gamma^{[i]} + a\pi_\gamma^{[i-\frac{1}{2}]})\right) dx \le 0, \tag{5.22}$$

and for $a \le \bar{a}$ and $\chi_{\tilde{\gamma}} < \delta_{\mathrm{N}} a$ we have

$$\partial_t \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx + \frac{1}{5}\int_{\mathbb{R}^n} \left(\pi_{\tilde{\gamma}}^{[0]} + \sum_{1\le i\le k} (\pi_{\tilde{\gamma}}^{[i]} + a\pi_{\tilde{\gamma}}^{[i-\frac{1}{2}]})\right) dx \le 0. \tag{5.23}$$

Proof. We only consider the case $\xi = \gamma$ since the proof is similar for the modified estimators $\xi = \tilde{\gamma}$. From the differential inequality (5.3) for $\gamma^{[i]}$, $1 \le i \le k \le l$, and the results of Proposition 5.2, we obtain that

$$\begin{aligned} \partial_t \int_{\mathbb{R}^n} \gamma^{[i]}\, dx + (1 - 2c_i\chi_\gamma)\int_{\mathbb{R}^n} \pi_\gamma^{[i]}\, dx &\le 2c_i\chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{1}{2}]} + \pi_\gamma^{[i-1]})\, dx \\ &\quad + \mathbf{1}_{i>2} 2c_i\chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx. \end{aligned} \tag{5.24}$$

Similarly, from the differential inequality (5.4), and the results of Proposition 5.3 we obtain that

$$\begin{aligned} \partial_t \int_{\mathbb{R}^n} \gamma^{[i-\frac{1}{2}]}\, dx + (1 - 2\epsilon_0 - 2c_i\chi_\gamma)\int_{\mathbb{R}^n} \pi_\gamma^{[i-\frac{1}{2}]}\, dx \le \left(\frac{\mathsf{c}_0^2}{4\epsilon_0} + 2c_i\chi_\gamma\right)\int_{\mathbb{R}^n} \pi_\gamma^{[i]}\, dx \\ + \left(\mathsf{c}_0' + \frac{\mathsf{c}_0'^2}{4\epsilon_0} + 2c_i\chi_\gamma\right)\int_{\mathbb{R}^n} \pi_\gamma^{[i-1]}\, dx + \mathbf{1}_{i>2} 2c_i\chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx. \end{aligned} \tag{5.25}$$

Forming (5.24)+a(5.25), we obtain after some algebra

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^n} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]})\, dx + \left(1 - 2c_i\chi_\gamma - a\left(\frac{\mathsf{c}_0^2}{4\epsilon_0} + 2c_i\chi_\gamma\right)\right) \int_{\mathbb{R}^n} \pi_\gamma^{[i]}\, dx \\
&\qquad + (a(1 - 2\epsilon_0 - 2c_i\chi_\gamma) - 2c_i\chi_\gamma) \int_{\mathbb{R}^n} \pi_\gamma^{[i-\frac{1}{2}]}\, dx \\
&\quad \le \left(a\left(\mathsf{c}_0' + \frac{\mathsf{c}_0'^2}{4\epsilon_0} + 2c_i\chi_\gamma\right) + 2c_i\chi_\gamma\right) \int_{\mathbb{R}^n} \pi_\gamma^{[i-1]}\, dx \\
&\qquad + \mathbf{1}_{i>2} 2(1+a)c_i\chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx.
\end{aligned} \tag{5.26}$$

Assuming then that

$$\begin{aligned}
&0 < a \le 1, \quad 2\epsilon_0 = \frac{1}{10}, \quad 2(\max_{1\le i\le k} c_i)\chi_\gamma \le \frac{a}{10}, \\
&\qquad a\frac{\mathsf{c}_0^2}{4\epsilon_0} \le \frac{1}{10}, \quad a\left(\mathsf{c}_0' + \frac{\mathsf{c}_0'^2}{4\epsilon_0}\right) \le \frac{1}{10},
\end{aligned}$$

that is, $a \le \bar{a}$ and $\chi_\gamma < \delta_{\mathrm{N}} a$ with

$$\bar{a} = \min\left(1, \frac{4\epsilon_0}{10\mathsf{c}_0^2}, \frac{4\epsilon_0}{10(\mathsf{c}_0'^2 + 4\epsilon_0\mathsf{c}_0')}\right), \quad \delta_{\mathrm{N}} = \frac{1}{20 \max\limits_{1\le i\le k} c_i},$$

we obtain that

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^n} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]})\, dx + \frac{7}{10} \int_{\mathbb{R}^n} (\pi_\gamma^{[i]} + a\pi_\gamma^{[i-\frac{1}{2}]})\, dx \\
&\qquad \le \frac{3}{10} \int_{\mathbb{R}^n} \pi_\gamma^{[i-1]}\, dx + \mathbf{1}_{i>2} \frac{2}{10}\chi_\gamma \int_{\mathbb{R}^n} (a\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx.
\end{aligned} \tag{5.27}$$

Summing for $1 \le i \le k$, and adding to the zeroth order inequality (5.14) we finally obtain (5.22) and the proof of (5.23) is similar. □

Corollary 5.8. *Let* (ρ, v, T) *be a smooth solution of the compressible Navier–Stokes equations* (3.1)–(3.5) *with regularity* (3.6), (3.7) *and let* $1 \le k \le l$. *There exist positive constants* C_0, $\bar{a} \le 1$, $\underline{b}$ *and* $\delta_{\mathrm{N}}'(k, n)$ *such that when* $\chi_\gamma < \delta_{\mathrm{N}}'$ *we have*

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^n} (\gamma^{[0]} + \cdots + \gamma^{[k]} \bar{a}(\gamma^{[\frac{1}{2}]} + \cdots + \gamma^{[k-\frac{1}{2}]}))\, dx \\
&\qquad + \underline{b} \int_{\mathbb{R}^n} \rho T^{1-\varkappa}(\gamma^{[1]} + \cdots + \gamma^{[k]})\, dx \le 0,
\end{aligned} \tag{5.28}$$

and when $\chi_{\tilde{\gamma}} < \delta_{\mathrm{N}}'$ *we have*

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^n} (\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]} + \bar{a}(\tilde{\gamma}^{[\frac{1}{2}]} + \cdots + \tilde{\gamma}^{[k-\frac{1}{2}]}))\, dx \\
&\qquad + \underline{b} \int_{\mathbb{R}^n} \rho T^{1-\varkappa}(\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]})\, dx \le 0.
\end{aligned} \tag{5.29}$$

Proof. This is a consequence of Theorem 5.7 in the special situation $a = \bar{a}$ letting $\delta'_{\rm N} = \delta_{\rm N}\bar{a}$, $\underline{b} = \bar{a}\min(1, \underline{\mathfrak{a}})/5(1 + c_v)$, and using $\rho T^{1-\varkappa} = g^2/\rho$. □

Theorem 5.7 and Corollary 5.8 show that the $(2k)$th order kinetic entropy estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ obey entropic principles. Upon integrating these inequalities (5.28) and (5.29), *a priori* estimates are obtained for the solutions of the compressible Navier–Stokes equations. These entropic inequalities and the related *a priori* estimates are also invariant — up to a multiplicative factor — by the change of scales (3.8) described in Remark 3.1 and naturally associated with the Navier–Stokes equations. Since we have formally $v/\sqrt{r_{\rm g}T} = \mathcal{O}(\mathrm{Ma})$, $\log(T/T_\infty) = \mathcal{O}(\mathrm{Ma})$, and $\log(\rho/\rho_\infty) = \mathcal{O}(\mathrm{Ma})$, where Ma denotes the Mach number, the constraint that χ_γ or $\chi_{\tilde{\gamma}}$ remain small may be interpreted as a small Mach number constraint, which is consistent with Enskog expansion.[34] These estimates also provide a thermodynamic interpretation of the corresponding weighted Sobolev norms involving either renormalized derivatives for $\Gamma^{[k]}$, or derivatives of the renormalized variable $\widetilde{\mathrm{z}}$ — which is also a normal variable[33,44] — for $\widetilde{\Gamma}^{[k]}$, and involving as well the dependence on density and temperature of the local mean free path through the factor h. This factor h ensures in particular that the operator $h\partial_x$ is scale invariant.

Many recent works have been devoted to the compressible Navier–Stokes equations and related *a priori* estimates. Estimates for smooth solutions are generally obtained upon deriving the governing equations, multiplying by the solution derivatives and integrating in space and time, whereas estimates for weak solutions are usually derived from energy and zeroth order entropy estimates as well as by using renormalized equations.

Danchin[15,16] has established the existence of global solutions around constant equilibrium states in critical hybrid Besov spaces with minimum regularity. In order to established this existence result, Danchin has derived *a priori* estimates for linearized equations in hybrid Besov spaces using Littlewood–Paley decompositions. Since Danchin only considered the scaling properties associated with incompressible models, some norms used in these papers do not appear to be scaling invariant with respect to the two-parameter family of transformations of Remark 3.1 but scaling invariance is easily recovered upon restoring the dependence of various constants on the state at infinity (ρ_∞, T_∞). The main difference with Danchin's estimates is that there are established for linearized equations and contain an explicit time dependence. They are also established with equations with constant transport coefficients and associated with the functions $(\rho - \rho_\infty)/\rho_\infty$, $v/\sqrt{T_\infty}$, $(T - T_\infty)/T_\infty$, whereas the entropic estimates mainly consider $\log(\rho/\rho_\infty)$, $v/\sqrt{T}$, and $\log(T/T_\infty)$. Another difference is that in Danchin's estimates, the velocity and the gradient of the density have the same regularity. This is a key point when dealing with data with low regularity which is not taken into account with the entropic estimates derived in this paper. In particular, Danchin's existence result is stronger than the example of application presented in the next section.

The general estimates of Kawashima[42,43] for symmetric hyperbolic-parabolic composite systems can be applied to the compressible flows equations and the results be compared with higher order entropic estimates. In particular, the extra correctors $\gamma^{[k-\frac{1}{2}]}$ or $\tilde{\gamma}^{[k-\frac{1}{2}]}$ are similar to the perturbed quadratic terms introduced by Kawashima for linearized equations around equilibrium states and decay estimates. The differences are that we are using renormalized variable as well as powers of h as extra weights factors in order to maintain scaling invariance and we also directly obtain differential inequalities associated with solutions of non linear equations.

Alazard[2] has investigated local existence of smooth solutions and the limit of small Mach numbers for a family of flows covering inviscid as well as viscous flows. Some norms used in this paper do not appear to be scaling invariant but invariance is easily recovered upon restoring the dependence of various constants in the state at infinity (p_∞, T_∞). The main difference with Alazard's estimates is that there are established for linearized equations and contain an explicit time dependence. They are also associated with the functions $\log(p/p_\infty)$, $v/\sqrt{T_\infty}$, $\log(T/T_\infty)$, whereas the entropic estimates mainly consider $\log(\rho/\rho_\infty)$, $v/\sqrt{T}$, and $\log(T/T_\infty)$. On the other hand, Alazard's estimates are established for a *family of flows* encompasing inviscid as well as viscous flows and are uniform with respect to the flow parameters.[2] In addition, the linearized equations are unstable because of the large temperature variations so that the estimates cannot be obtained by differentiating nor localizing in frequency spaces by means of littlewood-Paley operators.[2]

Hoff[37,38] has investigated the existence of discontinuous solutions around constant equilibrium states for $n = 2$ and $n = 3$. The transport coefficients are assumed to be constants in these studies and there is a constraint on the ratio κ/η. The various estimates are essentially associated with the energy and the zeroth order entropy inequalities. Hoff has shown in particular the importance of the effective viscous pressure $p^e = (\kappa + \frac{2(n-1)}{n}\eta)\partial_x\cdot v - (p - p_\infty)$ which naturally arises in the governing equations. This quantity is free of jump discontinuities, has weak continuity properties, and has also been used by Lions, Vaigan and Khazikhov, and Feireisl. This quantity scales like ρT so that the rescaled effective pressure $(1/\rho T)^2 h^{2k}|\partial^k p^e|^2$ can be estimated in terms of $\gamma^{[0]}+\cdots+\gamma^{[k+1]}$, but the effective viscous pressure does not seem to play a fundamental rôle for smooth solutions as it does for discontinuous or weak solutions.

Lions[49] has investigated the existence of global weak periodic solutions in dimensions $n = 2$ and $n = 3$ upon modifying the state law and the thermal conductivity coefficient. The pressure has been taken in the form $p = g(\rho)(T + \delta)$, where $\delta \geq 0$ and g is a continuous non-decreasing function with $g(0) = 0$, $\lim_{\rho\to\infty} g(\rho)\rho^{-a}$ exists and is positive with $a > 1$, and $\int_0^1 (g(s)/s^2)ds < \infty$. The thermal conductivity has been taken such that $\lambda = \lambda(T)$ is continuous for $T \geq 0$ and $\lim_{T\to\infty} \lambda(T)T^{-b}$ exists and is positive. In his pioneering books, Lions has established the existence of global weak solutions to the compressible Navier–Stokes equations for a and b large enough using the energy and zeroth order entropy estimates as well as compactness properties of weak sequences of approximated solutions.[49]

Feireisl[23,24] has investigated the existence of global weak solutions in dimensions $n = 2$ and $n = 3$. Feireisl has stabilized the governing equations at low/high temperatures and low/high dentities by introducing in particular a cold pressure. The pressure is in the form $p(\rho, T) = p_c(\rho) + Tp_\theta(\rho)$ where $p_c(0) = 0$, $p_c'(\rho) \geq a_1\rho^{\gamma-1} - b$ for $\rho > 0$, $p_c(\rho) \leq a_2\rho^\gamma + b$ for $\rho \geq 0$, and $p_\theta(0) = 0$, $p_\theta'(\rho) \geq 0$ for $\rho > 0$, and $p_\theta(\rho) \leq a_3\rho^\Gamma + b$ for $\rho \geq 0$, where $\gamma > n/2$, $\Gamma < \gamma/2$ if $n = 2$, $\Gamma = \gamma/3$ if $n = 3$, and a_1, a_2, a_3, and b and positive constants. The transport coefficients are assumed to depend on temperature and such that $0 < \underline{\mathfrak{a}} \leq \eta(T) \leq \overline{\mathfrak{a}}$, $|\kappa(T)| \leq \overline{\mathfrak{a}}$ for $T \geq 0$, and $0 < \underline{\mathfrak{a}}(1+T^\alpha) \leq \lambda(T) \leq \overline{\mathfrak{a}}(1+T^\alpha)$ for $T \geq 0$ where $\alpha \geq 2$. Such an assumption for $\lambda(T)$ yields in particular an L^2 estimate of $\partial_x T$ from the zeroth order entropic estimates. Feireisl's estimates are then essentially that of energy and zeroth order entropy combined with the notion of renormalized solutions and weak limits in L^1. Among the fundamental difficulties are density oscillations and concentrations in temperature.[23] Feireirsl has used in particular the weak continuity properties of the the effective viscous pressure $p^e = (\kappa + 2\eta - \frac{2}{n}\eta)\partial_x\cdot v - (p - p_\infty)$.

Bresch and Desjardins[8,9] have investigated the existence of global weak solutions for compressible Navier–Stokes equations. In their study, they have stabilized the governing equations at low/high temperatures and low/high dentities by introducing in particular a cold pressure. More specifically, the state law $p = p_c + Tp_\theta$ contain a cold pressure term p_c which may still vanish away from zero.[8,9] The transport coefficients $\kappa(\rho)$ and $\eta(\rho)$ only depend on the density ρ and are such that there exists a constraint in the form $\kappa(s) - \frac{2}{n}\eta(s) = 2(s\eta'(s) - \eta(s))$ (in our notation). We refer to Bresch and Desjardins[8,9] for the full set of assumptions of the state law and the transport coefficients. Under these assumptions, Bresch and Desjardin have obtained existence of global weak solutions by using a new entropy in the form (in our notation)

$$\rho T \left| \frac{v}{\sqrt{T}} + 2\frac{\eta'(\rho)}{\sqrt{T}}\frac{\partial_x \rho}{\rho} \right|^2,$$

which presents many similarities with the higher entropy correctors $\gamma^{[1]}$ and $\gamma^{[\frac{1}{2}]}$. These are the estimates which are closer in spirit to the higher order entropic inequalities investigated in this paper.

6. Global Solutions

Many results have been devoted to the existence of solutions to the compressible Navier–Stokes equations. Local existence of smooth solutions has been established by Nash[57] and global existence around equilibrium states by Matsumura and Nishida.[51] Kawashima has established global existence of smooth solutions around constant equilibrium states for composite hyperbolic-parabolic symmetric systems[42,43] which can also be applied to the situation of compressible flows. The cases of multicomponent flows with complex chemistry and ambipolar reactive plasmas have also been investigated.[32,33]

More recently, Danchin[15,16] has established global existence of solutions in critical hybrid Besov spaces with minimum regularity for the isentropic as well as the full compressible model around constant equilibrium states.

Hoff[37,38] has also investigated discontinous solutions with small data. Alazard,[2] Danchin,[17,18] and Feireisl and Novotnỳ[25] have further investigated the limit of small Mach number flows in various functional settings.

With respect to weak solutions, we mention the pioneering work of Lions[49] as well as the fundamental results of Feireisl,[23,24] Bresch and Desjardins,[8,9] Bresch, Desjardins, and Vallet,[10] and Mellet and Vasseur.[50]

Our aim in this section is more limited since we only want to illustrate higher order entropy estimates. Therefore, we investigate global existence of smooth solutions when the initial values $\log(\rho_0/\rho_\infty)$, $v_0/\sqrt{T_0}$, and $\log(T_0/T_\infty)$ are small enough in appropriate weighted spaces. Although the set of assumptions (3.11)–(3.12) on transport coefficents derived from the kinetic theory is new, we do not claim originality in these existence results — since it is well known that such smooth solutions exists — but in their proof which illutrates the use of higher order entropic estimates.

6.1. *Local existence*

We denote by z the combined unknown $\mathrm{z} = (\rho, v, T)$ and accordingly by z_∞ the equilibrium point $\mathrm{z}_\infty = (\rho_\infty, 0, T_\infty)$ with $\rho_\infty > 0$, $v_\infty = 0$ and $T_\infty > 0$. We denote by $\mathcal{O}_\mathrm{z} = (0,\infty) \times \mathbb{R}^n \times (0,\infty)$ the natural domain for the variable z.

Theorem 6.1. *Let $n \geq 1$ and $l \geq [n/2] + 3$ be integers and let $b > 0$ be given. Let $\mathcal{O}_0$ be an open bounded convex set such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_\mathrm{z}$, d_1 with $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_\mathrm{z})$, and define $\mathcal{O}_1 = \{\mathrm{z} \in \mathcal{O}_\mathrm{z};\ d(\mathrm{z}, \overline{\mathcal{O}}_0) < d_1\}$. There exists $\bar{t} > 0$ small enough, which only depends on $\mathcal{O}_0$, d_1 and b, such that for any z_0 with $\|\mathrm{z}_0 - \mathrm{z}_\infty\|_{W^{l,2}} < b$ and $\mathrm{z}_0 \in \overline{\mathcal{O}}_0$, there exists a unique local solution $\mathrm{z} = (\rho, v, T)$ to the system* (3.1)–(3.3) *with initial condition*

$$(\rho(0,x), v(0,x), T(0,x)) = (\rho_0(x), v_0(x), T_0(x)), \tag{6.1}$$

such that

$$(\rho(t,x), v(t,x), T(t,x)) \in \mathcal{O}_1, \tag{6.2}$$

and

$$\rho - \rho_\infty \in C^0([0,\bar{t}], W^{l,2}(\mathbb{R}^n)) \cap C^1([0,\bar{t}], W^{l-1,2}(\mathbb{R}^n)), \tag{6.3}$$

$$v, T - T_\infty \in C^0([0,\bar{t}], W^{l,2}(\mathbb{R}^n)) \cap C^1([0,\bar{t}], W^{l-2,2}(\mathbb{R}^n)) \cap L^2((0,\bar{t}), W^{l+1,2}(\mathbb{R}^n)). \tag{6.4}$$

In addition, there exists $C > 0$ which only depends on $\mathcal{O}_0$, d_1, and b, such that

$$\sup_{0\leq s\leq \bar{t}} \{\|\rho(s)-\rho_\infty\|^2_{W^{l,2}} + \|v(s)\|^2_{W^{l,2}} + \|T(s)-T_\infty\|^2_{W^{l,2}}\}$$
$$+ \int_0^{\bar{t}} \{\|\rho(s)-\rho_\infty\|^2_{W^{l,2}} + \|v(s)\|^2_{W^{l+1,2}} + \|T(s)-T_\infty\|^2_{W^{l+1,2}}\}\, ds$$
$$\leq C(\|\rho_0-\rho_\infty\|^2_{W^{l,2}} + \|v_0\|^2_{W^{l,2}} + \|T_0-T_\infty\|^2_{W^{l,2}}). \tag{6.5}$$

Proof. There are many proofs for local existence of solutions in various functional settings.[27,41–45,51,57,63] We refer the reader to Kawashima[42,43] for a general proof concerning hyperbolic–parabolic symmetric systems in normal form. This proof is also adapted to the parameter dependent case in Giovangigli and Graille.[32] □

6.2. *Properties of the solutions*

We establish in this section that the solutions constructed in Theorem 6.1 are as smooth as expected from initial data.

Theorem 6.2. *The solutions obtained in Theorem* 6.1 *inherit the regularity of* z_0, *that is, for any* $k \geq l$ *such that* $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{k,2}$, *we have*

$$\rho - \rho_\infty \in C^0([0,\bar{t}], W^{k,2}) \cap C^1([0,\bar{t}], W^{k-1,2}), \tag{6.6}$$

$$v, T - T_\infty \in C^0([0,\bar{t}], W^{k,2}) \cap C^1([0,\bar{t}], W^{k-2,2}) \cap L^2((0,\bar{t}), W^{k+1,2}). \tag{6.7}$$

In particular, z *is smooth when* $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{k,2}(\mathbb{R}^n)$ *for any* $k \in \mathbb{N}$.

Proof. Let $k \geq l$ be such that $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{k,2}$ and denote by $e^{[k]}$ the quantity $e^{[k]} = |\partial^k \rho|^2 + |\partial^k v|^2 + |\partial^k T|^2$. We have to estimate $e^{[k]}$ in order to establish (6.6), (6.7).

Assume first that the regularity properties (6.6), (6.7) hold. A balance equation for $e^{[k]}$ can easily be derived — and is simpler than that of $\gamma^{[k]}$ of $\tilde{\gamma}^{[k]}$ — and written in the form

$$\partial_t e^{[k]} + \partial_x\cdot(v e^{[k]}) + \partial_x\cdot \varphi_e^{[k]} + \pi_e^{[k]} + \Sigma_e^{[k]} + \omega_e^{[k]} = 0. \tag{6.8}$$

This equation holds in $\mathcal{D}'((0,\bar{t})\times\mathbb{R}^n)$ and $L^1((0,\bar{t}), W^{-1,1})$, $e^{[k]} \in C^0([0,\bar{t}], L^1)$, and $\varphi_e^{[k]}$, $\pi_e^{[k]}$, $\Sigma_e^{[k]}$, $\omega_e^{[k]} \in L^1((0,\bar{t}), L^1(\mathbb{R}^n))$. The term $\pi_e^{[k]}$ is given by

$$\pi_e^{[k]} = \frac{2\lambda}{\rho c_v}|\partial^{k+1}T|^2 + \frac{2\eta}{\rho}|\partial^{k+1}v|^2 + \frac{2(\frac{1}{3}\eta + \kappa)}{\rho}|\partial^k(\partial_x\cdot v)|^2, \tag{6.9}$$

and the term $\Sigma_e^{[k]}$ is of the form

$$\Sigma_e^{[k]} = \sum_{\sigma\nu\mu} c_{\sigma\nu\mu\phi} T^{a_{\nu\mu\phi}} \rho^{b_{\nu\mu\phi}} \partial_T^\sigma \phi \widehat{\Pi}_\nu^{(k+1)} \widehat{\Pi}_\mu^{(k+1)}, \tag{6.10}$$

where the sums are over $0 \le \sigma \le k$, $\phi \in \{\lambda, \eta, \kappa\}$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The quantities $a_{\nu\mu\phi}$ and $b_{\nu\mu\phi}$ are integers depending on ν, μ and ϕ. The products $\widehat{\Pi}_\nu^{(k+1)}$ are defined by

$$\widehat{\Pi}_\nu^{(k+1)} = \prod_{1\le|\alpha|\le k+1} (\partial^\alpha \rho)^{\nu_\alpha} (\partial^\alpha v)^{\nu'_\alpha} (\partial^\alpha T)^{\nu''_\alpha}, \tag{6.11}$$

where v denotes any of its components $v_1, \ldots, v_n$, and ν must be such that

$$\sum_{1\le|\alpha|\le k+1} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1} \nu_\alpha = 0,$$

so that there is a total of $k+1$ derivations and there is no derivative of order $k+1$ of density. In addition, we have $\sum_{|\alpha|=k+1}(\nu'_\alpha + \nu''_\alpha + \mu'_\alpha + \mu''_\alpha) \le 1$, so that there is at most one derivative of $(k+1)$th order in the product $\widehat{\Pi}_\nu^{(k+1)}\widehat{\Pi}_\mu^{(k+1)}$. Furthermore the term $\omega_e^{[k]}$ is given by

$$\omega_e^{[k]} = \sum_{\nu\mu} c_{\nu\mu} T^{a_{\nu\mu}} \rho^{b_{\nu\mu}} \widehat{\Pi}_\nu^{(k)} \widehat{\Pi}_\mu^{(k+1)} - \frac{2T}{c_v} \partial^k T \partial^k (\partial_x \cdot v) \tag{6.12}$$

$$+ \frac{2T}{\rho} \partial^k \rho \partial^k (\partial_x \cdot v) - 2\rho \partial^k \rho \partial^k (\partial_x \cdot v), \tag{6.13}$$

where we use similar notation for $\widehat{\Pi}_\nu^{(k)}$ as for $\widehat{\Pi}_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k, \qquad \sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = k+1,$$

so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = 0$ and there are always at least two derivative factors in the product $\widehat{\Pi}_\mu^{(k+1)}$. Finally the flux $\varphi_e^{[k]} = (\varphi_{e1}^{[k]}, \ldots, \varphi_{en}^{[k]})$ is in the form

$$\varphi_{el}^{[k]} = \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu\phi l} T^{a_{\nu\mu\phi}} \rho^{b_{\nu\mu\phi}} \partial_T^\sigma \phi \widehat{\Pi}_\nu^{(k)} \widehat{\Pi}_\mu^{(k+1)}. \tag{6.14}$$

After integrating Eq. (6.8) over $\mathbb{R}^n$ and using uniform lower bounds on $\lambda/\rho c_v$ and η/ρ, thanks to (6.2), we obtain that there exists a $\delta > 0$ with

$$\partial_t \int_{\mathbb{R}^n} e^{[j]}\, dx + 2\delta \int_{\mathbb{R}^n} (|\partial^{j+1} T|^2 + |\partial^{j+1} v|^2)\, dx \le c \int_{\mathbb{R}^n} (|\Sigma_e^{[k]}| + |\omega_e^{[k]}|)\, dx, \quad 1 \le j \le k.$$

Now regrouping all derivatives of order $k+1$ appearing in $\Sigma_e^{[k]}$ in the left member, using $xy \le \epsilon x^2 + 4y^2/\epsilon$, we only have to estimate the L^2 norm of multiple products with $k+1$ derivations $\widehat{\Pi}_\nu^{(k+1)}$ with at least two derivative factors or of multiple products with only k derivations $\widehat{\Pi}_\nu^{(k)}$. From Theorem 4.8 and since $\|\mathrm{z} - \mathrm{z}_\infty\|_{L^\infty}$, and $\|\partial \mathrm{z}\|_{L^\infty}$, are finite thanks to $l > n/2+1$, whenever the product $\widehat{\Pi}_\nu^{(k+1)}$ is split, we have estimates in the form

$$\|\widehat{\Pi}_\nu^{(k+1)}\|_{L^2}^2 \leq c(1 + \|\partial \mathrm{z}\|_{L^\infty})^{2(k-1)} \int_{\mathbb{R}^n} (e^{[1]} + \cdots + e^{[k]})\, dx,$$

where c only depends on $\|\mathrm{z}\|_{L^\infty}$. The products $\widehat{\Pi}_\nu^{(k)}$ are also estimated, thanks to Theorem 4.7. Combining these estimates, we obtain after some algebra that

$$\partial_t \int_{\mathbb{R}^n} e^{[j]}\, dx + \delta \int_{\mathbb{R}^n} (|\partial^{j+1} T|^2 + |\partial^{j+1} v|^2)\, dx \leq c \int_{\mathbb{R}^n} (e^{[1]} + \cdots + e^{[k]})\, dx, \quad 1 \leq j \leq k,$$

where δ and c depend on L^∞ estimates of z and ∂z. Upon summing these inequalities and using Gronwall lemma we deduce that $\int_{\mathbb{R}^n} e^{[k]}\, dx$ remain uniformly bounded over the whole time interval under consideration $[0, \bar{t}\,]$ and we thus have a uniform upper bound B for the sobolev norm $\|\mathrm{z} - \mathrm{z}_\infty\|_{W^{k,2}} \leq B$. This also implies that $\int_0^{\bar{t}} \int_{\mathbb{R}^n} |\partial^{j+1} T|^2\, dxdt$ and $\int_0^{\bar{t}} \int_{\mathbb{R}^n} |\partial^{j+1} v|^2\, dxdt$ are finite.

Now from the local existence theorem, there exists a positive time $0 < t' \leq \bar{t}$ constructed with the parameters $\mathcal{O}_0$, d_1 and $2B$, where a solution with regularity (6.6), (6.7) exists and coincide with z. The preceding estimates then show that the local existence theorem can be used repeatedly over $[0, \bar{t}\,]$ since we have the uniform bound $\|\mathrm{z} - \mathrm{z}_\infty\|_{W^{k,2}} \leq B$ over this interval so that finally (6.6), (6.7) hold over $[0, \bar{t}]$. Moreover, when $\mathrm{z}_0 - \mathrm{z}_\infty$ is in $W^{k,2}$ for any $k \geq 0$, $\mathrm{z} - \mathrm{z}_\infty$ is in $C^0([0, \bar{t}\,], W^{k,2})$ for any k, and we recover the regularity with respect to time from the governing equations so that z is smooth. □

In the next propositions, we reformulate for convenience the local existence theorem in terms of the combined unknown $\widetilde{\mathrm{z}} = (r, w, \tau)$ associated with the renormalized variables r, w and τ.

Lemma 6.3. *Denote by* $\mathcal{F} : (0,\infty) \times \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^{n+2}$ *the application defined by* $\mathcal{F}(\mathrm{z}) = \widetilde{\mathrm{z}}$, *that is,* $\mathcal{F}(\rho, v, T) = (r, w, \tau) = (\log \rho, v/\sqrt{T}, \log T)$. *Then* $\mathcal{F}$ *is a* C^∞ *diffeomorphism and its Jacobian matrix reads*

$$\partial_{\mathrm{z}} \mathcal{F} = \begin{pmatrix} \frac{1}{\rho} & 0 & 0 \\ 0 & \frac{I}{\sqrt{T}} & -\frac{1}{2}\frac{v}{T^{\frac{3}{2}}} \\ 0 & 0 & \frac{1}{T} \end{pmatrix}.$$

Moreover, for any $\mathrm{M}_r > 0$, $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, *defining* $\widetilde{\mathcal{O}} = (-\mathrm{M}_r, \mathrm{M}_r) \times (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau)$, *the corresponding open set* $\mathcal{O} = \mathcal{F}^{-1}(\widetilde{\mathcal{O}})$ *is convex.*

Proof. The proof is similar to that of the incompressible case.[29] □

Proposition 6.4. *Let* $\mathrm{M}_r > 0$, $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, *define*

$$\widetilde{\mathcal{O}}_0 = (-\mathrm{M}_r, \mathrm{M}_r) \times (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau),$$

and $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$. *Let* $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial \mathcal{O}_{\mathrm{z}})$, $\mathcal{O}_1 = \{\mathrm{z} \in \mathcal{O}_{\mathrm{z}};\ d(\mathrm{z}, \overline{\mathcal{O}}_0) < d_1\}$, *and select an arbitrary* $b > 0$. *From Theorem* 6.1 *we have a local solution built with the* *paramaters* $\mathcal{O}_0$, d_1, *and* b. *This solution is then such that*

$$r - r_\infty \in C^0([0,\bar{t}\,], W^{l,2}) \cap C^1([0,\bar{t}\,], W^{l-1,2}), \tag{6.15}$$

$$w,\tau - \tau_\infty \in C^0([0,\bar{t}\,], W^{l,2}) \cap C^1([0,\bar{t}\,], W^{l-2,2}) \cap L^2((0,\bar{t}), W^{l+1,2}), \tag{6.16}$$

and there exists $C > 0$ *which only depend on* $\mathcal{O}_0$, d_1, *and* b, *such that*

$$\begin{aligned} &\sup_{0\le s\le \bar{t}} \{\|r(s) - r_\infty\|_{W^{l,2}}^2 + \|w(s)\|_{W^{l,2}}^2 + \|\tau(s) - \tau_\infty\|_{W^{l,2}}^2\} \\ &\qquad + \int_0^{\bar{t}} \{\|r(s) - r_\infty\|_{W^{l,2}}^2 + \|w(s)\|_{W^{l+1,2}}^2 + \|\tau(s) - \tau_\infty\|_{W^{l+1,2}}^2\}\, ds \\ &\quad \le C(\|r_0 - r_\infty\|_{W^{l,2}}^2 + \|w_0\|_{W^{l,2}}^2 + \|\tau_0 - \tau_\infty\|_{W^{l,2}}^2). \end{aligned} \tag{6.17}$$

Moreover, the kinetic estimators are such that $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C([0,\bar{t}\,], L^1(\mathbb{R}^n))$.

Proof. The set $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$ is convex and from Theorem 6.1, there exists a local solution built with $\mathcal{O}_0$, d_1 and b. We then have estimates in the form

$$\underline{c}_{\mathrm{Z}}\|\widetilde{\mathrm{Z}} - \widetilde{\mathrm{Z}}_\infty\|_{W^{l,2}} \le \|\mathrm{Z} - \mathrm{Z}_\infty\|_{W^{l,2}} \le \overline{c}_{\mathrm{Z}}\|\widetilde{\mathrm{Z}} - \widetilde{\mathrm{Z}}_\infty\|_{W^{l,2}}, \tag{6.18}$$

where $\underline{c}_{\mathrm{Z}}$ and $\overline{c}_{\mathrm{Z}}$ only depend on $\mathcal{O}_1$ and l thanks to the classical estimates

$$\|\psi(\phi) - \psi(0)\|_{W^{k,2}} \le C_0\|\psi\|_{C^k(\overline{\mathcal{O}}_\phi)}(1 + \|\phi\|_{L^\infty})^{k-1}\|\phi\|_{W^{k,2}}, \tag{6.19}$$

where $\mathcal{O}_\phi$ is a convex open set with $\phi(x) \in \mathcal{O}_\phi$, $x \in \mathbb{R}^n$. Similarly, the regularity properties are direct consequences of the estimates

$$\|\psi(\phi) - \psi(\hat{\phi})\|_{W^{k,2}} \le C_0\|\psi\|_{C^{k+1}(\overline{\mathcal{O}}_\phi)}(1 + \|\phi\|_{W^{k,2}} + \|\hat{\phi}\|_{W^{k,2}})^k\|\phi - \hat{\phi}\|_{W^{k,2}}, \tag{6.20}$$

where $\mathcal{O}_\phi$ is a convex open set with $\phi(x) \in \mathcal{O}_\phi$, $\hat{\phi}(x) \in \mathcal{O}_\phi$, $x \in \mathbb{R}^n$, and k is such that $k \ge [n/2] + 1$. The properties $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C([0,\bar{t}\,], L^1(\mathbb{R}^n))$ are then straightforward to establish. □

6.3. *Global existence*

In this section, we investigate global existence of solutions for which the quantity $\chi_{\tilde{\gamma}} = \|r\|_{\mathrm{BMO}} + \|w\|_{L^\infty} + \|\tau\|_{\mathrm{BMO}} + \|h\partial_x r\|_{L^\infty} + \|h\partial_x w\|_{L^\infty} + \|h\partial_x\tau\|_{L^\infty} + \|h^2\partial_x^2\tau\|_{L^\infty}$ remains small. We investigate solutions with given bounds $\rho_{\min} < \rho < \rho_{\max}$ and $T_{\min} < T < T_{\max}$, where $\rho_{\min} < \rho_\infty < \rho_{\max}$ and $T_{\min} < T_\infty < T_{\max}$, and assume that C_0 has been chosen large enough as in Lemma 5.6. We will also use the results of Corollary 5.8 and assume that the fixed value $a = \bar{a}$ has been selected for the parameter a in this section.

Theorem 6.5. *Let* $n \ge 1$ *and* $l \ge [n/2]+3$ *be integers. Assume that the coefficients* λ, κ, *and* η *satisfy* (3.11), (3.12). *There exists* $\delta_\Gamma(l, n, T_{\min}, T_{\max}, \rho_{\min}, \rho_{\max}) > 0$

such that for ρ_0, v_0 *and* T_0 *satisfying* $T_{\min} < \inf_{\mathbb{R}^n} T_0$, $\sup_{\mathbb{R}^n} T_0 < T_{\max}$, $\rho_{\min} < \inf_{\mathbb{R}^n} \rho_0$, $\sup_{\mathbb{R}^n} \rho_0 < \rho_{\max}$, $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{l,2}$ *and*

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leq \delta_\Gamma, \tag{6.21}$$

where $\widetilde{\Gamma}_0^{[l]}$ *denotes the functional* $\widetilde{\Gamma}^{[l]}$ *evaluated at initial conditions, there exists a unique global solution* $\mathrm{z} = (\rho, v, T)$ *with initial conditions*

$$(\rho(0,x), v(0,x), T(0,x)) = (\rho_0(x), v_0(x), T_0(x)), \tag{6.22}$$

such that

$$\rho - \rho_\infty, r - r_\infty \in C^0([0,\infty), W^{l,2}) \cap C^1([0,\infty), W^{l-1,2}), \tag{6.23}$$

$$v, w, T - T_\infty, \tau - \tau_\infty \in C^0([0,\infty), W^{l,2}) \cap C^1([0,\infty), W^{l-2,2}), \tag{6.24}$$

$$\partial_x \rho, \partial_x r \in L^2((0,\infty), W^{l-1,2}) \quad \partial_x T, \partial_x \tau, \partial_x v, \partial_x w \in L^2((0,\infty), W^{l,2}), \tag{6.25}$$

and we have the estimates

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx + \underline{b} \int_0^t \int_{\mathbb{R}^n} \rho T^{1-\varkappa} (\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[l]})\, dx dt \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx. \tag{6.26}$$

Furthermore, we have

$$\lim_{t\to\infty} \|\mathrm{z}(t,\cdot) - \mathrm{z}_\infty\|_{L^\infty} = 0. \tag{6.27}$$

Proof. We investigate solutions such that $\rho_{\min} < \rho < \rho_{\max}$ and $T_{\min} < T < T_{\max}$. For such solutions, thanks to classical estimates in the form

$$\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{C_0^2} \leq c_0 \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l_0+2,2}},$$

where $l_0 = [n/2] + 1$ we have the inequalities

$$\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{L^\infty} + \chi_{\tilde{\gamma}} \leq c_\chi \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l_0+2,2}}$$

and

$$c_\Gamma \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l,2}}^2 \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx,$$

thanks to Lemma 5.6 where c_χ and c_Γ depend on $T_{\min}$, $T_{\max}$, $\rho_{\min}$, $\rho_{\max}$ and l. In order to obtain a value of δ_Γ small enough, so that the higher order entropic estimates of Theorem 5.7 hold, we will ensure that $\delta_\Gamma \leq c_\Gamma \delta_{\mathrm{N}}'^2 / 4c_\chi^2$ where δ_{N}' is defined in Corollary 5.8 and this value will indeed insure that $\chi_{\tilde{\gamma}} \leq \delta_{\mathrm{N}}'/2$. Corresponding to this value of δ_Γ, we have estimates in the forms $\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{L^\infty} \leq c_\chi (\delta_\Gamma / c_\Gamma)^{1/2}$ and

$\|\widetilde{\mathrm{z}}-\widetilde{\mathrm{z}}_\infty\|_{W^{l,2}} \leq (\delta_\Gamma/c_\Gamma)^{1/2}$. We now select $\mathrm{M}_r > 0$, $\mathrm{M}_w > 0$, and $\mathrm{M}_\tau > 0$, such that

$$\log(\rho_{\min}/\rho_\infty) < -\mathrm{M}_r < \mathrm{M}_r < \log(\rho_{\max}/\rho_\infty),$$

$$\log(T_{\min}/T_\infty) < -\mathrm{M}_\tau < \mathrm{M}_\tau < \log(T_{\max}/T_\infty),$$

and define

$$\widetilde{\mathcal{O}}_0 = (-\mathrm{M}_r, \mathrm{M}_r) \times (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau),$$

and for $\delta > 0$

$$\widetilde{\mathcal{O}}_\delta = \{z \in \mathbb{R}^{n+2}; \|z - \widetilde{\mathrm{z}}_\infty\| \leq c_\chi (\delta/c_\Gamma)^{1/2}\}.$$

For δ_0 small enough we have

$$\widetilde{\mathcal{O}}_{2\delta_0} = \{z \in \mathbb{R}^{n+2}; \|z - \widetilde{\mathrm{z}}_\infty\| \leq \sqrt{2} c_\chi (\delta_0/c_\Gamma)^{1/2}\} \subset \widetilde{\mathcal{O}}_0,$$

and we now set

$$\delta_\Gamma = \min\left(\frac{c_\Gamma \delta_{\mathrm{N}}'^2}{4c_\chi^2}, \delta_0\right).$$

The open set $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$ is convex and let $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathrm{Z}})$, and define $\mathcal{O}_1 = \{z \in \mathcal{O}_{\mathrm{Z}}; d(z, \mathcal{O}_0) < d_1\}$ and $\widetilde{\mathcal{O}}_1 = \mathcal{F}(\mathcal{O}_1)$. Now for functions taking their values in $\mathcal{O}_1$ we have inequalities in the form $\|\mathrm{z}-\mathrm{z}_\infty\|_{W^{k,2}} \leq \overline{c}_{\mathrm{Z}} \|\widetilde{\mathrm{z}}-\widetilde{\mathrm{z}}_\infty\|_{W^{k,2}}$ where $\overline{c}_{\mathrm{Z}}$ only depends on k and $\mathcal{O}_1$. We thus obtain the *a priori* estimate $\|\mathrm{z}-\mathrm{z}_\infty\|_{W^{l,2}} \leq \overline{c}_{\mathrm{Z}}(\delta_\Gamma/c_\Gamma)^{1/2}$. We now set $b = \overline{c}_{\mathrm{Z}}(\delta_\Gamma/c_\Gamma)^{1/2} + 1$ and from Theorem 6.1 and Proposition 6.4 we have local solutions over a time interval $[0, \bar{t}]$ built with the paramaters $\mathcal{O}_0$, d_1, and b.

Let now ρ_0, v_0, and T_0 satisfy $T_{\min} < \inf_{\mathbb{R}^n} T_0$, $\sup_{\mathbb{R}^n} T_0 < T_{\max}$, $\rho_{\min} < \inf_{\mathbb{R}^n} \rho$, $\sup_{\mathbb{R}^n} \rho < \rho_{\max}$, $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{l,2}$, and $\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leq \delta_\Gamma$. Then by construction $\mathrm{z}_0 \in \mathcal{O}_0$ and $\|\mathrm{z}_0 - \mathrm{z}_\infty\|_{W^{l,2}} < b$, and we have a local solution over the time interval $[0, \bar{t}]$. Letting

$$\begin{aligned}\chi_{\tilde{\gamma}}(t) = {} & \|r(t,\cdot)\|_{\mathrm{BMO}} + \|w(t,\cdot)\|_{L^\infty} + \|\tau(t,\cdot)\|_{\mathrm{BMO}} + \|h(t,\cdot)\partial_x r(t,\cdot)\|_{L^\infty} \\ & + \|h(t,\cdot)\partial_x w(t,\cdot)\|_{L^\infty} + \|h(t,\cdot)\partial_x \tau(t,\cdot)\|_{L^\infty} + \|h^2(t,\cdot)\partial_x^2 \tau(t,\cdot)\|_{L^\infty},\end{aligned}$$

we also have by construction $\chi_{\tilde{\gamma}}(0) \leq \delta_{\mathrm{N}}'/2$ and we claim that for any $t \in [0, \bar{t}]$ we also have $\chi_{\tilde{\gamma}}(t) \leq \delta_{\mathrm{N}}'/2$. We introduce the set

$$\mathcal{E} = \{s \in (0, \bar{t}]; \forall\ t \in [0, s], \chi_{\tilde{\gamma}}(t) \leq (2/3)\delta_{\mathrm{N}}',\ \mathrm{z}(t) \in \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_{2\delta_0})\},$$

which is not empty since $t \to \chi_{\tilde{\gamma}}(t)$ is continuous, $\chi_{\tilde{\gamma}}(0) \leq \delta_{\mathrm{N}}/2$, and $\widetilde{\mathrm{z}}(0) \in \widetilde{\mathcal{O}}_{\delta_0}$ so that $\mathrm{z}(0) \in \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_{\delta_0})$. Denoting $\mathsf{e} = \sup \mathcal{E}$ we have $\chi_{\tilde{\gamma}}(t) \leq (2/3)\delta_{\mathrm{N}}'$ over $[0, \mathsf{e}]$ so that the entropic estimates of Theorem 5.7 hold and we have

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leq \delta_\Gamma, \quad 0 \leq t \leq \mathsf{e}.$$

This now implies that $\chi_{\tilde{\gamma}}(t) \leq \delta_{\mathrm{N}}'/2$ and that $\rho_{\min} < \rho < \rho_{\max}$ and $T_{\min} < T < T_{\max}$ uniformly over $[0, \mathsf{e}]$ so that $\mathsf{e} = \bar{t}$. From the above *a priori* estimates, we also obtain that for $t \in [0, \bar{t}]$ we have $\|\widetilde{\mathrm{z}}(t) - \widetilde{\mathrm{z}}_\infty\|_{L^\infty} \le c_\chi(\delta_\Gamma/c_\Gamma)^{1/2}$, so that $\mathrm{z}(t) \in \mathcal{O}_0$, and $\|\mathrm{z}(t) - \mathrm{z}_\infty\|_{W^{l,2}} \le b - 1 < b$, in particular at $t = \bar{t}$. We may now use again the local existence theorem over $[\bar{t}, 2\bar{t}]$ and an easy induction shows that the solution is a global solution.

The asymptotic stability is obtained by letting $\Phi(t) = \int_{\mathbb{R}^n} (\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[l-1]})\, dx$ and establishing that

$$\int_0^\infty |\Phi(t)| dt + \int_0^\infty |\partial_t \Phi(t)| dt \le C \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx.$$

This shows that $\lim_{t\to\infty} \|\partial_x \widetilde{\mathrm{z}}(t, \cdot)\|_{W^{l-2,2}} = 0$, and using the interpolation inequality

$$\|\phi\|_{C^0} \le C_0 \|\partial_x^{l-1}\phi\|_{L^2}^{a} \|\phi\|_{L^2}^{1-a},$$

where $n/a = 2(l-1)$ we conclude that $\lim_{t\to\infty} \|\widetilde{\mathrm{z}}(t, \cdot) - \widetilde{\mathrm{z}}_\infty\|_{C^0} = 0$, and next that $\lim_{t\to\infty} \|\mathrm{z}(t, \cdot) - \mathrm{z}_\infty\|_{C^0} = 0$. □

7. Conclusion

Higher order entropic estimates have been established for compressible equations whenever the quantities $\|\log\rho\|_{\mathrm{BMO}}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{\mathrm{BMO}}$, $\|h\partial_x\rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$, are small enough. An asymptotic expansion of higher order entropies for small Mach and Knudsen numbers has also been performed.[30,31]

A first natural extension of this work would be to investigate other types of higher order entropy estimators where the velocity and the gradient of the density have the same regularity. This is indeed a key point for dealing with data with low regularity as shown by Danchin[16] and Bresch and Desjardins[8,9] and since such entropies are a natural reordering of the higher order entropies used in this paper in term of the Mach number.

Another natural extension would also be to investigate the situation dense gas where gradient entropies have already been used — albeit with some constraint between the visosities — in order to obtain weak solutions with large data.[8,9]

Appendix A. Derivation of the $\gamma^{[k]}$ Balance Equation

We derive the balance equation for the entropic correctors $\gamma^{[k]}$. The proof is lengthy and tedious but presents no serious difficulties.

To obtain more concise analytic expressions, it is convenient to define $a_k = 1 + k(1 - 2\varkappa)$ and $b_k = -1 + 2k$ in such a way that

$$\gamma^{[k]} = \frac{1}{T^{a_k - 1}\rho^{b_k}} \left(\frac{|\partial^k \rho|^2}{\rho^2} + \frac{|\partial^k v|^2}{T} + c_v \frac{|\partial^k T|^2}{T^2} \right).$$

In order to obtain a balance equation for $\gamma^{[k]}$ for smooth solutions we form its time differential $\partial_t \gamma^{[k]}$

$$\partial_t \gamma^{[k]} + \left(\frac{(a_k-1)|\partial^k \rho|^2}{T^{a_k}\rho^{b_k+2}} + \frac{a_k|\partial^k v|^2}{T^{a_k+1}\rho^{b_k}} + \frac{c_v(a_k+1)|\partial^k T|^2}{T^{a_k+2}\rho^{b_k}} \right) \partial_t T$$
$$+ \left(\frac{(b_k+2)|\partial^k \rho|^2}{T^{a_k-1}\rho^{b_k+3}} + \frac{b_k|\partial^k v|^2}{T^{a_k}\rho^{b_k+1}} + \frac{c_v b_k|\partial^k T|^2}{T^{a_k+1}\rho^{b_k+1}} \right) \partial_t \rho - 2c_v \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^\alpha T \partial^\alpha \partial_t T}{T^{a_k+1}\rho^{b_k}}$$
$$- 2 \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^\alpha \rho \partial^\alpha \partial_t \rho}{T^{a_k-1}\rho^{b_k+2}} - 2 \sum_{\substack{1\le i\le n\\ |\alpha|=k}} \frac{k!}{\alpha!} \frac{\partial^\alpha v_i \partial^\alpha \partial_t v_i}{T^{a_k}\rho^{b_k}} = 0, \tag{A.1}$$

and we use the governing equations in order to express $\partial_t T$, $\partial_t \rho$ and $\partial_t v$ in terms of spatial gradients

$$\partial_t \rho = -\rho \partial_x \cdot v - v \cdot \partial_x \rho, \tag{A.2}$$

$$\partial_t v_i = \frac{1}{\rho} \sum_{1\le j\le n} \partial_j \left(\eta \partial_j v_i + \eta \partial_i v_j + \left(\kappa - \frac{2}{n}\eta \right) \partial_x \cdot v \delta_{ij} \right) - \frac{1}{\rho} \partial_i(\rho T) - v \cdot \partial_x v_i, \tag{A.3}$$

$$\partial_t T = \frac{1}{\rho c_v} \sum_{1\le j\le n} \partial_j(\lambda \partial_j T) + \frac{\eta}{2\rho c_v}|d|^2 + \frac{\kappa}{\rho c_v}(\partial_x \cdot v)^2 - \frac{T}{c_v}\partial_x \cdot v - v \cdot \partial_x T. \tag{A.4}$$

We denote respectively by $\mathcal{T}^T$, $\mathcal{T}^\rho$, $\mathcal{T}^{\partial T}$, $\mathcal{T}^{\partial \rho}$ and $\mathcal{T}^{\partial v}$, the five sums appearing in the governing equation for $\partial_t \gamma^{[k]}$, keeping in mind that the time derivative terms $\partial_t \rho$, $\partial_t v$ and $\partial_t T$ have been replaced by their expressions (A.2)–(A.4). We first examine separately higher order derivative contributions associated with each sum $\mathcal{T}^T$, $\mathcal{T}^\rho$, $\mathcal{T}^{\partial T}$, $\mathcal{T}^{\partial \rho}$ and $\mathcal{T}^{\partial v}$. The lower order derivative terms of convective origin are examined altogether at the end.

The term in $\mathcal{T}^T$ associated with $|\partial^k \rho|^2 \lambda \Delta T$, which is not of the admissible form, is isolated in $\Sigma_\gamma^{[k]}$ whereas all terms associated with $|\partial^k \rho|^2 |\partial_x T|^2$, $|\partial^k \rho|^2 |d|^2$, and $|\partial^k \rho|^2 |\partial_x \cdot v|^2$ are of the admissible form, that is, in the form

$$\sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi} T^{\sigma-\varkappa} \partial_T^\sigma \phi \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)},$$

where $c_{\sigma\nu\mu\phi}$ are constants and the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = gh^{k+1} \prod_{1\le|\alpha|\le k+1} \left(\frac{\partial^\alpha \rho}{\rho} \right)^{\nu_\alpha} \left(\frac{\partial^\alpha v}{\sqrt{T}} \right)^{\nu'_\alpha} \left(\frac{\partial^\alpha T}{T} \right)^{\nu''_\alpha}.$$

The sums are over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \le \sigma \le k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and μ and ν must be such that $\sum_{1\le|\alpha|\le k+1} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k+1$, $\sum_{1\le|\alpha|\le k+1} (\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = k+1$, $\sum_{|\alpha|=k+1} (\nu_\alpha + \mu_\alpha) = 0$, $\sum_{|\alpha|=k+1} (\nu'_\alpha + \nu''_\alpha + \mu'_\alpha + \mu''_\alpha) \le 1$, so that there is no derivative of order $k+1$ of density and at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. In particular, one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split between two or more derivative factors.

Similarly, all terms of $\mathcal{T}^T$ in the form $|\partial^k T|^2|d|^2$, $|\partial^k T|^2|\partial_x\cdot v|^2$ $|\partial^k v|^2|d|^2$ and $|\partial^k v|^2|\partial_x\cdot v|^2$ are of the admissible form. On the other hand, the terms associated with $|\partial^k T|^2\partial_x\cdot(\lambda\partial_x T)$ and $|\partial^k v|^2\partial_x\cdot(\lambda\partial_x T)$ are integrated by parts. They yield flux contributions and source terms in the form

$$-\sum_{1\le l\le n}\partial_l\left(\frac{(a_k+1)|\partial^k T|^2}{T^{2+a_k}\rho^{b_k}}+\frac{a_k|\partial^k v|^2}{c_v T^{1+a_k}\rho^{b_k}}\right)\lambda\partial_l T,$$

which are easily rewritten as sums of terms like $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\lambda\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)$th order. All other contributions from $\mathcal{T}^T$ as well as all contributions from $\mathcal{T}^\rho$ and $\mathcal{T}^{\partial\rho}$ are of lower order type.

We now consider the term $\mathcal{T}^{\partial T}$ with each contribution at a time. The most important contribution in $\mathcal{T}^{\partial T}$ is that associated with

$$-2c_v\sum_{\substack{1\le l\le n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{a_k+1}\rho^{b_k}}\partial^\alpha\left(\frac{1}{\rho c_v}\partial_l(\lambda\partial_l T)\right).$$

We then write

$$\frac{1}{\rho}\partial_l(\lambda\partial_l T)=\partial_l\left(\frac{1}{\rho}\lambda\partial_l T\right)+\frac{\lambda\partial_l T\partial_l\rho}{\rho^2}$$

and the contributions associated with $\partial_l(\lambda\partial_l T/\rho c_v)$ are integrated by parts. This yields source terms in the form

$$+\,2\sum_{\substack{1\le l\le n\\|\alpha|=k}}\frac{k!}{\alpha!}\partial_l\left(\frac{\partial^\alpha T}{T^{1+a_k}\rho^{b_k}}\right)\partial^\alpha\left(\frac{\lambda\partial_l T}{\rho}\right).$$

After expanding the derivatives, using the differential identities of Sec. 4.1, the above sum can be written as

$$\begin{aligned}2\sum_{\substack{1\le l\le n\\|\alpha|=k}}&\frac{k!}{\alpha!}\left(\frac{\partial^\alpha\partial_l T}{T^{1+a_k}\rho^{b_k}}-(1+a_k)\frac{\partial^\alpha T\partial_l T}{T^{2+a_k}\rho^{b_k}}-b_k\frac{\partial^\alpha T\partial_l\rho}{T^{1+a_k}\rho^{1+b_k}}\right)\\&\times\frac{1}{\rho}\left(\lambda\partial^\alpha\partial_l T+\sum_{\tilde\alpha\nu\mu}c_{\alpha\tilde\alpha\nu\mu}T^\sigma\partial_T^\sigma\lambda\prod_\beta\left(\frac{\partial^\beta T}{T}\right)^{\nu_\beta}\prod_\beta\left(\frac{\partial^\beta\rho}{\rho}\right)^{\mu_\beta}\partial^{\alpha-\tilde\alpha}\partial_l T\right),\end{aligned}$$

where the summations and products extend over $1 \leq l \leq n$, $|\alpha| = k$, $0 \leq \tilde{\alpha} \leq \alpha$, $\tilde{\alpha} \neq 0$, $1 \leq \sigma \leq |\tilde{\alpha}|$, $\sum_\beta \beta(\nu_\beta + \mu_\beta) = \tilde{\alpha}$, $1 \leq |\beta| \leq |\tilde{\alpha}|$, and $\sum_\beta \nu_\beta = \sigma$. We can now extract for $\pi_\gamma^{[k]}$ the term in the form $\lambda(\partial^\alpha \partial_l T)^2$ which can be written as

$$2 \sum_{\substack{1 \leq l \leq n \\ |\alpha| = k}} \frac{k!}{\alpha!} \frac{(\partial^\alpha \partial_l T)^2}{T^{1+a_k} \rho^{b_k}} = 2 \sum_{|\alpha| = k+1} \frac{(k+1)!}{\alpha!} \frac{(\partial^\alpha T)^2}{T^{1+a_k} \rho^{b_k}},$$

thanks to the properties of multinomial coefficients.[14,60] All other terms are of admissible form for $\Sigma_\gamma^{[k]}$, i.e. in the form $c_{\sigma\nu\mu} T^{\sigma - \varkappa} \partial_T^\sigma \lambda \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order since $\sum_\beta |\beta| \nu_\beta + 1 + |\alpha - \tilde{\alpha}| = k+1$. More specifically, we can factorize T^{-a_k} in the first factors, $T^{1+\varkappa}$ in the parenthesis, and all the terms involving derivatives of $\partial_T^\sigma \lambda$ are multiplied and divided by T^σ thanks to $\sum_\beta \nu_\beta = \sigma$.

The contributions associated with $\lambda \partial_l T \partial_l \rho / \rho^2$ are integrated by parts thanks to a decomposition in the form $\alpha = \tilde{\alpha} + e_{i_\alpha}$ where $|\tilde{\alpha}| = k - 1$, as well as the contributions in $\mathcal{T}^{\partial T}$ associated with $\eta |d|^2 + \kappa (\partial_x \cdot v)^2$, and only yield admissible source terms. More specifically, we decompose each multi-index α with $|\alpha| = k$ into $\alpha = \tilde{\alpha} + e_{i_\alpha}$ where $|\tilde{\alpha}| = k - 1$, i_α is chosen arbitrarily with $\alpha_{i_\alpha} \neq 0$, and $e_1, \ldots, e_n$ denotes the canonical basis of $\mathbb{N}^n$, so that we have $\partial^\alpha = \partial^{\tilde{\alpha}} \partial_{i_\alpha}$. We can then integrate these terms by parts and obtain sources in the form

$$\sum_{\substack{1 \leq i,j \leq n \\ |\alpha| = k}} \partial_{i_\alpha} \left(\frac{\partial^\alpha T}{T^{1+a_k} \rho^{b_k}} \right) \partial^{\tilde{\alpha}} (\eta d_{ij}^2).$$

Upon expanding the derivatives with the help of the differential identities established the Sec. 4.1, all these terms are of admissible form for $\Sigma_\gamma^{[k]}$.

We now consider the sum $\mathcal{T}^{\partial v}$ and its most important contribution is that corresponding to $\partial^\alpha \partial_x \cdot (\eta d + \kappa \partial_x \cdot v I)$ which reads

$$-2 \sum_{\substack{1 \leq i,l \leq n \\ |\alpha| = k}} \frac{k!}{\alpha!} \frac{\partial^\alpha v_i}{T^{a_k} \rho^{b_k}} \partial^\alpha \left(\frac{1}{\rho} \partial_l \left(\eta \partial_l v_i + \eta \partial_i v_l + \left(\kappa - \frac{2}{n} \eta \right) \partial_x \cdot v \delta_{il} \right) \right),$$

where δ_{il} is the Kronecker symbol. We first consider the contribution associated with $\eta \partial_l v_i$ using the identity

$$\frac{1}{\rho} \partial_l (\eta \partial_l v_i) = \partial_l \left(\frac{1}{\rho} \eta \partial_l v_i \right) + \frac{\partial_l \rho \partial_i v_l}{\rho^2}$$

and focus on the contributions of the terms $\partial_l (\eta \partial_l v_i / \rho)$. The contributions associated with $\partial_l \rho \partial_i v_l$ are of admissible form for $\Sigma_\gamma^{[k]}$ after one integration by parts using $\alpha = \tilde{\alpha} + e_{i_\alpha}$ and the corresponding details are omitted. After integration by parts we obtain sources in the form

$$2 \sum_{\substack{1\le i,l\le n\\ |\alpha|=k}} \frac{k!}{\alpha!}\partial_l\left(\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\right)\partial^\alpha\left(\frac{\eta\partial_l v_i}{\rho}\right).$$

Expanding the derivatives, the sum is rewritten

$$\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}} \frac{k!}{\alpha!}\left(\frac{\partial^\alpha\partial_l v_i}{T^{a_k}\rho^{b_k}} - a_k\frac{\partial^\alpha v_i\partial_l T}{T^{a_k+1}\rho^{b_k}} - b_k\frac{\partial^\alpha v_i\partial_l\rho}{T^{a_k+1}\rho^{b_k+1}}\right)$$

$$\times\frac{1}{\rho}\left(\eta\partial^\alpha\partial_l v_i + \sum_{\tilde\alpha\nu\mu} c_{\alpha\tilde\alpha\nu\mu}T^\sigma\partial_T^\sigma\eta\prod_\beta\left(\frac{\partial^\beta T}{T}\right)^{\nu_\beta}\prod_\beta\left(\frac{\partial^\beta\rho}{\rho}\right)^{\mu_\beta}\partial^{\alpha-\tilde\alpha}\partial_l v_i\right),$$

where the summations and products extend over $1\le i,l\le n$, $|\alpha|=k$, $0\le\tilde\alpha\le\alpha$, $\tilde\alpha\ne 0$, $1\le\sigma\le|\tilde\alpha|$, $\sum_\beta\beta(\nu_\beta+\mu_\beta)=\tilde\alpha$, $1\le|\beta|\le|\tilde\alpha|$, and $\sum_\beta\nu_\beta=\sigma$. We can extract the term in the form $\eta(\partial^\alpha\partial_l v_i)^2$ for $\pi_\gamma^{[k]}$ which is rewritten as

$$2\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_i)^2}{T^{a_k}\rho^{b_k}} = 2\sum_{\substack{1\le i\le n\\ |\alpha|=k+1}}\frac{(k+1)!}{\alpha!}\frac{(\partial^\alpha v_i)^2}{T^{a_k}\rho^{b_k}},$$

thanks to the properties of multinomial coefficients. All the other terms are of admissible form for $\Sigma_\gamma^{[k]}$, that is, in the form $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\eta\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)$th order.

The contributions associated with $\eta\partial_i v_l$ is treated in an analogous way with the identity

$$\partial_l(\eta\partial_i v_l) = \partial_T\eta\partial_l T\partial_i v_l + \partial_i(\eta\partial_l v_l) - \partial_T\eta\partial_i T\partial_l v_i,$$

and yields a source term for $\pi_\gamma^{[k]}$ in the form

$$2\eta\sum_{\substack{1\le l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_l)^2}{T^{a_k}\rho^{b_k}}.$$

Finally, the terms $(\kappa-\frac{2}{n}\eta)\partial_x\cdot v\delta_{il}$ can be treated in a similar way and yields a source term for $\pi_\gamma^{[k]}$ in the form

$$2\left(\kappa-\frac{2}{n}\eta\right)\sum_{\substack{1\le l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_l)^2}{T^{a_k}\rho^{b_k}},$$

as well as contributions of the admissible form.

Lower order convective terms first yield the contributions

$$
\begin{aligned}
&-\left(\frac{(a_k-1)|\partial^k\rho|^2}{T^{a_k}\rho^{b_k+2}}+\frac{a_k|\partial^k v|^2}{T^{a_k+1}\rho^{b_k}}+\frac{c_v(a_k+1)|\partial^k T|^2}{T^{a_k+2}\rho^{b_k}}\right)v\cdot\partial_x T\\
&\quad-\left(\frac{(b_k+2)|\partial^k\rho|^2}{T^{a_k-1}\rho^{b_k+3}}+\frac{b_k|\partial^k v|^2}{T^{a_k}\rho^{b_k+1}}+\frac{c_v b_k|\partial^k T|^2}{T^{a_k+1}\rho^{b_k+1}}\right)v\cdot\partial_x\rho\\
&\quad+2c_v\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T\partial^\alpha(v\cdot\partial_x T)}{T^{a_k+1}\rho^{b_k}}+2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha\rho\partial^\alpha(v\cdot\partial_x\rho)}{T^{a_k-1}\rho^{b_k+2}}\\
&\quad-2\sum_{\substack{1\le i\le n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i\partial^\alpha(v\cdot\partial_x v_i)}{T^{a_k}\rho^{b_k}},
\end{aligned}
$$

and all terms proportional to v are easily recast in the form $v\cdot\partial_x\gamma^{[k]}$, so that the only remaining contributions are the sources

$$
2c_v\sum_{\substack{|\alpha|=k\\1\le l\le n}}\sum_{\substack{0\le\beta\le\alpha\\1\le|\beta|}}c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{a_k+1}\rho^{b_k}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_l T
$$

$$
2\sum_{\substack{|\alpha|=k\\1\le l\le n}}\sum_{\substack{0\le\beta\le\alpha\\1\le|\beta|}}c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha \rho}{T^{a_k-1}\rho^{b_k+2}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_l \rho
$$

$$
2\sum_{\substack{|\alpha|=k\\1\le i,l\le n}}\sum_{\substack{0\le\beta\le\alpha\\1\le|\beta|}}c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_l \rho
$$

which are easily rewritten in the form $c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}$.

The remaining first-order terms are then in the form

$$
\begin{aligned}
&-\left(\frac{(a_k-1)|\partial^k\rho|^2}{T^{a_k}\rho^{b_k+2}}+\frac{a_k|\partial^k v|^2}{T^{a_k+1}\rho^{b_k}}+\frac{c_v(a_k+1)|\partial^k T|^2}{T^{a_k+2}\rho^{b_k}}\right)\frac{T\partial_x\cdot v}{c_v}\\
&\quad-\left(\frac{(b_k+2)|\partial^k\rho|^2}{T^{a_k-1}\rho^{b_k+3}}+\frac{b_k|\partial^k v|^2}{T^{a_k}\rho^{b_k+1}}+\frac{c_v b_k|\partial^k T|^2}{T^{a_k+1}\rho^{b_k+1}}\right)\rho\partial_x\cdot v\\
&\quad+2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{a_k+1}\rho^{b_k}}\partial^\alpha(T\partial_x\cdot v)+2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha\rho}{T^{a_k-1}\rho^{b_k+2}}\partial^\alpha(\rho\partial_x\cdot v)\\
&\quad+2\sum_{\substack{1\le i\le n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\partial^\alpha\left(T\frac{\partial_i\rho}{\rho}+\partial_i T\right).
\end{aligned}
$$

The first two sums are easily recast in the admissible form $c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}$. In the last three sums, it is then important to separate admissible terms form unsplit ones, that is, to separate terms with three or more derivatives — which are then of the admissible form — from quadratic terms. The third and fourth terms yield the special source terms

$$+\,2\,\frac{\partial^k T \partial^k (\partial_x \cdot v)}{T^{a_k} \rho^{b_k}} + 2\frac{\partial^k \rho \partial^k (\partial_x \cdot v)}{T^{a_k-1} \rho^{b_k+1}}. \tag{A.5}$$

In the last sum, the contributions associated with $\partial_i T$ are integrated by parts and yield admissible terms plus the special term

$$-\,2\,\frac{\partial^k T \partial^k (\partial_x \cdot v)}{T^{a_k} \rho^{b_k}},$$

which compensates with the first term of (A.5). Finally, the special contributions associated with $T\partial_i \rho/\rho = T\partial_i \log \rho$ are integrated by parts and yields the source term

$$-2\frac{\partial^k \rho \partial^k (\partial_x \cdot v)}{T^{a_k-1} \rho^{b_k+1}},$$

which compensate with the second term of (A.5). This compensation of quadratic terms involving hyperbolic variables are the consequence of the symmetric structure of the system of partial differential equations.

Let now (ρ, v, T) be a smooth solution of the compressible Navier–Stokes equations (3.1)–(3.5) with regularity (3.6)–(3.7), and assume that $T \geq T_{\min}$ and that $\rho \geq \rho_{\min}$. The preceding derivation of the $\gamma^{[k]}$ balance equation can then be justified for $0 \leq k \leq l$ by using mollifiers and classical properties of commutators.[42,43,63]

Moreover, from classical interpolation inequalities the following lemmas ensure that $\varphi_\gamma^{[k]}$, $\pi_\gamma^{[k]}$, $\Sigma_\gamma^{[k]}$, $\omega_\gamma^{[k]} \in L^1((0,\bar{t}), W^{l-k,1})$.

Lemma A.1. *Let $i \geq 1$, α^j, $1 \leq j \leq i$, be multi-indices such that $|\alpha^j| \geq 1$, $1 \leq j \leq i$, and let $k = \sum_{1\leq j\leq i} |\alpha^j|$. Let $\mathsf{u}_1, \ldots, \mathsf{u}_i$, be such that there exist constants $\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{m,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ and assume that $1 \leq k \leq m$. There exists a constant $c = c(m,n)$ only depending on (m,n), such that*

$$\left\| \prod_{1\leq j\leq i} \partial^{\alpha^j} \mathsf{u}_j \right\|_{W^{m-k,2}} \leq c \|\mathsf{u} - \mathsf{u}_\infty\|_{L^\infty}^{i-1} (\|\partial^k \mathsf{u}\|_{L^2} + \cdots + \|\partial^m \mathsf{u}\|_{L^2}), \tag{A.6}$$

where

$$\|\mathsf{u} - \mathsf{u}_\infty\|_{L^\infty} = \sum_{1\leq j\leq i} \|\mathsf{u}_j - \mathsf{u}_{j,\infty}\|_{L^\infty},$$

$$\|\partial^m \mathsf{u}\|_{L^2}^2 = \sum_{1\leq j\leq i} \|\partial^m \mathsf{u}_j\|_{L^2}^2,$$

and the derivatives of $\prod_{1\leq j\leq i} \partial^{\alpha^j} \mathsf{u}_j$ can be evaluated by using Leibniz' formula.

Lemma A.2. *Let $i \geq 2$, α^j, $1 \leq j \leq i$, be multi-indices such that $|\alpha^j| \geq 1$, $1 \leq j \leq i$, and let $k = \sum_{1\leq j\leq i} |\alpha^j|$. Let $\mathsf{u}_1, \ldots, \mathsf{u}_i$, be such that there exist constants*

$\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{m,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ and assume that $2 \le k \le m+1$. There exists a constant $c = c(m,n)$ only depending on (m,n), such that

$$\Big\| \prod_{1\le j\le i} \partial^{\alpha^j} \mathsf{u}_j \Big\|_{W^{m+1-k,2}} \le c(\|\mathsf{u} - \mathsf{u}_\infty\|_{L^\infty} + \|\partial_x u\|_{L^\infty})^{i-1}(\|\partial^1 \mathsf{u}\|_{L^2} + \cdots + \|\partial^m \mathsf{u}\|_{L^2}), \tag{A.7}$$

where

$$\|\partial_x \mathsf{u}\|_{L^\infty} = \sum_{1\le j\le i} \|\partial_x \mathsf{u}_j\|_{L^\infty},$$

and the derivatives of $\prod_{1\le j\le i} \partial^{\alpha^j} \mathsf{u}_j$ can be evaluated by using Leibniz' formula.

These lemmas can be established by using classical interpolation inequalities[63] or by using Theorems 4.7 and 4.8 with a weight unity.

Lemma A.3. *Let $m \ge 0$, and a, $b \in W^{m,2}(\mathbb{R}^n)$. Then $ab \in W^{m,1}$ and there exists a constant $c(m,n)$ only depending on (m,n) such that*

$$\|ab\|_{W^{m,1}} \le c\|a\|_{W^{m,2}}\|b\|_{W^{m,2}}, \tag{A.8}$$

and the derivatives of ab can be evaluated by using Leibniz' formula.

This lemma is a direct consequence of Hölder inequality.

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