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Abstract

We are interested in approximating a multidimensional Itô process $(X_t)_{t\geq 0}$ killed when it leaves a smooth domain D: when the exit time is discretized along a regular mesh with time step h, we prove under a non characteristic boundary condition, that the discretization error is bounded from above by $C_1\sqrt{h}$, extending a previous result [Gob00] obtained in the Markovian case under uniform ellipticity assumptions.

In the case of hypoelliptic diffusion processes and when a discrete Euler scheme is additionally used as an approximation of X, we prove that the upper bound for the weak error is still valid and that a lower bound with the same rate $\sqrt{h}$ holds true, thus proving that the order of convergence is exactly $\frac{1}{2}$.

This provides a theoretical explanation of the well-known bias that we can numerically observe in that kind of procedure.

Key words: Weak approximation, Killed processes, Discrete exit time, Overshoot above the boundary.
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1 Statement of the problem

Let $(X_t)_{0\leq t\leq T}$ be a d-dimensional Itô process, whose dynamics is given by

$$X_t = x + \int_0^t b_s ds + \int_0^t \sigma_s dW_s \tag{1}$$

with a fixed initial data x and a fixed terminal time T. Here, W is a d'-dimensional standard Brownian motion (BM in short) defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, with the usual assumptions on the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. The progressively measurable coefficients $(b_s)_{s\geq 0}$ and $(\sigma_s)_{s\geq 0}$ are bounded. In this work, we are more specifically interested in the law of this Itô process, killed when it exits from some fixed domain D. Namely, for a measurable function f, we consider the quantity

$$\mathbb{E}[f(X_T)\mathbf{1}_{\tau>T}] \tag{2}$$

with $\tau := \inf\{t \geq 0 : X_t \notin D\}$, and we may focus on the impact of a discretization of the exit time in the above expectation: this is a pertinent question if we try to evaluate it by Monte Carlo simulations (which may be an especially appropriate approach if the dimension d is large). Actually, the numerical computation of this type of expectation is a well-known issue in finance since it is related to the pricing of barrier options (see [Gob00] for some references on the subject).

For this, consider a regular mesh of the interval $[0, T]$ with N time steps $(t_i = ih)_{0\leq i\leq N}$ ($h = T/N$ being the step size) and then, simply put $\tau^N := \inf\{t_i \geq 0 : X_{t_i} \notin D\}$. The associated weak error is then defined by

$$\mathrm{Err}_1(T, h, f, x) := \mathbb{E}[f(X_T)\mathbf{1}_{\tau^N>T}] - \mathbb{E}[f(X_T)\mathbf{1}_{\tau>T}].$$

The first main result of this work (Theorem 2) states that this error is at least of order $\frac{1}{2}$ w.r.t. h, provided that f satisfies a support condition and that for X_s on the boundary ∂D, σ_s is uniformly non degenerate in the normal direction at X_s (this is a so-called *non characteristic boundary* condition **(C)**). Note that this result holds true under a very weak non degeneracy condition and without any Markovian structure on the process $(X_t)_{t\geq 0}$ (analogous previous results were proved in [Gob00] in the diffusion case under ellipticity condition). Remark also that **(C)** is somehow a minimal condition to ensure a convergent approximation. Indeed, it easy to imagine a deterministic path which hits ∂D only at time $\tau = \chi T$ where χ is an irrational number in $]0, 1[$: for this, $\tau^N > T$ for any $N \geq 1$ and $\mathrm{Err}_1(T, h, f, x) = f(X_T)$ is constant. On the other hand, we prove that as soon as **(C)** is satisfied, the rate of convergence is at least 1/2.

To be able to say more about the exact rate of convergence, we may restrict to the Markovian structure of stochastic differential equations (SDE in short), with coefficients $b_s = b(X_s)$ and $\sigma_s = \sigma(X_s)$. To authorize a fully implementable simulation procedure, we consider also its Euler approximation defined by

$$X_t^N = x + \int_0^t b(X_{\phi(s)}^N)ds + \int_0^t \sigma(X_{\phi(s)}^N)dW_s \tag{3}$$

where $\phi(t) := \sup\{t_i : t_i \leq t < t_{i+1}\}$. Note that the values $(X_{t_i}^N)_{0\leq i\leq N}$ are straightforward to obtain using the simulations of the Brownian increments along the mesh. The so-called *discrete Euler scheme* corresponds to the killing time $\tau^N := \inf\{t_i \geq 0 : X_{t_i}^N \notin D\}$: thus, the random variable to simulate is simply given by $f(X_T^N)\mathbf{1}_{\tau^N>T}$ which can directly be derived from the realizations of $(X_{t_i}^N)_{0\leq i\leq N}$. The discretization error is now given by

$$\mathrm{Err}_2(T,h,f,x) := \mathbb{E}_x[f(X_T^N)\mathbf{1}_{\tau^N>T}] - \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau>T}]$$

(compared to $\mathrm{Err}_1(T,h,f,x)$, an additional error is considered, corresponding to replacing $(X_{t_i})_{0\leq i\leq N}$ by $(X_{t_i}^N)_{0\leq i\leq N}$).

The unpleasant feature of this procedure is that it likely overestimates the quantity of interest when f is non negative ($\mathrm{Err}_2(T,h,f,x) \geq 0$): this fact is clear if $X^N = X$ (as for the previous case of Itô processes, or as in the case of constant coefficients) since one has $\tau \leq \tau^N$ with probability 1. In the more general case where $X^N \neq X$, it is not so obvious but nevertheless, this has been observed numerically in many situations (see Rubinstein and Reiner [RR91]; Boyle and Lau [BL94]; Baldi [Bal95]). One of the purposes of this work is to prove that this bias is a systematic feature of this discrete killing procedure and to provide a deeper analysis of this phenomena.

Of course, many improvements of the above procedure are now available: the leading idea of these methods consists in performing additional simulations of the exit of some appropriately scaled Brownian bridge on each discretization interval. This has been introduced by Lerche and Siegmund [LS89] in the case of BM in two-dimensional smooth domains, and later generalized in arbitrary dimensions by Baldi [Bal95]. Some numerical studies with financial applications have been developed by Andersen and Brotherton-Ratcliffe [ABR96], Beaglehole, Dybvig and Zhou [BDZ97], Baldi, Caramellino and Iovino [BCI99] among others. The global error w.r.t. the time step h has been analyzed by the first author in [Gob00] (see also [Gob01] for a simplified and accurate procedure using half-space approximations): it essentially yields an order of convergence equal to 1. Hence, we recover the usual rate of convergence which is obtained without boundary conditions (see Bally and Talay [BT96a] and references therein).

We now go back to the analysis of the error $\mathrm{Err}_2(T,h,f,x)$ associated to the discrete Euler scheme, and in the sequel, we restrict to the case of non-negative functions f, in order to obtain a positive bias. We will consider rather weak assumptions on the SDE (assuming **H**ypoellipticity type conditions **(H)**), the case of smooth domain D (with a non **C**haracteristic boundary assumption **(C)**) and the **F**unction f will be taken to be only measurable with a support condition (see assumption **(F)**). Under these hypotheses, we prove in Theorem 9 that the weak error $\mathrm{Err}_2(T,h,f,x)$ is bounded from above by $C_2\sqrt{h}$ recovering in some sense the case of general Itô processes. Moreover, this Theorem states, and this will be our second main contribution to the problem, that the weak error is also bounded from below by $C_1\sqrt{h}$ (with $C_1 > 0$): this proves that the order of convergence is exactly $\frac{1}{2}$. This original estimate justifies the well-known overestimation which has been observed in numerical experiments.

In the error analysis below, an essential feature which comes up is that the main part of this error can be expressed as a suitable average with positive weights of the overshoot of the discretely killed process above the boundary (the overshoot being defined as the distance to the boundary of the process when it exits the domain). Hence, it provides a clear explanation of the main origin of the error: roughly speaking, the increments are of order $1/2$, and hence (but this is not so straightforward as it will be seen), the same estimate holds for the overshoot. The central role played by the overshoot in this problem has been identified in [RR91], [BL94] and analyzed in the context of lattice approximations for barrier options in [Gob99].

The derivation of an expansion of the $\mathrm{Err}_2(T,h,f,x)$ at the order $\frac{1}{2}$ would require the computation of the asymptotic law of this overshoot: this is a classical issue which is usually analyzed with the renewal theory for Markov chains. Unfortunately, in a multidimensional setting, the available results only hold under ergodicity type conditions (see Alsmeyer [Als94], Fuh and Lai [FL01] and references therein), which are never satisfied on the relevant process (i.e. the time-rescaled Euler scheme). Hence, we have been able to expand the error only in some simple situations, derived from the case of scalar random walks (see Siegmund and Yuh [SY82] and Siegmund [Sie79]).

Note also that in the one-dimensional case with constants coefficients (where the expansion at order $1/2$ holds true), a nice improvement of the procedure (the so-called *barrier correction*) is available (see Broadie, Glasserman and Kou [BGK99]): it simply consists in doing the simulations with the discrete Euler scheme, but with a shifted boundary from the quantity $C\sqrt{h}$ (for some appropriate explicit constant C). We will see that this *boundary correction* can be extended to a multidimensional setting.

Outline of the paper. General notation used throughout the paper will be defined at the end of this section. In Section 2, we consider the case of general Itô processes and prove that the discretization of the exit time induces an error of order $\frac{1}{2}$ w.r.t. h. For this, we will overcome two major difficulties. First, the process is not Markovian and we can not rely on an appropriate PDE solved by the expectation of interest to decompose the weak error (see [Gob00]): new ingredients are needed and this is a first attempt for the weak approximation in non Markovian situations. Second, a non-degeneracy condition is stated only on the boundary and this will force us to introduce localization techniques that are of interest for themselves. Some technical results, that will also be used in the Markovian case, will be also proved in this part. In section 3, we consider SDEs and their approximations using the discrete Euler scheme (to which the rest of the paper is devoted): we then give specific notations, define some assumptions and state the main results concerning the upper and lower bounds of the error $\mathrm{Err}_2(T,h,f,x)$. Then, we give their proofs, using some complementary technical results whose justifications are postponed to section 4. Section 4 puts together all the estimates related to the behavior of the discrete Euler scheme near the boundary: this is the technical core of the paper and justifies the origin of the order of convergence equal to $\frac{1}{2}$. Furthermore, some more or less standard Malliavin calculus computations are given and complete the proofs of technical results. Finally, in section 5, we give an expansion result in the case of drifted BM with constant diffusion coefficient, in the case of a half-space domain, and we improve the original procedure by a *boundary correction* technique: numerical experiments confirm its accuracy. In section 6, we conclude giving some easy extensions.

Notations

For smooth functions $g(t,x)$, we denote by $\partial_x^\alpha g(t,x)$ the derivative of g w.r.t. x according to the multi-index α, whereas time derivatives of g are denoted by $\partial_t g(t,x), \partial_t^2 g(t,x), \dots$. The notation $\nabla g(t,x)$ stands for the usual gradient w.r.t. x and $\frac{\partial g}{\partial n}(t,x) = \nabla g(t,x).n(x)$ is the normal derivative on the boundary. The Hessian matrix of g is denoted by H_g.

As usual, the index x in $\mathbb{E}_x$ and $\mathbb{P}_x$ refers to the initial value of a given process for which we compute the expectation or the probability: this will be clear from the context.

The distribution function of the standard normal law is denoted by Φ.

We will keep the same notation C (or C') for all finite, non-negative constants which will appear in our computations: they may depend on D, on T, on the coefficients which define the process X or on the function f, but they will not depend on the number of time steps N and the initial value x. We reserve the notation c and c' for constants also independent of x, T and f.

In the following, we consider a domain $D \subset \mathbb{R}^d$, i.e. an open connected set, which satisfies the following smoothness hypothesis (see [GT77]).

(D) The boundary ∂D is bounded and of class C^∞.

For some $x \in \partial D$, denote by $n(x)$ the unit inward normal vector at x.

For $r \geq 0$, set $V_{\partial D}(r) := \{z \in \mathbb{R}^d : d(z, \partial D) \leq r\}$ and $D(r) := \{z \in \mathbb{R}^d : d(z, D) \leq r\}$. $B(z,r)$ stands for the closed ball with center z and radius r.

We now recall standard facts on the distance to the boundary and the projection on D (see [GT77] pp. 381-384, [Gob01]).

Proposition 1 *Assume* **(D)**. *There is a constant $R > 0$ such that:*

i) for any $x \in V_{\partial D}(R)$, there are unique $s = \pi_{\partial D}(x) \in \partial D$ and $F(x) \in \mathbb{R}$ such that $x = \pi_{\partial D}(x) + F(x) n(\pi_{\partial D}(x))$.
ii) The function $x \longmapsto \pi_{\partial D}(x)$ is the normal projection of x on ∂D: this is a C^∞-function on $V_{\partial D}(R)$.
iii) The function $x \longmapsto F(x)$ is the signed normal distance of x to ∂D: this is a C^∞-function on $V_{\partial D}(R)$, which can be extended to a C^∞ function on $\mathbb{R}^d$ with bounded derivatives. This extension satisfies $F(x) \geq d(x, \partial D) \wedge R$ on D, $F(x) \leq -[d(x, \partial D) \wedge R]$ on D^c and $F = 0$ on ∂D.
iv) For $x \in V_{\partial D}(R)$, one has $\nabla F(x) = n(\pi_{\partial D}(x))$.

In the paper, the function f involved in the expectation of interest satisfy the following condition.

(F) f is a non-negative bounded measurable function with support strictly included in D: $d(\mathrm{Supp}(f), \partial D) \geq 2\varepsilon > 0$.

We may assume that $2\epsilon \leq R$ and that f is not identically equal to 0.

2 General Itô processes

In this section, we consider $(X_t)_{t \geq 0}$ solving (1) with *bounded* coefficients $(b_s)_{0 \leq s \leq T}$ and $(\sigma_s)_{0 \leq s \leq T}$: define the two stopping times $\tau := \inf\{t \geq 0 : X_t \notin D\}$ and $\tau^N := \inf\{t_i \geq 0 : X_{t_i} \notin D\}$.

For $s \in [0,T]$, define $\alpha_s = \nabla F(X_s).\sigma_s \sigma_s^* \nabla F(X_s)$. Remind that for $X_s \in V_{\partial D}(R)$, $\nabla F(X_s) = n(\pi_{\partial D}(X_s))$. We now state an appropriate *non characteristic boundary* condition.

(C) There is some constant $a_0 > 0$ such that almost surely, for $s \in [0,T]$ with $X_s \in V_{\partial D}(R)$ [1], one has $\alpha_s \geq a_0$.

[1] w.l.o.g., we can assume that the constant R is the same as the one defining the

This is of course weaker than uniform ellipticity, and ensures somehow that the paths of X are *a.s.* non tangent to ∂D. Some mild smoothness property on σ (some continuity in probability) will be also needed: the condition stated below is not restrictive at all and is fulfilled for instance as soon as $(\sigma_s)_{0\leq s\leq T}$ satisfies a Hölder property in $\mathbf{L}_p$-norm.

(S) For any $\delta > 0$, there is some function η_δ with $\lim_{h\to 0^+} \eta_\delta(h) = 0$ such that *a.s.*, for $s \in]t_i, t_{i+1}[$ with $X_s \in \partial D$, one has $\mathbb{P}(|\int_s^{t_{i+1}}(\sigma_u - \sigma_s)dW_u| \geq \delta\sqrt{t_{i+1}-s} \,|\, \mathcal{F}_s) \leq \eta_\delta(h)$.

The main result of this section states that the approximation error consisting in replacing τ by τ^N in $\mathbb{E}\big(f(X_T)\mathbf{1}_{\tau>T}\big)$ yields an error at least of order $\frac{1}{2}$ w.r.t. h, under the non characteristic boundary condition **(C)**.

Theorem 2 *Assume* **(C)**, **(D)**, **(F)** *and* **(S)**. *For some constant C, one has*

$$0 \leq \mathbb{E}[f(X_T)\mathbf{1}_{\tau^N>T}] - \mathbb{E}[f(X_T)\mathbf{1}_{\tau>T}] \leq C\frac{\|f\|_\infty}{1\wedge\varepsilon}\sqrt{h}.$$

The error is non negative since *a.s.*, one has $\tau^N \geq \tau$. Note that a similar result was proved by the first author in [Gob00] under much stronger assumptions, namely in the Markovian setting and under a uniform ellipticity condition.
One may wonder if a lower bound is available with the same rate: the answer is NO if no extra conditions are imposed. Indeed, even under **(C)**, it is possible that $\mathbb{P}(\tau \leq T) = 0$ and the error equals 0. It is also possible that the support of f does not intersect that of the law of X_T. So, we leave this issue of a lower bound in the non-Markovian setting and will handle it in the diffusion case (Section 3).

PROOF. Since $\tau^N \geq \tau$, one has $\mathbb{E}[f(X_T)\mathbf{1}_{\tau^N>T}] - \mathbb{E}[f(X_T)\mathbf{1}_{\tau>T}] = \mathbb{E}[\mathbf{1}_{T\leq\tau}\mathbb{E}[f(X_T)\mathbf{1}_{\tau^N>T} \mid \mathcal{F}_\tau]]$ and thus, it is enough to derive

$$\mathcal{E} := \mathbb{E}[f(X_{T'})\mathbf{1}_{\tau^{N'}>T'}] \leq C\frac{\|f\|_\infty}{1\wedge\varepsilon}\sqrt{h}, \tag{4}$$

for an initial point $x \in \partial D$, for a shifted time mesh defined by $\{t_i : 0 \leq i \leq N'\}$ with $t_0 = 0$, $0 < t_1 \leq h$, $t_{i+1} = t_i + h$ $(i \geq 1)$, for a new terminal time $T' = t_{N'}$ and a modified exit time $\tau^{N'} = \inf\{t_i \geq t_1 : X_{t_i} \notin D\}$. The constant C in (4) has to be uniform in T' in a compact set, in N' and in x. For convenience, we still denote N for N' and T for T'.

We recall some basic estimates, which will be useful in this section but also in the next sections: they only exploit the boundedness of the coefficients.

signed distance to the boundary in Proposition 1.

Lemma 3 (Bernstein's type inequality) *Consider two stopping times S, S' upper bounded by T with $0 \leq S' - S \leq \Delta \leq T$. Then for any $p \geq 1$ and $c' > 0$, there are some constants $c > 0$ and C, such that for any $\eta \geq 0$, one has a.s.:*

$$\begin{aligned}
\mathbb{P}[\sup_{t\in[S,S']} \|X_t - X_S\| \geq \eta \mid \mathcal{F}_S] \leq& C \exp\left(-c\frac{\eta^2}{\Delta}\right), \\
\mathbb{E}[\sup_{t\in[S,S']} \|X_t - X_S\|^p \mid \mathcal{F}_S] \leq& C\Delta^{p/2}, \\
\mathbb{E}[\exp\left(-c'\frac{d^2(X_{S'}, \partial D)}{\Delta}\right) \mid \mathcal{F}_S] \leq& C \exp\left(-c\frac{d^2(X_S, \partial D)}{\Delta}\right).
\end{aligned}$$

We omit the proof of the first inequality which is standard and refer the reader to Lemma 4.1 in [Gob00] for instance. The two other ones easily follow from the first one, see also Lemma 4.1 in [BT96b] for the last one.

The next lemma is crucial in our analysis: its rather technical proof is postponed to the end of this section.

Lemma 4 *Assume* **(C)**, **(D)** *and* **(S)**. *There are some positive constants C and N_0 such that for $N \geq N_0$, for any $i \in \{0, \cdots, N-1\}$, one has for $X_{t_i} \in D$*

$$\mathbb{P}[\exists t \in [t_i, t_{i+1}] : X_t \notin D \mid \mathcal{F}_{t_i}] \leq C\, \mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{t_i}].$$

We now turn to the proof of (4) when $x \in \partial D$ and for this, we introduce an auxiliary non negative process defined by

$$V_t = \mathbb{E}\Big(\mathbf{1}_{T<\tau_t} f(X_T) \mid \mathcal{F}_t\Big), \tag{5}$$

with $\tau_t = \inf\{s \geq t : X_s \notin D\}$. This definition has some similarities with $v(t, X_t)$ later defined in section 3 in the Markovian case (v solving some PDE), except that under the sole non characteristic boundary condition, there is no guarantee that v is smooth (and in general, it is not).

First, note that $V_t = 0$ if $X_t \notin D$, hence $V_0 = 0$ ($x \in \partial D$). Since $\tau_T \geq T$, one has clearly $V_T = f(X_T)$ for $X_T \in D$. Hence, the error to analyze can be easily decomposed as follows:

$$\begin{aligned}
\mathcal{E} =& \mathbb{E}(\mathbf{1}_{T<\tau^N} V_T) = \sum_{i=0}^{N-1} \mathbb{E}(V_{t_{i+1}\wedge\tau^N} - V_{t_i\wedge\tau^N}) \\
=& \sum_{i=0}^{N-1} \mathbb{E}(\mathbf{1}_{t_i<\tau^N}[(V_{t_{i+1}} - V_{t_{i+1}\wedge\tau_{t_i}}) + (V_{t_{i+1}\wedge\tau_{t_i}} - V_{t_i})]).
\end{aligned} \tag{6}$$

First, it is easy to check that $\mathbf{1}_{t_i<\tau^N}(V_{t_{i+1}} - V_{t_{i+1}\wedge\tau_{t_i}}) = \mathbf{1}_{t_i<\tau^N}\mathbf{1}_{\tau_{t_i}<t_{i+1}}V_{t_{i+1}}$. Second, we assert that $\mathbb{E}(\mathbf{1}_{t_i<\tau^N}\Big(V_{t_{i+1}\wedge\tau_{t_i}} - V_{t_i}\Big)|\mathcal{F}_{t_i}) = 0$. Indeed, by definition of $V_{t_{i+1}\wedge\tau_{t_i}}$, one gets

$$\mathbb{E}(\mathbf{1}_{t_i<\tau^N} V_{t_{i+1}\wedge\tau_{t_i}}|\mathcal{F}_{t_i}) = \mathbb{E}(\mathbf{1}_{t_i<\tau^N}\mathbf{1}_{T<\tau_{t_{i+1}\wedge\tau_{t_i}}} f(X_T)|\mathcal{F}_{t_i}). \tag{7}$$

On the event $\{\tau_{t_i} < t_{i+1}\}$, one has $t_{i+1}\wedge\tau_{t_i} = \tau_{t_i} = \tau_{t_{i+1}\wedge\tau_{t_i}} < t_{i+1} \leq T$: hence, $\mathbf{1}_{T<\tau_{t_{i+1}\wedge\tau_{t_i}}} = 0$. On the complementary event $\{t_{i+1} \leq \tau_{t_i}\}$, one has $\{t_{i+1} \leq \tau_{t_i}\} \cap \{T < \tau_{t_{i+1}\wedge\tau_{t_i}}\} = \{t_{i+1} \leq \tau_{t_i}\} \cap \{T < \tau_{t_{i+1}}\} = \{T < \tau_{t_i}\}$ using the definition of τ_t. We have proved that the r.h.s. of (7) equals $\mathbb{E}(\mathbf{1}_{t_i<\tau^N} V_{t_i}|\mathcal{F}_{t_i})$ and this completes our assertion.
Plugging previous equalities into (6) leads to

$$\mathcal{E} = \sum_{i=0}^{N-1} \mathbb{E}(\mathbf{1}_{t_i<\tau^N}\mathbf{1}_{\tau_{t_i}<t_{i+1}} V_{t_{i+1}}). \tag{8}$$

To complete the analysis, we need the following lemma that will be proved later.

Lemma 5 *Assume* **(C)**, **(D)**, **(F)** *and* **(S)**. *There is some constant C such that for any $t \in [0,T]$, one has a.s.*

$$V_t \leq C\frac{\|f\|_\infty}{1\wedge\varepsilon}[F(X_t)]_+. \tag{9}$$

Thus, it follows that

$$\begin{aligned}
\mathbb{E}(\mathbf{1}_{t_i<\tau^N}\mathbf{1}_{\tau_{t_i}<t_{i+1}} V_{t_{i+1}}) \leq& C\frac{\|f\|_\infty}{1\wedge\varepsilon}\sqrt{h}\mathbb{E}\Big(\mathbf{1}_{t_i<\tau^N}\mathbf{1}_{\tau_{t_i}<t_{i+1}}\Big)\\
\leq& C\frac{\|f\|_\infty}{1\wedge\varepsilon}\sqrt{h}\mathbb{P}\Big(t_i<\tau^N; X_{t_{i+1}}\notin D\Big)\\
=& C\frac{\|f\|_\infty}{1\wedge\varepsilon}\sqrt{h}\mathbb{P}\Big(t_{i+1}=\tau^N\Big)
\end{aligned}$$

where we used at the first line, $\mathbf{L}_1$ estimates on the increments $[F(X_{t_{i+1}})]_+ = [F(X_{t_{i+1}}) - F(X_{\tau_{t_i}})]_+$ (see Lemma 3), and Lemma 4 at the second line. Combining the previous estimate with (8) yields $\mathcal{E} \leq C\frac{\|f\|_\infty}{1\wedge\varepsilon}\sqrt{h}\mathbb{P}\Big(\tau^N \leq T\Big)$ and thus, completes the proof of (4).

PROOF of Lemma 4 We adapt some ideas from [Gob00]: in the cited paper, a uniform ellipticity condition was assumed, and this enabled to use a Gaussian type lower bound for the conditional density of $X_{t_{i+1}}$ w.r.t. the Lebesgue measure, together with some computations related to a cone exterior to D. Here, under **(C)**, the conditional law of $X_{t_{i+1}}$ may be degenerate and our proof rather exploits the scaling invariance of the cone and of the Brownian increments.

It is enough to prove that $a.s$ on $\{t_i < \tau_{t_i} < t_{i+1}\}$, one has

$$\mathbb{P}(X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}) \geq \frac{1}{C}. \tag{10}$$

Indeed, it follows that $\mathbb{P}(X_{t_{i+1}} \notin D \mid \mathcal{F}_{t_i}) = \mathbb{E}(\mathbf{1}_{\tau_{t_i} \leq t_{i+1}} \mathbb{P}(X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}) \mid \mathcal{F}_{t_i}) \geq \frac{\mathbb{P}(\tau_{t_i} \leq t_{i+1} \mid \mathcal{F}_{t_i})}{C}$ and Lemma 4 is proved.
To get (10), write $X_{t_{i+1}} = X_{\tau_{t_i}} + \sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}}) + R_i$ where $R_i = \int_{\tau_{t_i}}^{t_{i+1}} b_u du + \int_{\tau_{t_i}}^{t_{i+1}} (\sigma_u - \sigma_{\tau_{t_i}}) dW_u$. The domain D is of class C^2, and thus satisfies a uniform exterior sphere condition with radius $R/2$ (R defined in Proposition 1): for any $z \in \partial D$, $B(z - \frac{R}{2} n(z), \frac{R}{2}) \subset D^c$. In particular, if we define for $\theta \in]0, \pi/2[$ the cone $\mathcal{K}(\theta, z) = \{y \in \mathbb{R}^d : (y - z).[-n(z)] \geq \|y - z\| \ \cos(\theta)\}$, then one has $\mathcal{K}(\theta, z) \cap B(z, R(\theta)) \subset B(z - \frac{R}{2} n(z), \frac{R}{2}) \subset D^c$ for some appropriate choice of the *strictly positive* function $R(.)$. Then, it follows that

$$\begin{aligned}
&\mathbb{P}(X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}) \geq \mathbb{P}(X_{t_{i+1}} \in \mathcal{K}(\theta, X_{\tau_{t_i}}) \cap B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}) \\
&\geq \mathbb{P}(X_{t_{i+1}} \in \mathcal{K}(\theta, X_{\tau_{t_i}}) \mid \mathcal{F}_{\tau_{t_i}}) - \mathbb{P}(X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}) \\
&\geq \mathbb{P}((X_{t_{i+1}} - X_{\tau_{t_i}}).(-n(X_{\tau_{t_i}})) \geq \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \geq \|X_{t_{i+1}} - X_{\tau_{t_i}}\| \cos(\theta) \mid \mathcal{F}_{\tau_{t_i}}) \\
&\quad - \mathbb{P}(X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}) \geq A_1 - A_2(\theta) - A_3(\theta),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\text{where } \ A_1 &= \mathbb{P}((X_{t_{i+1}} - X_{\tau_{t_i}}).(-n(X_{\tau_{t_i}})) \geq \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}), \\
A_2(\theta) &= \mathbb{P}(\sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} < \|X_{t_{i+1}} - X_{\tau_{t_i}}\| \cos(\theta) \mid \mathcal{F}_{\tau_{t_i}}), \\
A_3(\theta) &= \mathbb{P}(X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}).
\end{aligned}$$

<u>Term A_1.</u> Clearly, one has $A_1 \geq \mathbb{P}((-n(X_{\tau_{t_i}})).\sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}}) \geq 2\sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}) - \mathbb{P}(|n(X_{\tau_{t_i}}).R_i| \geq \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}) := A_{11} - A_{12}$. The random variable $(-n(X_{\tau_{t_i}})).\sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}})$ is conditionally to $\mathcal{F}_{\tau_{t_i}}$ a centered Gaussian variable with variance $\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})$, and thus $A_{11} = \Phi(-2) > 0$. Owing to the condition **(S)** and since $\alpha_{\tau_{t_i}} \geq a_0$ $a.s$, it is easy to see that the contribution A_{12} converges uniformly to 0 when h goes to 0, and thus for $h = T/N$ enough small, one has $A_1 \geq \frac{A_{11}}{2} > 0$.
<u>Term $A_2(\theta)$.</u> From Markov's inequality, $A_2(\theta) \leq \frac{\mathbb{E}(\|X_{t_{i+1}} - X_{\tau_{t_i}}\|^2 \cos^2(\theta) \mid \mathcal{F}_{\tau_{t_i}})}{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \leq C \cos^2(\theta)$ using **(C)** and estimates of Lemma 3. In particular, taking θ close to $\pi/2$ ensures that $A_2(\theta) \leq \frac{A_{11}}{6}$.
<u>Term $A_3(\theta)$.</u> Using Lemma 3, one readily gets $A_3(\theta) \leq C \exp\left(-c \frac{R^2(\theta)}{h}\right) \leq \frac{A_{11}}{6}$ for h small enough ($R(\theta) > 0$).

Putting together estimates for $A_1, A_2(\theta)$ and $A_3(\theta)$ into (11) give $\mathbb{P}(X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}) \geq \frac{A_{11}}{6}$. This proves (10).

PROOF of Lemma 5. Since $V_t = 0$ for $X_t \notin D$, it is enough to prove the

estimate for $X_t \in D \cap V_{\partial D}(R \wedge \varepsilon/2)$ for which $0 < F(X_t) \leq R \wedge \varepsilon/2$. Denote $\tau_t^R = \inf\{s \geq t : F(X_s) \leq R\}$ and split V into two parts $V_t = V_t^1 + V_t^2$ with $V_t^1 = \mathbb{E}\Big(\mathbf{1}_{T<\tau_t}\mathbf{1}_{T<\tau_t^R} f(X_T) \mid \mathcal{F}_t\Big)$ and $V_t^2 = \mathbb{E}\Big(\mathbf{1}_{T<\tau_t}\mathbf{1}_{T\geq\tau_t^R} f(X_T) \mid \mathcal{F}_t\Big)$.
Before estimating separately each contribution, we set some standard notations related to time-changed Brownian martingales. Define the increasing continuous process $\mathcal{A}_s = \int_t^s \alpha_u du$ (from $[t, +\infty[$ into $\mathbb{R}^+$) and its increasing right-continuous inverse $\mathcal{C}_s = \inf\{u \geq t : \mathcal{A}_u > s\}$ (from $\mathbb{R}^+$ into $[t, +\infty[$) (see section V.1 in Revuz-Yor [RY94]) and put $M_s = \int_t^{\mathcal{C}_s} \nabla F(X_u).\sigma_u dW_u$, $Z_s = F(X_{\mathcal{C}_s})$. From the Dambis-Dubins-Schwarz theorem, M coincides with a standard BM β (defined on a possibly enlarged probability space) for $s < \int_t^\infty \alpha_u du$ and it is easy to check that β is independent of $\mathcal{F}_t$ (see the arguments in the proof of Theorem 1.7 in [RY94]).
Owing to the assumption **(C)**, $\mathcal{A}$ and $\mathcal{C}$ are strictly increasing on $[t, \tau_t^R]$ and $[0, \int_t^{\tau_t^R} \alpha_u du]$. Thus, for $s \in [0, \int_t^{\tau_t^R} \alpha_u du]$, one easily obtains

$$Z_s = F(X_t) + \beta_s + \int_0^s \lambda_v dv$$

where $\lambda_v = \{[\nabla F(X_u).b_u + \frac{1}{2}\mathrm{Tr}(H_F(X_u)\sigma_u\sigma_u^*)]|_{u=\mathcal{C}_v}\}\frac{1}{\alpha_{\mathcal{C}_v}}$ is bounded by $\|\lambda\|_\infty$.
Define

$$Z'_s = F(X_t) + \beta_s + \|\lambda\|_\infty s \geq Z_s. \tag{12}$$

Finally, put $\tau_0^Z = \inf\{s \geq 0 : Z_s \leq 0\}$, $\tau_R^Z = \inf\{s \geq 0 : Z_s \geq R\}$ and analogously $\tau_0^{Z'}$, $\tau_R^{Z'}$ for Z'.

Estimation of V^1. Let us first prove that for any stopping time $S \in [t, T]$, one has

$$\begin{aligned}\mathbb{E}\Big(f(X_T) \mid \mathcal{F}_S\Big) &\leq \|f\|_\infty \mathbb{P}(F(X_T) \geq 2\varepsilon \mid \mathcal{F}_S)\\ &\leq C\|f\|_\infty \exp\Big(-c\frac{(2\varepsilon - F(X_S))_+^2}{T-S}\Big) \ a.s.. \end{aligned}\tag{13}$$

The first inequality simply results from the support of f included in $D \backslash V_{\partial D}(2\varepsilon)$ (assumption **(F)**). To justify the second one, note that $\{F(X_T) \geq 2\varepsilon\} \subset \{|F(X_T) - F(X_S)| \geq 2\varepsilon - F(X_S)\} \subset \{|F(X_T) - F(X_S)| \geq (2\varepsilon - F(X_S))_+\}$ and the proof of (13) is complete using Lemma 3 applied to the Itô process $(F(X_s))_{s\geq 0}$ with bounded coefficients.
We now turn to the evaluation of V_t^1. On $\{T < \tau_t^R\}$, using the notation with the time change above, one has $T = \mathcal{C}_{\mathcal{A}_T} \geq \mathcal{C}_{a_0(T-t)}$ and $T - \mathcal{C}_{a_0(T-t)} \leq \frac{1}{a_0}(\mathcal{A}_T - a_0(T-t)) \leq \frac{\|\alpha\|_\infty}{a_0}(T-t)$. Thus, one obtains

$$\begin{aligned}V_t^1 &\leq \mathbb{E}\Big(\mathbf{1}_{\mathcal{C}_{a_0(T-t)}<\tau_t}\mathbf{1}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\mathbf{1}_{T-\mathcal{C}_{a_0(T-t)}\leq\frac{\|\alpha\|_\infty}{a_0}(T-t)}\mathbb{E}\Big(f(X_T) \mid \mathcal{F}_{\mathcal{C}_{a_0(T-t)}}\Big) \mid \mathcal{F}_t\Big)\\
&\leq C\|f\|_\infty \mathbb{E}\Big(\mathbf{1}_{\mathcal{C}_{a_0(T-t)}<\tau_t}\mathbf{1}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R} \exp\Big(-c'\frac{(2\varepsilon - F(X_{\mathcal{C}_{a_0(T-t)}}))_+^2}{T-t}\Big) \mid \mathcal{F}_t\Big)\\
&\leq C\|f\|_\infty \mathbb{E}\Big(\mathbf{1}_{a_0(T-t)<\tau_0^{Z'}}\mathbf{1}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R} \exp\Big(-c'\frac{(2\varepsilon - Z'_{a_0(T-t)})_+^2}{T-t}\Big) \mid \mathcal{F}_t\Big)\end{aligned}$$

where one has applied at the second line the estimate (13) with $S = \mathcal{C}_{a_0(T-t)}$ (here $c' = c\frac{a_0}{\|\alpha\|_\infty}$), at the third one $\{\mathcal{C}_{a_0(T-t)} < \tau_t\} = \{\forall s \in [t, \mathcal{C}_{a_0(T-t)}] : F(X_s) > 0\} = \{\forall u \in [0, a_0(T-t)] : Z_u > 0\} = \{a_0(T-t) < \tau_0^Z\} \subset \{a_0(T-t) < \tau_0^{Z'}\}$ and $(2\varepsilon - F(X_{\mathcal{C}_{a_0(T-t)}}))_+ = (2\varepsilon - Z_{a_0(T-t)})_+ \geq (2\varepsilon - Z'_{a_0(T-t)})_+$. Reminding the law of β, one finally gets that $V_t^1 \leq C\|f\|_\infty \Phi_1(a_0(T-t), F(X_t))$ with $\Phi_1(r,z) = \mathbb{E}\Big(\mathbf{1}_{\forall u\in[0,r]: z+\beta_u+\|\lambda\|_\infty u>0} \exp\Big(-a_0 c' \frac{(2\varepsilon - z - \beta_r - \|\lambda\|_\infty r)_+^2}{r}\Big)\Big)$. With clear notations involving the smooth transition density of the killed drifted BM (see Sections 3 and 5) and Gaussian type estimates of its gradient (see [LSU68] Theorem 16.3), one has $\Phi_1(r,z) = \int_0^\infty q_r(z,y) \exp\Big(-a_0 c' \frac{(2\varepsilon-y)_+^2}{r}\Big) dy$ and

$$|\partial_z \Phi_1(r,z)| \leq C \int_0^\infty \frac{1}{r} \exp(-c\frac{(z-y)^2}{r}) \exp\Big(-a_0 c' \frac{(2\varepsilon - y)_+^2}{r}\Big) dy.$$

We now justify that $|\partial_z \Phi_1(r,z)| \leq \frac{C}{1\wedge\varepsilon}$ for $0 \leq z \leq \epsilon/2$ and for this, we may split the domain of integration into two parts. For $y < \varepsilon$, $(2\varepsilon - y)_+^2 \geq \varepsilon^2$ and the corresponding contribution for the integral is bounded by $\int_0^\infty \frac{1}{\sqrt{r}} \exp(-c\frac{(z-y)^2}{r})$ $\Big[\frac{1}{\sqrt{r}} \exp\Big(-a_0 c' \frac{\varepsilon^2}{r}\Big)\Big] dy \leq \frac{C}{1\wedge\varepsilon}$. For $y \geq \varepsilon$ and $0 \leq z \leq \varepsilon/2$, $(z-y)^2 \geq \varepsilon^2/4$ and the integral is bounded by $\int_0^\infty \frac{1}{\sqrt{r}} \exp(-\frac{c}{2}\frac{(z-y)^2}{r}) \frac{1}{\sqrt{r}} \exp(-\frac{c}{2}\frac{\varepsilon^2}{4r}) dy \leq \frac{C}{1\wedge\varepsilon}$. Since $\Phi_1(r,0) = 0$, one gets $\Phi_1(r,z) \leq \frac{C}{1\wedge\varepsilon} z$ for $z \in [0, \varepsilon/2]$ and this proves that $V_t^1 \leq C\frac{\|f\|_\infty}{1\wedge\varepsilon} F(X_t)$.

Estimation of V^2. Clearly, one has $V_t^2 \leq \|f\|_\infty \mathbb{P}\Big(\tau_t^R < \tau_t \mid \mathcal{F}_t\Big)$. Note that $\{\tau_t^R < \tau_t\} = \{\tau_R^Z < \tau_0^Z\} \subset \{\tau_R^{Z'} < \tau_0^{Z'}\}$ because of (12). Hence, one has $V_t^2 \leq \|f\|_\infty \Phi_2(z)$ where $\Phi_2(z) = \mathbb{P}((z + \beta_u + \|\lambda\|_\infty u)_{u\geq 0}$ hits R before $0)$. It is well-known that $\Phi_2(z) = \frac{1-\exp(-2\|\lambda\|_\infty z)}{1-\exp(-2\|\lambda\|_\infty R)} \leq Cz$ (see Section 5.5 in [KS91] e.g.) and this proves that $V_t^2 \leq C\|f\|_\infty F(X_t)$.

Combining estimates for V^1 and V^2 gives the result of Lemma 5.

3 Diffusion processes

In this section we consider $(X_t)_{t\geq 0}$ solving the SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \ X_0 = x \in D. \tag{14}$$

When needed, we will use the usual notation $X^{t_0,x}$ for the solution starting from x at time t_0. We approximate this diffusion by its Euler scheme $(X_t^N)_{t\geq 0}$ defined as in (3).

Additional notations and assumptions. In the following $O_{pol}(h)$ (resp.

$O(h)$) stands for every quantity $R(h)$ such that $\forall n \in \mathbf{N}$, for some $C>0$, one has $|R(h)| \leq Ch^n$ (resp. $|R(h)| \leq Ch$) (uniformly in x).

The notation L stands for the infinitesimal generator of the diffusion, $Lg(x) = b(x).\nabla g(x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(x)H_g(x))$. We additionally define for all $z \in \mathbb{R}^d$ the operator $\mathcal{L}_z$ by $\mathcal{L}_z g(x) = b(z).\nabla g(x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(z)H_g(x))$, which can locally be interpreted as the generator of the Euler approximation.

We introduce:

(S') b and σ of class C_b^∞ (bounded with bounded derivatives).

We also require the coefficients to satisfy the *strong* Hörmander assumption (see Remark 6). Identifying the coefficients $(\sigma_i)_{1\leq i\leq d'}$ with the vector fields which they define, we denote by $L_M(x)$ the set of the Lie brackets of length lower or equal to M of these vector fields taken at point x.

(H) $\exists M \in \mathbf{N}, \exists C>0, \forall x \in \mathbb{R}^d,\ \forall z \in \mathbb{R}^d,\ \sum_{Y\in L_M(x)} \langle Y, z\rangle^2 \geq C|z|^2.$

We now define the analogue to **(C)** by setting $\alpha(x) := \nabla F(x).\sigma\sigma^*(x)\nabla F(x)$.

(C') $\exists a_0 > 0, \forall x \in V_{\partial D}(R),\ \alpha(x) \geq a_0$.

For $(t,x) \in [0,T]\times\mathbb{R}^d$ we set $v(t,x) := \mathbb{E}_x[\mathbf{1}_{\tau>T-t} f(X_{T-t})]$. It is known, see [Cat91], that under **(C')**, **(D)**, **(H)**, **(S')** for every bounded measurable function f, $v(t,x) = \int_D q_{T-t}(x,y)f(y)dy$ where q_{T-t} denotes the transition density of the killed process at time $T-t$. The function q is $C^\infty((0,T]\times\bar{D}\times\bar{D},\mathbb{R})$, satisfies Kolmogorov's equations and for any multi-index α there exist constants $c>0, \zeta>0$ and C s.t.

$$\forall (s,x,y) \in (0,T]\times\bar{D}\times\bar{D},\ |\partial^\alpha q_s(x,y)| \leq \frac{C}{s^\zeta}\exp(-c\frac{\|x-y\|^2}{s}). \tag{15}$$

These estimates easily follow from Proposition 3.44 in [Cat91] and the arguments used in the proof of Proposition 1.12 in [Cat90]. Thus, v belongs to $C^\infty([0,T)\times\bar{D},\mathbb{R})$ and satisfies the mixed Cauchy-Dirichlet parabolic PDE

$$\partial_t v + Lv = 0 \text{ on } [0,T]\times\bar{D},\ v(t,x) = 0 \text{ on } [0,T]\times D^c,\ v(T,x) = f(x) \text{ on } D,$$

with the estimates $\sup_{x\in\bar{D}} |\partial_x^\alpha v(t,x)| \leq C\|f\|_\infty/(T-t)^\zeta$, for some constants depending on α. If we additionally assume **(F)**, as a consequence of (15), v and its derivatives near ∂D are uniformly bounded and exponentially decreasing when $t\to T$: for all multi-index α there exist constants $c>0, \zeta>0$ and C

such that

$$\forall (t,x) \in [0,T) \times V_{\partial D}(\varepsilon),\ |\partial_x^\alpha v(t,x)| \le C \frac{\|f\|_\infty}{1 \wedge \varepsilon^\zeta} \exp(-c \frac{\varepsilon^2}{T-t}). \qquad (16)$$

Since the function f is non-negative and not identically equal to 0, we have the following property, which will be used for the lower bound:

(P) Under **(C')**, **(D)**, **(F)**, **(H)**, **(S')**, we have $v(t,x) > 0$ on $[0,T) \times D$.

PROOF. We are reduced to check the strict positivity of $q_t(x,y)$ on $(0,T] \times D \times D$. But this property follows from the arguments used for Lemma 5.37 in [Cat92], that can be adapted to our case. Under a uniform ellipticity condition, see [Fri64] Theorem 11 p 44. For SDEs in the whole space under **(H)**, see [BL91] and references therein.

Remark 6 *The strong Hörmander condition* **(H)** *is needed only for the lower bound in Theorem 9. In particular, Theorem 8 and the upper bound in Theorem 9 are valid under the weak Hörmander condition* **(H')** *(see [Nua95] p.111 for a precise definition). We would like to thank P. Cattiaux [Cat03] for having suggested us the following example of a linear SDE which never hits the boundary of the domain* $D =]-\pi, 2\pi[$. *Set* $X_0 = \pi/2$, $b(x) = \cos(x)$, $\sigma(x) = \sin(x)$: **(H')** *holds true,* **(H)** *does not, and* $(X_t)_{t \ge 0}$ *is living in* $[0,\pi]$. *Thus, the killing boundary has no effect in that case and one could prove that* $\mathrm{Err}_2(T,h,f,x) = O(h)$ *(see [BT96a]), avoiding a possible lower bound with rate* $\sqrt{h}$.

Before stating our main results, we mention that Assumption **(C')** is sufficient to guarantee the convergence to 0 of the error.

Proposition 7 ***(Weak error convergence).*** *Under* **(C')**, *if* D *is of class* C^2 *with a compact boundary and the coefficients in (14) are Lipschitz continuous, for every bounded continuous function* f *we have:*

$$\lim_{h \longrightarrow 0} \mathrm{Err}_2(T,h,f,x) = 0.$$

PROOF. According to Proposition 1.1 in [Gob00] it suffices to satisfy the condition $\mathbb{P}_x[\exists t \in [0,T] : X_t \notin D;\ \forall t \in [0,T],\ X_t \in \bar{D}] = \mathbb{P}_x[M = 0] = 0$, where $M := \inf_{s \in [0,T]} F(X_s)$. For this, we verify the Nualart-Vives criterion for the local absolute continuity of the law of M around 0 (see Proposition 2.1.3, to check $M \in \mathbb{D}^{1,2}$, and Corollary 2.1.1 in [Nua95]): namely, it is enough to prove that $\|DF(X_t)\|_{L^2[0,T]} > 0$ a.s. for $t \in \mathcal{M}_T := \{s \in [0,T],\ F(X_s) = M\} \subset]0,T]$ on the event $|M| \le \frac{R \wedge F(x)}{2}$. But for $t \in \mathcal{M}_T$ and $|M| \le R/2,\ X_t \in V_{\partial D}(R)$ and thus $\|D_t F(X_t)\|^2 = \alpha(X_t) \ge a_0 > 0$: by continuity of $s \in [0,t] \mapsto D_s F(X_t)$, it easily follows $\|DF(X_t)\|_{L^2([0,T])} > 0$ a.s..

Theorem 8 *Under* **(C')**, **(D)**, **(F)**, **(H)**, **(S')**, *we have:*

$$\mathrm{Err}_2(T,h,f,x) = \frac{1}{2}\mathbb{E}_x[\int_0^T \frac{\partial v}{\partial n}(s,X_s^N)dL^0_{s\wedge\tau^N}(F(X^N))] + O(h).$$

Theorem 9 *Under* **(C')**, **(D)**, **(F)**, **(H)**, **(S')**, *for h small enough (depending on $d(x,\partial D)>0$), we have*

$$C_1\sqrt{h} \le \mathrm{Err}_2(T,h,f,x) \le C_2\sqrt{h}$$

for two constants $C_1>0$ and $C_2>0$.

Proposition 10 *Under* **(C')**, **(D)**, **(S')**, *for some $c_0>0$ one has:*

$$\sup_{N,s\in[0,T]} \mathbb{E}_x[\exp(c_0[h^{-1/2}F^-(X^N_{s\wedge\tau^N})]^2)] < \infty.$$

Hence, the sequence of random variables $(h^{-1/2}F^-(X^N_{s\wedge\tau^N}))_{N\ge 1}$ is uniformly tight on $[0,T]$.

The first theorem exhibits the relevant term of the error, i.e. the one that has to be developed in order to give an expansion of the error. The second one states that the leading term is really of order $\frac{1}{2}$. Moreover, the main term can be interpreted in terms of Tanaka's formula as a suitable average of the overshoot. Indeed, we show, see Lemma 13, that $\frac{1}{2}\mathbb{E}_x[L^0_{s\wedge\tau^N}(F(X^N))] = \mathbb{E}_x[F^-(X^N_{s\wedge\tau^N})] + O(h)$. Therefore, the last proposition is somehow the first step for a future expansion of the error.

3.2 Proof of the main results

We first state several technical lemmas whose proofs are postponed to section 4. Since f is non negative, v is non negative and since v vanishes on ∂D, clearly $\frac{\partial v}{\partial n}(t,x)\ge 0$: actually, the inequality is strict (thanks to Property **(P)**).

Lemma 11 (Positivity of the inner normal derivative). *Under* **(C')**, **(D)**, **(F)**, **(H)**, **(S')**, *for any $(t,x)\in[0,T[\times\partial D$, we have $\frac{\partial v}{\partial n}(t,x)>0$.*

Lemma 12 (Bounds for the expectation of the local time on the boundary). *Under* **(C')**, **(D)**, **(H)**, **(S')**, *for h small enough (depending on $d(x,\partial D)>0$), we have*

$$C_1\sqrt{h} \le \mathbb{E}_x[L^0_{T/2\wedge\tau^N}(F(X^N))] \le \mathbb{E}_x[L^0_{T\wedge\tau^N}(F(X^N))] \le C_2\sqrt{h},$$

with $C_2 > C_1 > 0$.

Lemma 13 (A sharp control for the integral of the exit probability). *Under* **(C')**, **(D)**, **(S')**, *we have* $\int_0^T \mathbb{P}_x[t \le \tau^N, X_t^N \notin D]dt \le Ch$.

A key step of the proof of Theorem 8 and Theorem 9 consists in controlling the derivatives of the function v. Equation (16) gives a point-wise estimate on these derivatives close to the boundary. Lemma 14 below gives controls of the expectations of these derivatives far from the boundary. From now on we denote by ψ a cutting function near ∂D such that: $\psi \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$, $\mathbf{1}_{V_{\partial D}(\varepsilon/2)} \le 1 - \psi \le \mathbf{1}_{V_{\partial D}(\varepsilon)}$ and $\|\partial^\alpha \psi\|_\infty \le \dfrac{C_{|\alpha|}}{1 \wedge \varepsilon^\alpha}$ for all multi-index α. In the following two lemmas we introduce $\tau_R^N := \inf\{t \ge 0 : X_t^N \notin D(R)\}$.

Lemma 14 (Expectation of the derivatives "far" from the boundary). *Under* **(C')**, **(D)**, **(F)**, **(H)**, **(S')**, *for all multi-indices* α, α', *all function* $g \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$, *there exist constants* ξ *and* C *such that* $\forall i \in [\![0, N-1]\!], t \in [t_i, T-h]$:

$$\left|\mathbb{E}_x\left[\mathbf{1}_{t<\tau^N\wedge\tau_R^N} g(X_t^N)\partial_x^\alpha(v\partial^{\alpha'}\psi)(t, X_t^N)\right]\right| \le C\frac{\|f\|_\infty}{1\wedge\varepsilon^\xi},$$

$$\left|\mathbb{E}_x\left[\mathbf{1}_{t<\tau^N\wedge\tau_R^N} g(X_{t_i}^N)\partial_x^\alpha(v\partial^{\alpha'}\psi)(t, X_t^N)\right]\right| \le C\frac{\|f\|_\infty}{1\wedge\varepsilon^\xi}.$$

Lemma 15 (Control of the last time step). *Under* **(C')**, **(D)**, **(F)**, **(H)**, **(S')**, *there exist* C, β *such that:*

$$|\mathbb{E}_x[v(T\wedge\tau^N\wedge\tau_R^N, X_{T\wedge\tau^N\wedge\tau_R^N}^N) - v((T-h)\wedge\tau^N\wedge\tau_R^N, X_{(T-h)\wedge\tau^N\wedge\tau_R^N}^N)]| \le \frac{C}{1\wedge\varepsilon^\beta}h.$$

We are now in position to prove our main results.

PROOF of Theorem 9. As a consequence of Theorem 8, the lower bound follows from Lemmas 11 and 12, while the upper bound is derived from equation (16) and Lemma 12.

PROOF of Theorem 8. Let us first write: $\mathrm{Err}_2(T, h, f, x) := E_1(h) + E_2(h) + E_3$ where $E_1(h) := \mathbb{E}_x[v(\tau^N \wedge \tau_R^N \wedge (T-h), X_{\tau^N\wedge\tau_R^N\wedge(T-h)}^N)] - v(0, x)$, $E_2(h) = \mathbb{E}_x[v(\tau^N \wedge \tau_R^N \wedge T, X_{\tau^N\wedge\tau_R^N\wedge T}^N)] - \mathbb{E}_x[v(\tau^N \wedge \tau_R^N \wedge (T-h), X_{\tau^N\wedge\tau_R^N\wedge(T-h)}^N)]$, $E_3 := \mathbb{E}_x[\mathbf{1}_{\tau_R^N < T < \tau^N} f(X_T^N)]$. Lemma 3 gives $E_3 = O_{pol}(h)$ and Lemma 15 states that $E_2(h) = O(h)$. Concerning $E_1(h)$, we use the semi-martingale decomposition of $v(t \wedge \tau_R^N, X_{t\wedge\tau_R^N}^N)$ as in [Gob00] Corollary 3.1. This term writes:

$$
\begin{aligned}
E_1(h) = \mathbb{E}_x \Bigg[& \int_0^{(T-h)\wedge\tau^N\wedge\tau_R^N} \mathbf{1}_{X_t^N \in D} \partial_t v(t, X_t^N) dt + \frac{1}{2}\frac{\partial v}{\partial n}(t, X_t^N) dL_t^0(F(X^N)) \\
& + \mathbf{1}_{X_t^N \in D} \left(\nabla v(t, X_t^N).dX_t^N + \frac{1}{2}\mathrm{Tr}(H_v(t, X_t^N) d\langle X^N \rangle_t) \right) \\
& + \mathbf{1}_{X_t^N \notin D} \left(\nabla v(t, \pi_{\partial D}(X_t^N)).dX_t^{N,\partial D} + \frac{1}{2}\mathrm{Tr}(H_v(t, \pi_{\partial D}(X_t^N)) d\langle X^{N,\partial D} \rangle_t) \right) \Bigg]
\end{aligned}
$$

where $(X_t^{N,\partial D})_{t\geq 0}$ is an Itô process with bounded coefficients, see Proposition 3.1 in [Gob00] for a complete description.

Estimates (16) and Lemma 13 directly yield that the contribution with $X_t^N \notin D$ is bounded by Ch. Terms with $X_t^N \in D$ can be treated as in [Gob00] using estimates (16) and Lemma 14: they are also bounded by Ch. We mention that the controls obtained therein for the terms outside the domain were not sharp enough to derive our current main results.
At last, the difference between the integrals w.r.t. the local time stopped in $(T-h)\wedge\tau^N\wedge\tau_R^N$ and $T\wedge\tau^N$ is a $O_{pol}(h)$: this is easy to prove using (16), Lemma 3 and the local $\mathbf{L}_p$ boundedness of the local time, we omit details.

PROOF of Proposition 10. We only have to prove that there exist constants $c > 0$ and C s.t: $\forall A \geq 0, \sup_N \mathbb{P}_x[F^-(X_{t\wedge\tau^N}^N) \geq Ah^{1/2}] \leq C\exp(-cA^2)$ for $t \in [0,T]$, then any choice of $c_0 < c$ is valid. We write:

$$
\begin{aligned}
\mathbb{P}_x[F^-(X_{t\wedge\tau^N}^N) \geq Ah^{1/2}] = & \sum_{i=1}^{\phi(t)/h} \mathbb{E}_x[\mathbf{1}_{\tau^N > t_{i-1}} \mathbf{1}_{\tau_{t_{i-1}}^N < t_i} \mathbb{P}[F^-(X_{t_i}^N) \geq Ah^{1/2} | \mathcal{F}_{\tau_{t_{i-1}}^N}]] \\
& + \mathbb{P}_x[F^-(X_t^N) \geq Ah^{1/2}, \tau^N > t] := A_t + B_t
\end{aligned}
$$

where we define $\tau_{t_{i-1}}^N := \inf\{t \geq t_{i-1} : X_t^N \notin D\}$. B_t is directly estimated applying Lemma 3. This Lemma also enables to develop A_t as follows:

$$
\begin{aligned}
A_t \leq & \sum_{i=1}^{\phi(t)/h} \mathbb{E}_x[\mathbf{1}_{\tau^N > t_{i-1}} \mathbf{1}_{\tau_{t_{i-1}}^N < t_i} C\exp\left(-c\frac{A^2 h}{t_i - \tau_{t_{i-1}}^N}\right)] \\
\leq & C\exp(-cA^2)h^{-1}\int_0^T dt \mathbb{P}_x[\tau^N > \phi(t), \phi(t) + h > \tau_{\phi(t)}^N] \\
\leq & C\exp(-cA^2)h^{-1}\int_0^T dt \mathbb{E}_x[\mathbf{1}_{\tau^N > \phi(t)} \exp\left(-c\frac{d^2(X_{\phi(t)}^N, \partial D)}{h}\right)].
\end{aligned}
$$

In the proof of Lemma 13, we show that the last integral is bounded by Ch, which completes the proof.

4 Proof of technical lemmas

This section is dedicated to the proofs of Lemmas 11 to 15.

4.1 Hopf boundary point lemma

Lemma 11 is a direct consequence of the following Lemma applied to the function v: indeed, the martingale property for U^x below easily follows from the one for $(v(t\wedge\tau, X_{t\wedge\tau}) = \mathbb{E}(f(X_T)\mathbf{1}_{\tau>T}|\mathcal{F}_t))_{0\le t\le T}$ (Markov property); since $v(t_0,x_0) = 0$ for $(t_0,x_0) \in [0,T[\times\partial D$, Property **(P)** provides the required strict lower bound for v; at last, since v is smooth, the lim inf below equals the normal derivative of v.

Actually, the type of result stated in Lemma 16 is known in the PDE theory as the *Hopf boundary point* lemma: in the *uniformly parabolic* case, see [Fri64]; for *partially degenerate* elliptic operators see [Lie85]. We give here a variant of this result, using a probabilistic proof under the sole assumption **(C')** and without smoothness properties on u, which seems to be new.

Lemma 16 *Assume* **(C')**, **(D)** *and* **(S)**. *Consider* $(t_0,x_0) \in \mathbb{R}^+\times\partial D$ *and the time-space set* $\mathcal{D} = [t_0, t_0+\delta]\times(D\cap V_{\partial D}(R))$ *(with* $\delta > 0$ *and* R *defined as in Proposition 1). If* u *is a bounded continuous function defined on* $\overline{\mathcal{D}}$ *such that* $U^x = (U^x_s = u(s\wedge\tau_{\mathcal{D}}, X^{t_0,x}_{s\wedge\tau_{\mathcal{D}}}))_{s\ge t_0}$ *(with* $\tau_{\mathcal{D}} = \inf\{s \ge t_0 : (s, X^{t_0,x}_s) \notin \mathcal{D}\} \le t_0+\delta$*) defines a super-martingale and* $u(t,x) > u(t_0,x_0)$ *for* $(t,x)\in\mathcal{D}$, *then one has*

$$\liminf_{\lambda\downarrow 0} \frac{u(t_0, x_0+\lambda n(x_0)) - u(t_0,x_0)}{\lambda} > 0.$$

PROOF. The main idea is to consider a closed subset $A \subset \overline{\mathcal{D}}$ containing the points $((t_0, x_0+\lambda n(x_0))_{0\le\lambda\le\lambda_0}$ ($\lambda_0 > 0$ small enough) and a $C^\infty_b(\overline{\mathcal{D}})$ function w with the four following requirements: **i)** $w(t_0,x_0) = 0$ **ii)** $\partial_t w + Lw \ge 0$ on A; **iii)** $\frac{\partial w}{\partial n}(t_0,x_0) > 0$; **iv)** $u \ge u(t_0,x_0) + \varepsilon_0 w$ on ∂A for some $\varepsilon_0 > 0$.
Then for such A and w, if we set τ_A for the exit time of $(s, X^{t_0,x}_s)_{s\ge t_0}$ from A (for (t_0,x) in A), we easily deduce by **ii)** that $(Z_s := u(s\wedge\tau_A, X^{t_0,x}_{s\wedge\tau_A}) - u(t_0,x_0) - \varepsilon_0 w(s\wedge\tau_A, X^{t_0,x}_{s\wedge\tau_A}))_{s\ge 0}$ is a super-martingale, and thus using **iv)** and **i)** $0 \le \mathbb{E}[Z_{\tau_A}] \le Z_{t_0} = u(t_0,x) - u(t_0,x_0) - \varepsilon_0(w(t_0,x) - w(t_0,x_0))$. Take $(t_0,x) = (t_0, x_0+\lambda n(x_0)) \in A$ with $\lambda\downarrow 0$ to get the result considering **iii)**.
Now, we turn to the construction of A, w and ε_0. Assumption **(C')** is here crucial. Up to modifying u for $t < t_0$, we can assume that $\mathcal{D}$ is of the form $\mathcal{D} = [t_0-\delta, t_0+\delta]\times(D\cap V_{\partial D}(R))$. Under **(D)**, x_0 satisfies an interior sphere condition in D that permits to construct a time-space ball $B := B(P^*, \mathcal{R}) \subset \overline{\mathcal{D}}$ (w.l.o.g. $\mathcal{R} < \delta\wedge R/2$), $P^* = (t_0, x^*)$ s.t. $x^* - x_0 = \mathcal{R}n(x_0)$ and $B\cap$

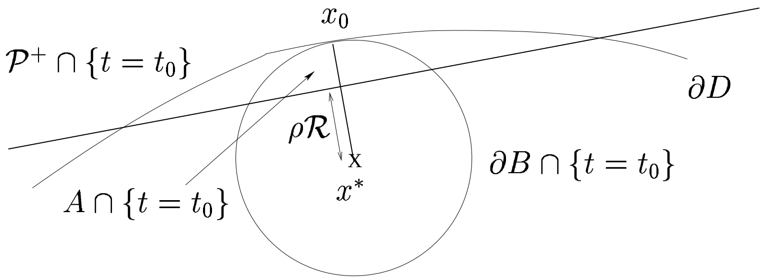

Figure 1. Space representation of A at $t = t_0$.

$\partial\mathcal{D} = \{(t_0, x_0)\}$ (see Figure 1). Now, introduce the time cylindrical half-space $\mathcal{P}^+ := [t_0 - \delta, t_0 + \delta] \times \{z \in \mathbb{R}^d : (x^* - z).n(x_0) \geq \rho\mathcal{R}\}$ for $\rho \in (0, 1)$ and denote $A := B \cap \mathcal{P}^+$ the expected set. For $\alpha > 0$, we define $w_\alpha(t, x) = \exp(-\alpha r^2) - \exp(-\alpha\mathcal{R}^2)$ where $r^2 := \|x - x^*\|^2 + (t - t_0)^2$: easily, we get $[\partial_t + L]w_\alpha(t, x) \geq \exp(-\alpha r^2)\,(2\langle\sigma\sigma^*(x)(x - x^*), x - x^*\rangle\alpha^2 - C\alpha)$ for $(t, x) \in A$. Since by continuity $\lim_{x\to x_0}\langle\sigma\sigma^*(x)(x - x^*), x - x^*\rangle \geq a_0\mathcal{R}^2$ under **(C')**, it is clear that we can choose the cutting-level ρ close enough to 1 to ensure $[\partial_t + L]w_\alpha \geq \exp(-\alpha r^2)(a_0\mathcal{R}^2\alpha^2 - C\alpha) \geq 0$ on A for α big enough: for such α, $w = w_\alpha$ satisfies **iii)**. Statements **i)** and **ii)** are straightforward to check. It remains to exhibit $\varepsilon_0 > 0$ in **iv)**: since $w = 0$ on ∂B, we may consider only $(t, x) \in \overline{\partial A \backslash \partial B}$. But on this compact set, $u > u(t_0, x_0)$ and thus, **iv)** holds true for ε_0 small enough.

4.2 Boundary estimates

We prove in this section various boundary estimates under mild assumptions, namely **(C')**, **(D)** , **(S')**. This significantly improves the previous results [Gob00] obtained in the uniformly elliptic case.

We first state a preliminary bound for the integral of the exit probability.

Lemma 17 *Under* **(C')**, **(D)**, **(S')**, *we have* $\int_0^T \mathbb{P}_x[X_t^N \notin D, \tau^N > t]dt \leq C\sqrt{h}$.

PROOF. Applying twice Lemma 3 first with $S' = t$, $S = \phi(t)$ and second with $S' = \phi(t) + h$, $S = t$, we easily get:

$$\begin{aligned}\int_0^T \mathbb{P}_x[t < \tau^N, X_t^N \notin D]dt \leq& Ch\sum_{i=0}^{N-1}\mathbb{E}_x\left[\mathbf{1}_{\tau^N > t_i}\exp\left(-c\frac{d^2(X_{t_i}^N, \partial D)}{h}\right)\right]\\ \leq& C\int_0^T \mathbb{E}_x\left[\mathbf{1}_{\tau^N > t}\exp\left(-c\frac{d^2(X_t^N, \partial D)}{h}\right)\right]dt + Ch.\end{aligned}$$

We now wish to apply the occupation times formula and use for this a localization argument. Namely, under **(C')**, it is clear that for some $r_0 > 0$ (w.l.o.g.

$r_0 \leq R/2$):

$$\forall z \in V_{\partial D}(R/2), \forall y \in B(z, r_0), \langle \sigma\sigma^*(z)\nabla F(y), \nabla F(y)\rangle \geq a_0/2. \qquad (17)$$

Thus, for $X^N_{\phi(t)} \in V_{\partial D}(R/2)$ and $X^N_t \in B(X^N_{\phi(t)}, r_0)$, we have $X^N_t \in V_{\partial D}(R)$, $d^2(X^N_t, \partial D) = F^2(X^N_t)$ and thus $d < F(X^N) >_t = \|\sigma(X^N_{\phi(t)})^* \nabla F(X^N_t)\|^2 dt \geq a_0/2dt$. The occupation times formula gives:

$$\begin{aligned}
&\int_0^T \mathbb{P}_x[t < \tau^N, X^N_t \notin D]dt \\
&\leq C\int_0^T \mathbb{E}_x\left[\mathbf{1}_{\tau^N > t, X^N_{\phi(t)} \in V_{\partial D}(R/4), X^N_t \in B(X^N_{\phi(t)}, r_0)} \exp\left(-c\frac{d^2(X^N_t, \partial D)}{h}\right)\right] dt + Ch \\
&\leq \frac{2C}{a_0}\int_{-R/4}^{R/4} dy \exp(-c\frac{y^2}{h})\mathbb{E}_x[L^y_{T\wedge\tau^N}(F(X^N))] + Ch \qquad (18)
\end{aligned}$$

where the discarded events in the second inequality are neglected using Lemma 3. Note that $\mathbb{E}_x[L^y_{T\wedge\tau^N}(F(X^N))] \leq C$ uniformly in $y \in [-R/4, R/4]$: the proof is complete.

The preliminary bound from Lemma 17 helps now to prove the upper bound stated in Lemma 12.

Lemma 18 *Under* **(C')**, **(D)**, **(S')**, *we have* $\mathbb{E}_x[L^0_{T\wedge\tau^N}(F(X^N))] \leq C\sqrt{h}$.

PROOF. As a consequence of Tanaka's formula and after taking the expectation, we have:

$$\left|\mathbb{E}_x[\frac{1}{2}L^0_{T\wedge\tau^N}(F(X^N)) - F^-(X^N_{T\wedge\tau^N})]\right| \leq C\int_0^T \mathbb{P}_x[X^N_t \notin D, \tau^N > t]dt. \qquad (19)$$

Hence using Lemma 17 it remains to control the expectation of the overshoot:

$$\begin{aligned}
\mathbb{E}_x[F^-(X^N_{T\wedge\tau^N})] &= \sum_{i=1}^N \mathbb{E}_x[F^-(X^N_{t_i})\mathbf{1}_{\tau^N = t_i}] = \sum_{i=1}^N \mathbb{E}_x[\mathbf{1}_{\tau^N > t_{i-1}} \mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})]] \\
&= \sum_{i=1}^N \mathbb{E}_x[\mathbf{1}_{\tau^N > t_{i-1}} \mathbf{1}_{X^N_{t_{i-1}} \in V_{\partial D}(R/2)} \mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})]] + O_{pol}(h). \qquad (20)
\end{aligned}$$

On the set $\{X^N_{t_{i-1}} \in V_{\partial D}(R/2)\}$ we have to upper-bound: $\mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})] = \mathbb{E}_{X^N_{t_{i-1}}}[\mathbf{1}_{\tau^N_{t_{i-1}} \leq t_i} \mathbb{E}[F^-(X^N_{t_i}) \mid \mathcal{F}_{\tau^N_{t_{i-1}}}]]$ with $\tau^N_{t_{i-1}} := \inf\{t > t_{i-1} : X^N_t \notin D\}$. Remind that F^- is Lipschitz so $\mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})] \leq C\sqrt{h}\mathbb{P}_{X^N_{t_{i-1}}}[\tau^N_{t_{i-1}} \leq t_i]$. We conclude the proof using Lemma 19 and summing over i.

Lemma 19 *Under* **(C')**, **(D)**, **(S')**, *for h small enough, we have for $x \in V_{\partial D}(R/2)$:*

$$\mathbb{P}_x[\tau_0^N \leq h] \leq C\mathbb{P}_x[X_h^N \notin D] + O_{pol}(h).$$

PROOF. We restrict to $X^N_{\tau_0^N} \in B(x, r_0)$, noting thanks to Lemma 3 that $\mathbb{P}_x[X^N_{\tau_0^N} \notin B(x, r_0)] = O_{pol}(h)$. The rest of the proof is similar to the one of Lemma 4.

The control of order 1 stated in Lemma 13 is then a direct consequence of (18) and Lemma 20 below.

Lemma 20 *Under* **(C')**, **(D)**, **(S')**, *we have for $y \in [-R/4, R/4]$*

$$\mathbb{E}_x[L^y_{T\wedge\tau^N}(F(X^N))] \leq C(|y| + h^{1/2}).$$

PROOF. Tanaka's formula gives:

$$\begin{aligned}\mathbb{E}_x[L^y_{T\wedge\tau^N}(F(X^N))] =& 2\mathbb{E}_x[(F(X^N_{T\wedge\tau^N}) - y)^- - (F(x) - y)^-] \\ &+ 2\mathbb{E}_x[\int_0^T \mathbf{1}_{F(X_t^N)\leq y}\mathbf{1}_{\tau^N>t}d(F(X_t^N))]. \qquad (21)\end{aligned}$$

Using Lemmas 18, 17 and estimates (19), we obtain that the first term of the r.h.s. above is upper bounded by $2(\mathbb{E}_x[F^-(X^N_{T\wedge\tau^N})] + |y|) \leq Ch^{1/2} + 2|y|$. For the other term it is enough to prove that $\omega(y) := \mathbb{E}_x[\int_0^T \mathbf{1}_{F(X_t^N)\leq y}\mathbf{1}_{\tau^N>t}dt] \leq C(\sqrt{h} + |y|)$. Since ω is increasing, for $y \leq 0$ one has $\omega(y) \leq \omega(0) \leq C\sqrt{h}$ by Lemma 17. For $y > 0$, it is enough to upper bound $\omega(y) - \omega(0)$ by $C(y + \sqrt{h})$: write $\omega(y) - \omega(0) = \mathbb{E}_x[\int_0^T \mathbf{1}_{0<F(X_t^N)\leq y}\mathbf{1}_{\tau^N>t}\mathbf{1}_{X^N_{\phi(t)}\in V_{\partial D}(R/2)}dt] + O_{pol}(h)$ using Lemma 3 (with $|y| \leq R/4$). The localization technique of Lemma 17 associated to the occupation times formula gives:

$$\mathbb{E}_x[\int_0^T \mathbf{1}_{0<F(X_t^N)\leq y}\mathbf{1}_{\tau^N>t}dt] \leq C\int_0^y du\,\mathbb{E}_x[L^u_{T\wedge\tau^N}(F(X^N))] + O_{pol}(h). \qquad (22)$$

The expected local time in the above integral is uniformly bounded in $u \in [0, R/4]$, and this gives $\omega(y) - \omega(0) \leq Cy + O_{pol}(h)$.

It remains to prove the lower bound from Lemma 12.

Lemma 21 *Under* **(C')**, **(D)**, **(H)**, **(S')**, *we have for h small enough (depending on $d(x, \partial D) > 0$)*

$$\mathbb{E}_x[L^0_{T/2\wedge\tau^N}(F(X^N))] \geq C\sqrt{h}$$

with $C > 0$.

PROOF. For sake of simplicity, we prove the result for the local time at time T instead of $T/2$. Set $L^N = \mathbb{E}_x[L^0_{T\wedge\tau^N}(F(X^N))]$; starting from (19), (20) and using Lemma 13, one has:

$$L^N \geq 2\sum_{i=1}^{N}\mathbb{E}_x[\mathbf{1}_{\tau^N>t_{i-1},X^N_{t_{i-1}}\in V_{\partial D}(c_0h^{1/2})}\mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})]] - Ch \qquad (23)$$

where c_0 denotes a constant to be fixed later on. If we write $F(X^N_{t_i}) = F(X^N_{t_{i-1}})+\nabla F(X^N_{t_{i-1}}).\sigma(X^N_{t_{i-1}})(W_{t_i}-W_{t_{i-1}})+R^N_i$, then $\mathbb{E}_{X^N_{t_{i-1}}}[|R^N_i|] \leq Ch$ and thus $\mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})] \geq \mathbb{E}_{X^N_{t_{i-1}}}[(\nabla F(X^N_{t_{i-1}}).\sigma(X^N_{t_{i-1}})(W_{t_i}-W_{t_{i-1}})+F(X^N_{t_{i-1}}))^-]-Ch$. A direct computation gives:

$$\mathbb{E}_{X^N_{t_{i-1}}}[F^-(X^N_{t_i})] \geq \alpha(X^N_{t_{i-1}})h^{1/2}g\left(\frac{F(X^N_{t_{i-1}})}{\alpha(X^N_{t_{i-1}})h^{1/2}}\right) - Ch \qquad (24)$$

where $g(z) := \dfrac{\exp(-\frac{z^2}{2})}{(2\pi)^{1/2}} - z\Phi(-z)$ is a positive decreasing function on $\mathbb{R}^+$. Note that for h small enough ($c_0h^{1/2} \leq R$), one has $\alpha(x) \geq a_0$ for $x \in V_{\partial D}(c_0h^{1/2})$ (Assumption **(C')**); thus, plugging (24) into (23) it comes:

$$L^N \geq 2a_0h^{1/2}\sum_{i=1}^{N}\mathbb{E}_x\left[\mathbf{1}_{\tau^N>t_{i-1}}\mathbf{1}_{F(X^N_{t_{i-1}})\in(0,c_0h^{1/2}]}g\left(\frac{F(X^N_{t_{i-1}})}{a_0h^{1/2}}\right)\right] - S(h)$$

with $S(h) = Ch\sum_{i=1}^{N}\mathbb{P}_x\left[\tau^N > t_{i-1}, F(X^N_{t_{i-1}}) \in (0, c_0h^{1/2}]\right]$. Assume for a while that $S(h) \leq Ch^{3/4}$ and consider the other contribution. Use that $\forall i \in [\![1, N]\!]$, $t \in [t_{i-1}, t_i]$, $\mathbf{1}_{t_i<\tau^N} \leq \mathbf{1}_{t<\tau^N} \leq \mathbf{1}_{t_{i-1}<\tau^N}$, and that g does not vanish on the compact sets of $\mathbb{R}^+$, to obtain:

$$\begin{aligned}
&\mathbb{E}_x\left[\mathbf{1}_{\tau^N>t_i}\mathbf{1}_{F(X^N_{t_i})\in(0,c_0h^{1/2}]}g\left(\frac{F(X^N_{t_i})}{a_0h^{1/2}}\right)\right]\\
&\geq C_1\mathbb{E}_x\left[\mathbf{1}_{\tau^N>t}\mathbf{1}_{F(X^N_t)\in[c_0h^{1/2}/4,3c_0h^{1/2}/4]}g\left(\frac{F(X^N_t)}{a_0h^{1/2}}\right)\mathbb{P}\left[F(X^N_{t_i}) \in (0, c_0h^{1/2}]|\mathcal{F}_t\right]\right].
\end{aligned}$$

where $C_1 > 0$. On $\{F(X^N_t) \in [c_0h^{1/2}/4, 3c_0h^{1/2}/4]\}$, we easily conclude by Lemma 3: $\mathbb{P}[F(X^N_{t_i}) \notin (0, c_0h^{1/2}]|\mathcal{F}_t] \leq C\exp(-cc_0^2/16)$, so that $\mathbb{P}[F(X^N_{t_i}) \in (0, c_0h^{1/2}]|\mathcal{F}_t] \geq 1/2$ for c_0 large enough. We have obtained:

$$\begin{aligned}
L^N &\geq a_0C_1h^{-1/2}\int_0^T\mathbb{E}_x[\mathbf{1}_{\tau^N>t}\mathbf{1}_{F(X^N_t)\in(c_0h^{1/2}/4,3c_0h^{1/2}/4]}g\Big(\frac{F(X^N_t)}{a_0h^{1/2}}\Big)]dt - Ch^{3/4}\\
&\geq \frac{a_0C_1h^{-1/2}}{\|\sigma^*\|^2_\infty\|\nabla F\|^2_\infty}\int_{c_0h^{1/2}/4}^{3c_0h^{1/2}/4} g\left(\frac{y}{a_0h^{1/2}}\right)\mathbb{E}_x[L^y_{T\wedge\tau^N}(F(X^N))]dy - Ch^{3/4}
\end{aligned}$$

where the occupation times formula is once again the key tool for the last inequality (we do not need here the rather tedious localization procedure of Lemmas 17 and 13, and only use σ's boundedness). Lemma 22 below and equation (21) then yield: $\mathbb{E}_x[L^y_{T\wedge\tau^N}(F(X^N))] = 2(\mathbb{E}_x[(F(X^N_{T\wedge\tau^N}) - y)^-] - (F(x) - y)^-) - Ch = 2\mathbb{E}_x[(F(X^N_{T\wedge\tau^N}) - y)^-] - Ch$ for y small enough (namely $y \leq F(x)$). If we put $C_2 = \frac{2a_0 C_1}{\|\sigma^*\|^2_\infty \|\nabla F\|^2_\infty} > 0$, it follows that

$$
\begin{aligned}
L^N \geq & C_2 h^{-1/2} \int_{c_0 h^{1/2}/4}^{3c_0 h^{1/2}/4} g\left(\frac{y}{a_0 h^{1/2}}\right) \mathbb{E}_x[(F(X^N_{T\wedge\tau^N}) - y)^-] dy - Ch^{3/4} \\
\geq & C_2 \int_{c_0/4}^{3c_0/4} g\left(\frac{z}{a_0}\right) \mathbb{E}_x[(zh^{1/2} - F(X^N_{T\wedge\tau^N}))\mathbf{1}_{zh^{1/2} > F(X^N_{T\wedge\tau^N})}] dz - Ch^{3/4} \\
\geq & C_2 h^{1/2} \Big(\int_{c_0/4}^{3c_0/4} z g(z/a_0) dz\Big) \mathbb{P}_x[\tau^N \leq T] - Ch^{3/4}
\end{aligned}
$$

noting that $(zh^{1/2} - F(X^N_{T\wedge\tau^N}))\mathbf{1}_{zh^{1/2} > F(X^N_{T\wedge\tau^N})} \geq zh^{1/2}\mathbf{1}_{0 \geq F(X^N_{T\wedge\tau^N})} = zh^{1/2}\mathbf{1}_{\tau^N \leq T}$.
To conclude the proof, note that $\mathbb{P}_x[\tau^N \leq T] \geq \mathbb{P}_x[X^N_T \notin D]$ which converges uniformly (see [BT96a]) in $x \in D$ to $\mathbb{P}_x[X_T \notin D] \geq \inf_{x \in D} \mathbb{P}_x[X_T \notin D]$: under **(H)**, this last quantity is strictly positive (see [BL91]).
It remains to estimate $S(h)$: for this, remark that $\mathbb{P}_x\left[\tau^N > t_i, F(X^N_{t_i}) \in (0, c_0 h^{1/2}]\right] \leq \mathbb{P}_x\left[\tau^N > t, F(X^N_t) \in (-c_0 h^{3/8}, c_0 h^{3/8}]\right] + O_{pol}(h)$ for $t_{i-1} \leq t < t_i$ and this provides the way to transform the sum over i in an integral over t: we conclude using Lemma 22.

Lemma 22 *Under* **(C')**, **(D)**, **(S')**, *we have for* $y \leq R/4$

$$
\int_0^T \mathbb{P}_x[F(X^N_t) \leq y, \tau^N > t] dt \leq C(h + y^2).
$$

PROOF. The contribution associated to $y \leq 0$ is already controlled by Lemma 13. For $y \in (0, R/4]$, by (22), write $\int_0^T \mathbb{P}_x[F(X^N_t) \leq y, \tau^N > t] dt \leq C\int_0^y du \mathbb{E}_x[L^u_{T\wedge\tau^N}(F(X^N))] + Ch \leq C(h + y\sqrt{h} + y^2)$ using Lemma 20.

4.3 Proof of Lemmas 14 and 15

One would prove Lemma 14 using the same techniques as in Lemma 15 which is the trickiest and the only one to be developed. The main ideas involved in the proof come from [Cat91], for the conditional Malliavin calculus, and [BT96a] for the localization techniques that allow the integrations by part in order to get rid of the derivatives of v: nevertheless, the proof of Lemma 4.3 in [BT96a] seems to be incomplete. We provide extra arguments that justify the result. For Malliavin calculus computations, we used standard notations from Nualart [Nua95].

We denote $\psi v(.,x) := \psi(x)v(.,x)$ and recall $1-\psi(x)\neq 0 \Longrightarrow x \in V_{\partial D}(\varepsilon)$. As a consequence of **(F)**, (16) and Lemma 3 the last term writes:

$$\begin{aligned}
&\mathbb{E}_x[v(T\wedge\tau^N\wedge\tau_R^N, X^N_{T\wedge\tau^N\wedge\tau_R^N}) - v((T-h)\wedge\tau^N\wedge\tau_R^N, X^N_{(T-h)\wedge\tau^N\wedge\tau_R^N})]\\
=&\mathbb{E}_x[\mathbf{1}_{\tau^N>T-h}(\psi v(T,X_T^N)-\psi v(T-h,X^N_{T-h}))]+O_{pol}(h)\\
=&-\mathbb{E}_x[\mathbf{1}_{\tau^N\le T-2h}\mathbb{E}_{\tau^N,X^N_{\tau^N}}(\psi v(T,X_T^N)-\psi v(T-h,X^N_{T-h}))]\\
&+\mathbb{E}_x[\psi v(T,X_T^N)-\psi v(T-h,X^N_{T-h})]+O_{pol}(h) := -A_1+A_2+O_{pol}(h).
\end{aligned}$$

The choice of $T-2h$ in the last equation will be justified later on. We detail the control of A_1 that is the less usual term, we would treat A_2 in the same way. For sake of simplicity, denote $\mathbb{E}_. = \mathbb{E}_{\tau^N,X^N_{\tau^N}}$. In order to use classical expansion techniques for smooth functions, we write $\mathbb{E}_.[\psi v(T,X_T^N)-\psi v(T-h,X^N_{T-h})] = A_3(m)+R_m$ with

$$A_3(m) = \mathbb{E}_.[\psi v_m(T,X_T^N)-\psi v_m(T-h,X^N_{T-h})] \tag{25}$$

where we put $v_m(t,x) := \mathbb{E}_x[f_m(X_{T-t})\mathbf{1}_{\tau>T-t}]$ for $f_m \in C_0^\infty(\mathbb{R}^d)$ and $R_m := \mathbb{E}_.[(\psi f-\psi f_m)(X_T^N)]+\mathbb{E}_.[(\psi v_m-\psi v)(T-h,X^N_{T-h})]$. By a density argument, we can choose $(f_m)_{m\ge 0}$ s.t. for all $m\ge 0$, $\|f_m\|_\infty \le 2\|f\|_\infty$, $d(\mathrm{supp}(f_m),\partial D)\ge 3/2\varepsilon$ and $f_m \underset{m\longrightarrow\infty}{\overset{L^1(\mu^N)}{\longrightarrow}} f$, where $\mu^N(dy) := \mathbb{E}_x[q_h(X^N_{T-h},y)]dy+\mathbb{P}\circ(X^N_T(x))^{-1}(dy)$. Hence, for m large enough $\mathbb{E}_x[|R_m|]\le Ch$. It is enough to prove $|A_3(m)|\le Ch$ uniformly in m, τ^N and $X^N_{\tau^N}\in V_{\partial D}(R)$.
Since ψv_m is smooth, one would like to develop $A_3(m)$ with Itô's formula and then use standard Malliavin integrations by parts; this last step can not be so direct because the variables of interest may be degenerate in the Malliavin sense. To circumvent this difficulty, we introduce a family of perturbed processes $(X_s^{N,\lambda})_{s\in[\tau^N,T]} := (X_s^N+\lambda h(\tilde W_s-\tilde W_{\tau^N}))_{s\in[\tau^N,T]}$ $(\lambda\in[0,1])$ starting from $X^N_{\tau^N}$ at time τ^N, where $\tilde W$ is a standard d-dimensional BM independent from W. We also consider the diffusion $(X_s)_{s\ge\tau_N}$ starting at τ^N from $X^N_{\tau^N}$: in the following, estimates will be uniform in $\tau^N\le T-2h$ and $X^N_{\tau^N}\in V_{\partial D}(R)$ and we omit from now on to indicate this dependence.
The next Malliavin calculus computations will be performed w.r.t. the $(d'+d)-$dimensional BM $(W,\tilde W)$ after time τ^N: $\|Z\|_{\mathbf{L}_p,.}$ and $\|Z\|_{\mathbb{D}^{j,p},.}$ stand for the associated $\mathbf{L}_p$ and Sobolev norms of Z. We denote γ_s the Malliavin covariance matrix of X_s and $\hat\gamma_s := \det(\gamma_s)$ its determinant. The same notations indexed by N(resp. N,λ) stand for X^N_s(resp. $X_s^{N,\lambda}$). We recall some classical controls (see [BT96a]); under the above assumptions, one has for any $p>1$ and $j\ge 1$

$$\|X_s^{N,\lambda}\|_{\mathbb{D}^{j,p},.}\le C,\quad \|\hat\gamma_s\|_{\mathbf{L}_p,.}\le \frac{C}{(s-\tau^N)^\zeta},\quad \|X_s-X_s^{N,\lambda}\|_{\mathbb{D}^{j,p},.}\le C\sqrt{h}, \tag{26}$$

for some constants, uniform in $\lambda\in[0,1]$, $\tau^N\le s\le T$ and $X^N_{\tau^N}\in V_{\partial D}(R)$.

At last, we state an integration by parts result derived from Propositions 4.3 and 4.4 in [BT96b] that turns out to be crucial in the rest of the proof. For $G \in \mathbb{D}^\infty$, $F \in (\mathbb{D}^\infty)^d$ satisfying the partial non degeneracy condition $\hat\gamma_F^{-1}\mathbf{1}_{G\neq 0} \in \bigcap_{p\geq 1}\mathbf{L}_p$, one has

$$|\mathbb{E}[\partial^\alpha \varphi(F)G]| \leq C\|\varphi(F)\|_{\mathbf{L}_2}\|\hat\gamma_F^{-1}\mathbf{1}_{G\neq 0}\|_{\mathbf{L}_p}^{q_3}\|F\|_{\mathbb{D}^{j_1,q_1}}^{q_4}\|G\|_{\mathbb{D}^{j_2,q_2}} \tag{27}$$

for some universal constants (depending in α) and for any smooth function φ with polynomial growth. From (25), $A_3(m)$ is equal to

$$\begin{aligned}&\mathbb{E}[\psi v_m(T,X_T^N) - \psi v_m(T,X_T^{N,1})] + \mathbb{E}[\psi v_m(T,X_T^{N,1}) - \psi v_m(T-h,X_{T-h}^{N,1})]\\ &+\mathbb{E}[\psi v_m(T-h,X_{T-h}^{N,1}) - \psi v_m(T-h,X_{T-h}^{N})] := (A_4+A_5+A_6)(m).\end{aligned}$$

For $A_4(m), A_6(m)$ we have to check that the difference between the Euler scheme and the perturbed process is negligible. For $A_5(m)$, since $X^{N,1}$ satisfies the non degeneracy condition we can use Itô's formula associated with integrations by parts techniques.

Control of $A_4(m), A_6(m)$. We only detail $A_4(m)$, the other term can be handled in the same way thanks to the restriction to $\tau^N \leq T-2h$. Let η_T be a $\mathbb{D}^\infty$ $[0,1]$-valued random variable, satisfying **(C1)**: $\mathbb{P}[\eta_T \neq 1] \leq C\frac{h^2}{(T-\tau^N)^\zeta}$ and **(C2)**: $\eta_T \neq 0 \Rightarrow \forall\lambda\in[0,1],\ \hat\gamma_T^{N,\lambda} \geq \hat\gamma_T/4$. It follows from **(C2)** and (26) that $\|(\hat\gamma_T^{N,\lambda})^{-1}\mathbf{1}_{\eta_T\neq 0}\|_{\mathbf{L}_{p},.} \leq \frac{C}{(T-\tau^N)^\zeta}$ for $\lambda\in[0,1]$. A Taylor expansion yields:

$$\begin{aligned}A_4(m) :=&\mathbb{E}[(\psi v_m(T,X_T^N) - \psi v_m(T,X_T^{N,1}))(1-\eta_T)]\\ &-h\int_0^1 \mathbb{E}[\nabla\psi v_m(T,X_T^{N,\lambda}).(\tilde W_T - \tilde W_{\tau^N})\eta_T]d\lambda := (A_{41}+A_{42})(m).\end{aligned}$$

From the support property of f_m, Lemma 3 and **(C1)** we easily deduce $|A_{41}(m)| \leq C\exp(-c\frac{\varepsilon^2}{T-\tau^N})\mathbb{E}[1-\eta_T]^{1/2} \leq C\frac{h}{1\wedge\varepsilon^\zeta}$. Taking additionally into account (27) and **(C2)** yields $|A_{42}(m)| \leq C\frac{h}{1\wedge\varepsilon^\zeta}$.
We now turn to the construction of η_T. To satisfy **(C2)** we will choose η_T as a mollified indicator function of the sets where $\gamma_T^{N,\lambda}$ is close enough to γ_T uniformly in $\lambda\in[0,1]$. Remark that $A(\lambda) := \gamma_T^{N,\lambda} = \gamma_T^N + \lambda^2h^2(T-\tau^N)I_d$ is $a.s.$ invertible for $\lambda > 0$. The function $a(\lambda) := \det(A(\lambda))$ is differentiable in λ and its derivative is given by (see Theorem A.98 from [RT99]) $a'(\lambda) = \mathrm{Tr}(\mathrm{Cof}(A(\lambda))A'(\lambda)) = 2\lambda h^2(T-\tau^N)\mathrm{Tr}(\mathrm{Cof}(\gamma_T^{N,\lambda}))$. Simple computations yield $|a'(\lambda)|^2 \leq Ch^4\Big(\int_{\tau^N}^T\{\|D_tX_T^N\|^2+h^2\}dt\Big)^2 := R_N$ so that $|\hat\gamma_T^{N,\lambda} - \hat\gamma_T^N|^2 \leq R_N$ for $\lambda\in[0,1]$. Introduce now an even function $\eta\in C_b^\infty(\mathbb{R})$ s.t. $\mathbf{1}_{[0,1/4]}(x) \leq \eta(x) \leq \mathbf{1}_{[0,1/2]}(x)$ for $x\geq 0$, and put $\eta_T^1 := \eta((\hat\gamma_T - \hat\gamma_T^N)/\hat\gamma_T)$, $\eta_T^2 := \eta(8R_N/[\hat\gamma_T]^2)$: we set $\eta_T = \eta_T^1\eta_T^2$. Indeed, **(C2)** is fulfilled: $\eta_T^1\eta_T^2 \neq 0 \Rightarrow \hat\gamma_T^N \geq \hat\gamma_T/2,\ R_N \leq [\hat\gamma_T]^2/16$ and thus $\hat\gamma_T^{N,\lambda} \geq \hat\gamma_T/4$ for $\lambda\in[0,1]$. To check **(C1)**, write: $\mathbb{E}[1-\eta_T^1\eta_T^2] \leq$

$\mathbb{P}.[\eta_T^1 \neq 1] + \mathbb{P}.[\eta_T^2 \neq 1]$. Using Markov's inequality and (26), one readily gets

$$\mathbb{P}.[\eta_T^1 \neq 1] \leq \sqrt{\mathbb{E}.[4^q |\hat{\gamma}_T^N - \hat{\gamma}_T|^q]}\sqrt{\mathbb{E}.[\hat{\gamma}_T^{-q}]} \leq C_q \frac{h^{q/4}}{(T-\tau^N)^{\zeta_q}} \tag{28}$$

for any q. An analogous estimate is available for $\mathbb{P}.[\eta_T^2 \neq 1]$.

Control of $A_5(m)$. Using Itô's formula $A_5(m)$ writes as a finite sum of terms $\int_{T-h}^T \mathbb{E}.[\partial_x^\alpha[(\partial_x^{\alpha'}\psi)v_m](s, X_s^{N,1}) g_{\alpha,\alpha'}(X_s^{N,1}, X_{T-h}^{N,1})]ds$ where $|\alpha| \leq 2, |\alpha| + |\alpha'| \leq 2$ and $g_{\alpha,\alpha'}$ is a bounded function that only depends on b, σ in (14). Combining (27) with estimates (16) (written for f_m) and Lemma 3 give $|A_5(m)| \leq C\frac{\|f\|_\infty}{1 \wedge \varepsilon^2}\int_{T-h}^T \exp\Big(-c\frac{\varepsilon^2}{s-\tau^N}\Big)\|(\hat{\gamma}_s^{N,1})^{-1}\|_{\mathbf{L}_{p,\cdot}}^q ds$. To complete the proof, we assert that $\|(\hat{\gamma}_s^{N,1})^{-1}\|_{\mathbf{L}_{p,\cdot}} \leq \|(\hat{\gamma}_s^{N,1})^{-1}\mathbf{1}_{\hat{\gamma}_s^{N,1} \geq \hat{\gamma}_s/2}\|_{\mathbf{L}_{p,\cdot}} + \|(\hat{\gamma}_s^{N,1})^{-1}\mathbf{1}_{\hat{\gamma}_s^{N,1} < \hat{\gamma}_s/2}\|_{\mathbf{L}_{p,\cdot}} \leq \frac{C}{(s-\tau^N)^\zeta}$. Indeed, the first term readily satisfies the required upper bound if we apply (26). For the second, note that since $\hat{\gamma}_s^{N,1} \geq ((s-\tau^N)h^2)^d$, it is enough to get that $\mathbb{P}.[\hat{\gamma}_s^{N,1} < \hat{\gamma}_s/2] \leq C_p\frac{h^p}{(s-\tau^N)^{\zeta_p}}$ for p large enough. This last estimate can be proved as (28). We omit further details.

5 Expansion result: an example

The aim of this section is to present an expansion result for d-dimensional processes of the form $X_s := x + \mu s + \sigma W_s$ (W is a standard d-dimensional BM) when the domain $D = \{x \in \mathbb{R}^d : a.x < b\}$ ($a \neq 0$) is a half space; we assume $\sigma\sigma^*$ to be positive definite.
We first state that the error is related to the one with $\sigma = \mathrm{I}_d$ with a new boundary, orthogonal to the first axis. Indeed, using a clear change of probability measure and a rotation of coordinates using an orthogonal matrix U (with a first row equal to the transpose of $\frac{\sigma^* a}{\|\sigma^* a\|}$) preserving the Wiener measure, one easily obtains

$$\mathrm{Err}_1(T,h,f,x) = \mathrm{Err}_2(T,h,f,x) = \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau_{D_0}^N > T} - f_0(W_T)\mathbf{1}_{\tau_{D_0} > T}] \tag{29}$$

with $f_0(y) = \exp(U\sigma^{-1}\mu.y - \frac{1}{2}\|\sigma^{-1}\mu\|^2 T)f(x + \sigma U^* y)$, $D_0 = \{y \in \mathbb{R}^d : y_1 < b_0\}$, $b_0 = \frac{b-a.x}{\|\sigma^* a\|} > 0$ (since $x \in D$), $\tau_{D_0} = \inf\{t \geq 0 : W_t \notin D_0\}$ and $\tau_{D_0}^N = \inf\{t_i \geq 0 : W_{t_i} \notin D_0\}$. This transformation illustrates that the problem is essentially one-dimensional.

To keep going with this connection, we introduce some notations related to one dimensional random walk techniques (see Siegmund [Sie79]), which will be used in the sequel. Let us define $s_0 := 0, \forall n \geq 1, s_n := \sum_{i=1}^{n} G^i$, where the G^i are i.i.d. standard centered normal variables. We introduce the stopping times $\bar{\tau}_a := \inf\{n > 0 : s_n > a\}$ for $a \geq 0$, $\tau^+ := \bar{\tau}_0$ and define $H(x) := (\mathbb{E}_0[s_{\tau^+}])^{-1} \int_0^x dy \mathbb{P}_0[s_{\tau^+} > y]$.

Lemma 23 (Asymptotic independence of the overshoot and the discrete exit time - Equivalence of the expectation of the local time.) *Let W be a standard linear BM. Put $x > 0$ and consider the domain $D :=]-\infty, x[$. We have for any $y \geq 0$*

$$\lim_{h \longrightarrow 0} \mathbb{P}_0[\tau^N \leq t, (W_{\tau^N} - x) \leq y\sqrt{h}] = \mathbb{P}_0[\tau \leq t] H(y) \tag{30}$$

$$\frac{1}{2}\mathbb{E}_0[L^x_{t\wedge\tau^N}(W)] = \sqrt{h}\frac{\mathbb{E}_0[s^2_{\tau^+}]}{2\mathbb{E}_0[s_{\tau^+}]}\mathbb{P}_0[\tau \leq t] + o(\sqrt{h}). \tag{31}$$

Both limits are uniform in $t \in [0, T]$.

One knows from [Sie79] that $\dfrac{\mathbb{E}_0[s^2_{\tau^+}]}{2\mathbb{E}_0[s_{\tau^+}]} = 0.5823\ldots$

PROOF. Equality (30) is a direct consequence of Lemma 3 in [Sie79] for a fixed t. We derive the uniformity on $[0, T]$ using Dini-like arguments noting that the l.h.s. of (30) defines a sequence of (discontinuous) increasing functions and that the simple limit is continuous (see e.g. problem 7.2.3 in [Die71]).
To prove (31), use Tanaka's formula and Lemma 13 to get

$$\frac{1}{2}\mathbb{E}_0[L^x_{t\wedge\tau^N}(W)] = \sqrt{h}\mathbb{E}_0[h^{-1/2}(W_{\tau^N} - x)\mathbf{1}_{\tau^N \leq t}] + \mathbb{E}_0[(W_t - x)^+ \mathbf{1}_{\tau^N > t}] + O(h)$$

uniformly in $t \in [0, T]$. The second term in the r.h.s. above can be easily estimated as $\mathbb{E}_0[(W_t - x)^+ \mathbf{1}_{\tau^N > t}] \leq C\sqrt{h}\mathbb{E}_0\Big(\exp(-c\frac{(W_{\phi(t)} - x)^2}{h})\Big) \leq Ch$ where we finally used a uniform upper bound (w.r.t. $\phi(t)$) for the density of $W_{\phi(t)}$ around $x \neq 0$. To deal with the first term, put $\Psi_N(y, t) = \mathbb{P}_0[h^{-1/2}(W_{\tau^N} - x) \geq y, \tau^N \leq t]$: it converges owing to (30) to $\Psi(y, t) := \mathbb{P}_0[\tau \leq t](1 - H(y))$ uniformly on $[0, T]$. Proposition (10) guarantees that the sequence $(\Psi_N(., t))_N$ is uniformly integrable, uniformly in $t \in [0, T]$. Thus, the dominated convergence theorem gives $\int_{\mathbb{R}^+} \Psi_N(y, t) dy \underset{N}{\longrightarrow} \int_{\mathbb{R}^+} \Psi(y, t) dy = \dfrac{\mathbb{E}_0[s^2_{\tau^+}]}{2\mathbb{E}_0[s_{\tau^+}]}\mathbb{P}_0[\tau \leq t]$ for each t, and using again Dini-like arguments, uniformly on $[0, T]$.

We now state the expansion result, using the previous notation with $v_0, D_0 \cdots$: the constant C_1 below could also be expressed with quantities related to the original problem, but the current formulation will be more useful in the following.

Theorem 24 (Error expansion for the Brownian motion in a half-space). *Let $X_s = x+\mu s+\sigma W_s$ for $s \geq 0$ where W is a standard d-dimensional BM, $\sigma\sigma^*$ is positive and $\mu \in \mathbb{R}^d$. For $D := \{x \in \mathbb{R}^d : x.a < b\}, b \in \mathbb{R},\ a \neq 0$, assume* **(F)***, the error writes*

$$\mathrm{Err}_1(T,h,f,x) = C_1\sqrt{h} + o(\sqrt{h})$$

with $C_1 = \dfrac{\mathbb{E}_0\left[s_{\tau^+}^2\right]}{2\mathbb{E}_0\left[s_{\tau^+}\right]}\mathbb{E}_0\left[\mathbf{1}_{\tau_{D_0}\leq T}(-\partial_{y_1}v_0(\tau_{D_0}, W_{\tau_{D_0}}))\right]$.

PROOF. Thanks to the equality (29), we are reduced to the analysis of the error with W in D_0: the careful reader could object that ∂D_0 is not compact and f_0 is not bounded, but however, Theorem 8 is still valid in this case (see extensions in Section 6) and it gives $\mathrm{Err}_1(T,h,f,x) = \frac{1}{2}\mathbb{E}_0\left(\int_0^T Y_s dL^{b_0}_{s\wedge\tau^N_{D_0}}(W^1_.)\right)+ O(h)$, with $Y_s = -\partial_{y_1}v_0(s,(b_0, W_s^2, \cdots, W_s^d))$. Note that $dY_s = y_s ds$+Brownian martingale and that Y (and $(y_s)_s$) is independent of W^1 (and hence of τ_{D_0} and $\tau^N_{D_0}$). Exploiting these independence properties and using twice the integration by parts combined with equality (31), one obtains

$$\begin{aligned}
\mathbb{E}_0\left(\int_0^T Y_s dL^{b_0}_{s\wedge\tau^N_{D_0}}(W^1)\right) &= \mathbb{E}_0\left(Y_T L^{b_0}_{T\wedge\tau^N_{D_0}}(W^1)\right) - \mathbb{E}_0\left(\int_0^T L^{b_0}_{s\wedge\tau^N_{D_0}}(W^1) y_s ds\right)\\
&= \sqrt{h}\frac{\mathbb{E}_0\left[s_{\tau^+}^2\right]}{\mathbb{E}_0\left[s_{\tau^+}\right]}\left(\mathbb{P}_0[\tau_{D_0}\leq T]\mathbb{E}_0(Y_T) - \int_0^T \mathbb{P}_0[\tau_{D_0}\leq s]\mathbb{E}_0(y_s)ds\right) + o(\sqrt{h})\\
&= \sqrt{h}\frac{\mathbb{E}_0\left[s_{\tau^+}^2\right]}{\mathbb{E}_0\left[s_{\tau^+}\right]}\mathbb{E}_0\left(Y_{\tau_{D_0}}\mathbf{1}_{\tau_{D_0}\leq T}\right) + o(\sqrt{h})
\end{aligned}$$

and the result follows.

5.3 The shifting boundary correction

We present a multidimensional extension of the Broadie-Glasserman-Kou correction [BGK99] which aims at improving the accuracy of the numerical procedure by removing the term of order $\frac{1}{2}$ in the error.
For this, the simulation of $(X_{t_i})_{0\leq i\leq N}$ is performed in a modified domain,

namely $D^h = \{x \in \mathbb{R}^d : x.a < b - \frac{\mathbb{E}_0[s^2_{\tau+}]}{2\mathbb{E}_0[s_{\tau+}]}\|\sigma^* a\|\sqrt{h}\}$ and we denote $\tau^N_{D^h}$ (resp. τ_{D^h}) the discrete (resp. continuous) exit time from this domain D^h. The following result states that the rate of convergence is now greater than $\frac{1}{2}$.

Theorem 25 *With the notations and assumptions of Theorem 24, we have*

$$\mathrm{Err}_3(T,h,f,x) := \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau^N_{D^h}>T}] - \mathbb{E}_x[f(X_T)\mathbf{1}_{\tau>T}] = o(\sqrt{h})$$

PROOF. We use again the transformation from the beginning of this section to get $\mathrm{Err}_3(T,h,f,x) = \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau^N_{D_0^h}>T}] - \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau_{D_0}>T}] = \mathrm{Err}_{31}(T,h,f,x) + \mathrm{Err}_{32}(T,h,f,x)$ where $D_0^h = \{y \in \mathbb{R}^d : y_1 < b_0 - \frac{\mathbb{E}_0[s^2_{\tau+}]}{2\mathbb{E}_0[s_{\tau+}]}\sqrt{h}\}$, $\mathrm{Err}_{31}(T,h,f,x) = \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau^N_{D_0^h}>T}] - \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau_{D_0^h}>T}]$ and $\mathrm{Err}_{32}(T,h,f,x) = \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau_{D_0^h}>T}] - \mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau_{D_0}>T}]$.
The first contribution can be analyzed applying Theorem 24, except that the domain D_0^h depends on h: however, an easy (but long) verification shows that the estimates are locally uniform in b_0, we omit the details. Hence, $\mathrm{Err}_{31}(T,h,f,x) = \frac{\mathbb{E}_0[s^2_{\tau+}]}{2\mathbb{E}_0[s_{\tau+}]}\sqrt{h}\mathbb{E}_0[\mathbf{1}_{\tau_{D_0}\leq T}(-\partial_{y_1}v_0(\tau_{D_0}, W_{\tau_{D_0}}))] + o(\sqrt{h})$.
To conclude the proof with the estimation of $\mathrm{Err}_{32}(T,h,f,x)$, note that it is enough to get $\partial_{b_0}v_0(0,0) = \partial_{b_0}\mathbb{E}_0[f_0(W_T)\mathbf{1}_{\tau_{D_0}>T}] = \mathbb{E}_0[\mathbf{1}_{\tau_{D_0}\leq T}(-\partial_{y_1}v_0(\tau_{D_0}, W_{\tau_{D_0}}))]$. To justify this equality, we exploit the explicit form of the killed transition densities for the linear BM (see [KS91] p.97-98). To simplify, put $W'_t = (W^2_t, \cdots, W^d_t)$, define $g_t(z) = \frac{e^{-z^2/(2t)}}{\sqrt{2\pi t}}$ and $a(b_0,t) = \frac{b_0}{t}g_t(b_0) = -g'_t(b_0)$ the density at time t of τ_{D_0}. Clearly, by independence of W^1 and W', one has $v_0(t,y) = \int_{-\infty}^{b_0}(g_{T-t}(z-y_1) - g_{T-t}(z+y_1-2b_0))\mathbb{E}(f_0(z,W'_T)|W'^2_t = y_2, \cdots, W'^d_t = y_d)dz$, from which it is easy to derive

$$\begin{aligned}
-\partial_{y_1}v_0(t,(b_0,W'_t)) &= \int_{-\infty}^{b_0} 2a(b_0-z,T-t)\mathbb{E}(f_0(z,W'_T)|W'_t)dz,\\
\partial_{b_0}v_0(0,0) &= 2\int_{-\infty}^{b_0} a(2b_0-z,T)\mathbb{E}_0(f_0(z,W'_T))dz,\\
\mathbb{E}_0[\mathbf{1}_{\tau_{D_0}\leq T}(-\partial_{y_1}v_0(\tau_{D_0},W_{\tau_{D_0}}))] &= \int_0^T a(b_0,t)\mathbb{E}_0(-\partial_{y_1}v_0(t,(b_0,W'_t)))dt,\\
&= 2\int_{-\infty}^{b_0}\mathbb{E}_0(f_0(z,W'_T))\Big(\int_0^T a(b_0,t)a(b_0-z,T-t)dt\Big)dz.
\end{aligned}$$

The convolution integral w.r.t. t simply reduces to $a(2b_0-z,T)$ (Markov property on the hitting times, see [KS91] p.197) and this proves our assertion.

We now provide a numerical example taken from financial applications. Consider a two-dimensional risky asset following the Black-Scholes-Merton dynamic, $S_t^1 = S_0^1 \exp(\sigma_1 W_t^1 + (r - \frac{\sigma_1^2}{2})t)$, $S_t^2 = S_0^2 \exp(\sigma_2 \rho W_t^1 + \sigma_2 \sqrt{1-\rho^2} W_t^2 + (r - \frac{\sigma_2^2}{2})t)$, where $W = (W^1, W^2)$ is a standard two dimensional BM. For a fixed final time T, given level B and strike K, put $D := \{s \in \mathbb{R}^2 : s_1 \geq B\}$, we are interested in computing $\mathbb{E}[e^{-rT}\mathbf{1}_{\tau > T}\mathbf{1}_{(S_T^1 \wedge S_T^2) \geq K}]$ related to the price of a digital barrier option. Let us remark that assumption **(F)** is satisfied as soon as $K > B$. For $r = .04, \sigma_1 = \sigma_2 = .3, \rho = .5, S_0^1 = S_0^2 = K = 100, B = 90, T = 1$ we compute the standard Monte-Carlo approximation, the Romberg extrapolation (see [TL90]) and the previously described correction with 10^6 paths: the width of the 95%-confidence interval is essentially equal to 1.5 10^{-3}. The reference value has been computed with the usual Brownian bridge techniques (see references in the introduction) for 10^8 paths.
Note (see Figure 2) the positive bias for the standard procedure as proved before. What appears is that the shifting boundary correction is more accurate than the Romberg extrapolation: it is promising since the computational time is also lower. It is not hopeless to extend this simple correction to less specific domains and this will concern further investigations.

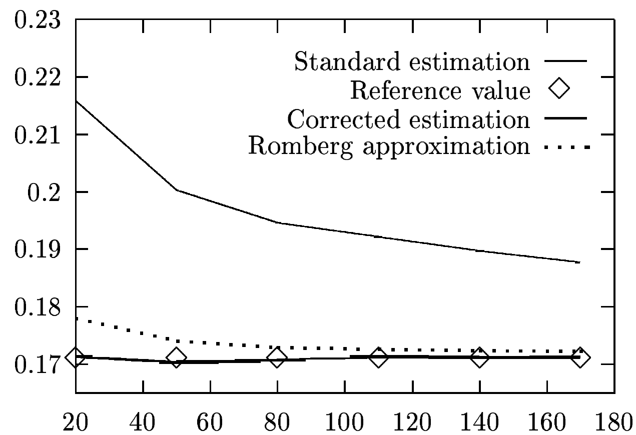

Figure 2. Convergence results w.r.t. the number of steps N.

6 Conclusion

We conclude giving some easy extensions of our previous results. For all our results, the compactness assumption on ∂D may be removed in the half space case, where the boundedness assumption on f can also be relaxed into $f(x) \leq C \exp(c|x|)$ since the coefficients in (14) are bounded. Concerning the smoothness property of D, results from section 2 concerning general Itô processes hold true if D is of class C^2, as well as the boundary results of section 4. In the uniformly elliptic case, we can under smoothness assumptions on b, σ, D, see Theorem 5.2 in [LSU68], weaken the support condition on f provided that it is smooth. One possible framework may be that $f \in H^l(D), l > 3$ (see [LSU68] pp 7,8 for the definitions of those functional spaces), f satisfies the compatibility conditions $f|_{\partial D} = Lf|_{\partial D} = 0$, D is of class H^l and $b, \sigma \in H^{l-2}(D)$. Unfortunately, it seems difficult to get rid of the support assumption in **(F)** in the general case, because we are not able to deal with exploding derivatives of v near ∂D.

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