ECOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11
http://www.cmap.polytechnique.fr/

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Discretization and simulation for a class of SPDEs with applications to Zakai and McKean-Vlasov equations

Emmanuel GOBET* Gilles PAGÈS† Huyên PHAM‡
Jacques PRINTEMS§

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Abstract

This paper is concerned with numerical approximations for a class of nonlinear stochastic partial differential equations: Zakai equation of nonlinear filtering problem and McKean-Vlasov type equations. The approximation scheme is based on the representation of the solutions as weighted conditional distributions. We first accurately analyse the error caused by an Euler type scheme of time discretization. Sharp error bounds are calculated: we show that the rate of convergence is in general of order $\sqrt{\delta}$ (δ is the time step), but in the case when there is no correlation between the signal and the observation for the Zakai equation, the order of convergence becomes δ. This result is obtained by carefully employing techniques of Malliavin calculus. In a second step, we propose a simulation of the time discretization Euler scheme by a quantization approach. This formally consists in an approximation of the weighted conditional distribution by a conditional discrete distribution on finite supports. We provide error bounds and rate of convergence in terms of the number N of the grids of this support. These errors are minimal at some optimal grids which are computed by a recursive method based on Monte Carlo simulations. Finally, we illustrate our results with some numerical experiments arising from correlated Kalman-Bucy filter and Burgers equation.

Key words: Stochastic partial differential equations, nonlinear filtering, McKean-Vlasov equations, Euler scheme, quantization, Malliavin calculus.

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*CMAP, Ecole Polytechnique, emmanuel.gobet@polytechnique.fr
†LPMA, Université Paris 6-Paris 7, gpa@ccr.jussieu.fr
‡LPMA, Université Paris 6-Paris 7, pham@math.jussieu.fr
§LAMA, Université Paris 12, printems@univ-paris12.fr

1 Introduction

We are interested in numerical approximation for the measure-valued process V governed by the following nonlinear stochastic partial differential equations (SPDE) written in weak form: for all test functions $f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} < V_t, f > &= < \mu_0, f > + \int_0^t < V_s, L(V_s) f > ds \\ &\quad + \int_0^t < V_s, h(., V_s) f + \gamma^\intercal(., V_s) \nabla f > .dW_s, \end{aligned} \tag{1.1}$$

where μ_0 is an initial probability measure. Here, for any $V \in \mathcal{M}(\mathbb{R}^d)$, set of finite signed measures on $\mathbb{R}^d$, $L(V)$ is the second-order differential operator:

$$L(V) f(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x, V) \partial^2_{x_i x_j} f(x) + \sum_{i=1}^d b_i(x, V) \partial_{x_i} f(x),$$

W is a q-dimensional Brownian motion, $a = (a_{ij})$ is a $d \times d$ matrix-valued, $\gamma = (\gamma_{il})$ is a $d \times q$ matrix-valued, $b = (b_i)$ is a $\mathbb{R}^d$-vector valued, and $h = (h_l)$ is a $\mathbb{R}^q$-vector valued function defined on $\mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d)$, in the form:

$$\begin{aligned} a(x, V) &= \sigma(x, V) \sigma^\intercal(x, V) + \gamma(x, V) \gamma^\intercal(x, V), \\ b(x, V) &= \beta(x, V) + \gamma(x, V) h(x, V), \end{aligned}$$

for some $d \times d$ matrix-valued function $\sigma = (\sigma_{ij})$ and $\mathbb{R}^d$-vector valued function $\beta = (\beta_i)$ on $\mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d)$. The transpose and the scalar product are respectively denoted by $^\intercal$ and a dot. The Euclidean norm of a vector is denoted $|.|$ and one uses the norm $|\sigma| = \sqrt{\mathrm{Tr}(\sigma\sigma^\intercal)}$ for a matrix σ.

When the distribution V_t admits a density $v(t, x)$, one may usually rewrite (1.1) in the form:

$$\begin{aligned} dv(t, x) &= \left(\frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} [a_{ij}(x, v(t, .)) v(t, x)] - \sum_{i=1}^d \partial_{x_i} [b_i(x, v(t, .)) v(t, x)] \right) dt \\ &\quad + \left(h^\intercal(x, v(t, .)) v(t, x) - \nabla [\gamma(x, v(t, .)) v(t, x)] \right) dW_t. \end{aligned} \tag{1.2}$$

Under appropriate conditions, it is proved in [19], that the solution V to (1.1) can be characterized through the following system of diffusions:

$$X_t = X_0 + \int_0^t \beta(X_s, V_s) ds + \int_0^t \sigma(X_s, V_s) dB_s + \int_0^t \gamma(X_s, V_s) dW_s, \tag{1.3}$$

$$X_0 \rightsquigarrow \mu_0,$$

$$\xi_t = \exp(Z_t) = \exp\left(\int_0^t h(X_s, V_s).dW_s - \frac{1}{2} \int_0^t |h(X_s, V_s)|^2 ds \right), \tag{1.4}$$

$$< V_t, f > = E_{W} [f(X_t) \xi_t], \tag{1.5}$$

where B is a $\mathbb{R}^d$-Brownian motion independent of W, and E_{W} denotes the conditional expectation given W. We also denote P_{W} the corresponding conditional probability.

In this paper, we shall focus on the two following main applications of SPDE (1.1):

1.1 Case A: Zakai equation of nonlinear filtering with correlated noise

This corresponds to equation (1.1) where all coefficients σ, β, h and γ are independent of V. More precisely, let X be the d-dimensional signal given by

$$dX_t \;=\; \beta(X_t)dt + \sigma(X_t)dB_t + \gamma(X_t)dW_t, \quad X_0 \rightsquigarrow \mu_0$$

and W the q-dimensional observation process given by:

$$W_t \;=\; \int_0^t h(X_s)ds + U_t,$$

on a probability space $(\Omega, \mathcal{F}, P)$ equipped with filtration $(\mathcal{F}_t)$ under which B and U are independent Brownian motions. The nonlinear filtering problem consists in estimating the conditional distribution of X given W, i.e. we want to compute the measure-valued process π_t characterized by:

$$< \pi_t, f > \;=\; E[f(X_t)|\mathcal{F}_t^W],$$

where $\mathcal{F}_t^W$ is the filtration generated by the whole observation of W until t. Under suitable conditions, there exists a reference probability measure Q, such that:

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} \;=\; \xi_t \;=\; \exp\left(\int_0^t h(X_s).dW_s - \frac{1}{2}\int_0^t |h(X_s)|^2 ds\right),$$

and (B, W) are two independent Brownian motions under Q. By the Kallianpur-Striebel formula, we have

$$< \pi_t, f > \;=\; \frac{< V_t, f >}{< V_t, 1 >},$$

where

$$< V_t, f > \;=\; E_W^Q[f(X_t)\xi_t].$$

Moreover, the measure-valued process V solves the so-called Zakai equation

$$< V_t, f > \;=\; < \mu_0, f > + \int_0^t < V_s, Lf > ds + \int_0^t < V_s, hf + \gamma^\intercal \nabla f > .dW_s. \quad (1.6)$$

1.2 Case B: stochastic McKean-Vlasov equation

This corresponds to equation (1.1) with $h = 0$ so that ξ in (1.4) is constant equal to one. All other coefficients depend on V through Lipschitz kernels $\tilde{\sigma}(x, y)$, $\tilde{\beta}(x, y)$, $\tilde{\gamma}(x, y)$:

$$\begin{aligned}
\beta(x, V) &= \int \tilde{\beta}(x, y)V(dy),\\
\sigma(x, V) &= \int \tilde{\sigma}(x, y)V(dy),\\
\gamma(x, V) &= \int \tilde{\gamma}(x, y)V(dy).
\end{aligned}$$

When there is no W (or when $\gamma = 0$), the measure-valued process V is deterministic and is solution of the classical McKean-Vlasov equation:

$$< V_t, f > \;=\; < \mu_0, f > + \int_0^t < V_s, L(V_s)f > ds. \tag{1.7}$$

V_t is characterized as the distribution of the solution X_t to:

$$dX_t \;=\; \beta(X_t, V_t)dt + \sigma(X_t, V_t)dB_t, \quad X_0 \rightsquigarrow \mu_0.$$

The general stochastic McKean-Vlasov equation is:

$$< V_t, f > \;=\; < \mu_0, f > + \int_0^t < V_s, L(V_s)f > ds + \int_0^t < V_s, \gamma^{\intercal}(., V_s)\nabla f > .dW_s, \tag{1.8}$$

and V_t is characterized as the conditional distribution given W of the solution X_t to:

$$dX_t \;=\; \beta(X_t, V_t)dt + \sigma(X_t, V_t)dB_t + \gamma(X_t, V_t)dW_t, \quad X_0 \rightsquigarrow \mu_0.$$

1.3 A short discussion of related literature

Numerical approximations of SPDEs have been extensively studied in the literature. We cite the survey paper [16] and the references therein. Roughly summarizing, one may classify between the following approaches:

- Approximations based on the analytic expression (1.2) vary from finite difference of finite elements methods, splitting up methods or Galerkin's approximation. We cite for instance for the finite difference method the papers of [29] for Zakai equation and [1] for the stochastic Burger equation. For the splitting up method of Zakai equation, see [6], [13].

- Another point of view, studied in [21] and [8], is based on a Wiener chaos decomposition of the solution to the Zakai equation. We mention also Wong-Zakai type approximations considered in [17].

- The third approach is based on the probabilistic representation (1.5) of the solution as a weighted (or unnormalized) conditional distribution. For the Zakai equation of nonlinear filtering problem, papers [20] and [12] develop approximation methods by replacing the signal process by a finite state Markov chain on an uniform grid prescribed *a priori*. This method is somewhat equivalent to the finite difference method. Another popular method is based on particle approximation of the conditional distribution, see for instance [10], [9], for the nonlinear filtering problem and [7] for the McKean-Vlasov equation.

1.4 Contribution and organization of the paper

The first contribution of our work consists in accurately estimating the error due to time discretizations on the conditional expectation (1.5). Without conditioning, classical results yield an error at most linear w.r.t. the time step δ (see for instance [3], [2]). Here, the situation is unusual because of the conditional expectation and our analysis makes clear the role of the correlation factor between the underlying process X and the observation process W. Regarding the proof, we use Malliavin calculus computations, but to leave W unchanged, extra technicalities are needed.

In a second part, we propose a simulation algorithm for the SPDE (1.1) based on an optimal quantization approach. Basically, this means a spatial discretization of the dynamics of the Euler time-discretization (X_k, V_k) of (1.3)-(1.5) optimally fitted to its probabilistic features. To be more specific, we first recall some short background on optimal quantization of a random vector. Let $X : (\Omega, \mathcal{F}, P) \to \mathbb{R}^d$ be a random vector and let $\Gamma = \{x^1, \dots, x^N\}$ be a subset (or *grid*) of $\mathbb{R}^d$ having N elements. We approximate X by one of its Borel closest neighbour projection $\widehat{X}^\Gamma := \mathrm{Proj}_\Gamma(X)$ on Γ. Such a projection is canonically associated to a Voronoi tessellation $(C_i(\Gamma))_{1\le i\le N}$ that is a Borel partition of $\mathbb{R}^d$ satisfying for any $i = 1, \dots, N$:

$$C_i(\Gamma) \quad\subset\quad \left\{\xi \in \mathbb{R}^d : |\xi - x^i| = \min_j |\xi - x^j|\right\}.$$

Hence

$$\widehat{X}^\Gamma \quad=\quad \mathrm{Proj}_\Gamma(X) \;:=\; \sum_{i=1}^N x^i \mathbf{1}_{\{X \in C_i(\Gamma)\}}.$$

As soon as $X \in L^p(\Omega, P, \mathbb{R}^d)$ the induced L^p*-quantization error* is given by

$$\|X - \widehat{X}^\Gamma\|_{_p} = \left(E \min_{1\le i\le N} |X - x^i|^p\right)^{\frac{1}{p}} < \infty.$$

The L^p-optimal N-quantization problem for X consists in finding a grid Γ^* which achieves the lowest L^p-quantization error among all grids of size at most N. Such an optimal grid does exist (see [15]), its size is exactly N if the support of X is infinite; it is generally not unique (except in 1-dimension where uniqueness holds when the distribution $P_{_X}$ of X has a log-concave density). The rate of convergence of the lowest L^p-quantization error as $N \to +\infty$ is ruled by the so-called Zador theorem (see [15]). For historical reasons, this theorem is usually stated with the p^{th} power of the L^p-quantization error, known as the L^p*-distortion.*

Theorem 1.1 *Assume that $X \in L^{p+\eta}(\Omega, P, \mathbb{R}^d)$ for some $\eta > 0$. Let f denote the probability density of the absolutely continuous part of its distribution $P_{_X}$ (f is possibly 0). Then,*

$$\lim_N \left(N^{\frac{p}{d}} \min_{|\Gamma|\le N} \|X - \widehat{X}^\Gamma\|_{_p}^p\right) = J_{p,d}\|f\|_{\frac{d}{d+p}}.$$

The constant $J_{p,d}$ corresponds to the uniform distribution over $[0,1]^d$ and in that case the above $\lim_{_N}$ also holds as an infimum.

The constant $J_{p,d}$ is unknown as soon as $d \ge 3$ although one knows that $J_{p,d} \sim (d/(2\pi e))^{\frac{p}{2}}$ as $d \to \infty$. This theorem says that the lowest L^p-quantization error goes to 0 at a $N^{-\frac{1}{d}}$-rate when $N \to \infty$. For more details about these results, we refer to [15] and the references therein.

From a computational viewpoint, no closed form is available for optimal quantization grids Γ^* except in some very specific 1-dimensional distributions like the uniform one. Several algorithms can be implemented to compute these optimal (or at least some efficient locally optimal) grids. Several of them rely on the differentiability of the L^p-distortion function as a function of the grid (viewed as a N-tuple of $(\mathbb{R}^d)^N$): if P_X is continuous, it is differentiable at any grid of size N and its gradient admits an integral representation with respect to the distribution of X. Consequently one may search for optimal grids by implementing a Newton-Raphson procedure (in 1-dimension) or a stochastic gradient descent (in d-dimension). These numerical aspects have been extensively investigated in [27] with a special attention to the d-dim normal distribution. Efficient grids for these distributions are now available for many sizes in dimensions $d = 1$ up to 10 (can be downloaded at `www.proba.jussieu.fr/pageperso/pages.html`); the extension to the quantization of Markov chains, including its numerical aspects, has already been discussed in several papers for various fields of applications like American option pricing, nonlinear filtering, or stochastic control (see *e.g.* [4], [24], [26] or [25]).

We now briefly explain in this introduction how to apply vector quantization method to the case of SPDE (1.1). In the case of Zakai equation, the process (X_k) is simply a time-discretization of a diffusion independent of V. In particular, given an observation W, (X_k) can be easily simulated and the idea is to quantize optimally at each time step k, the random vector X_k by a finite distribution $\hat{X}_k$. This provides in turn an approximation of (V_k) as the conditional distribution of $\hat{X}_k$ weighted by ξ_k. In the case of McKean-Vlasov equation, the diffusion X depends through its coefficients on its (conditional to W) distribution V. Hence, in order to simulate X_k at each time k, we use an approximation $\hat{V}_{k-1}$ of V_{k-1} based on an optimal quantization $\hat{X}_{k-1}$ of X_{k-1} (initially, $\hat{V}_0$ is the distribution of $\hat{X}_0$). Then, we can devise an optimal quantization of X_k and so provide an approximation of V_k.

The rest of this paper is organized as follows. Section 2 is devoted to the time discretization error of the SPDE (1.1). We prove that in general the rate of convergence is of order $\sqrt{\delta}$ but in the case where $\gamma = 0$, the order of convergence is improved to δ. We describe precisely in Section 3 the optimal quantization algorithm for the Zakai equation and we analyse the resulting error. The same structure is presented in Section 4 for the McKean-Vlasov equation. Finally, we illustrate our results in Section 5 with several simulations concerning the Zakai equation in the linear case and the Burger equation.

2 Time discretization error

In this section, we study the error caused by a time discretization of the system (1.3)-(1.4)-(1.5) characterizing the solution to the SPDE (1.1) on a finite time interval $[0, T]$. We consider regular discretization times $t_k = k\delta$, $k = 0, \ldots, n$, where $\delta = T/n$ is the time step, and we denote $\phi(t) = \sup\{t_k : t_k \leq t\}$. We then use an Euler scheme as follows:

$$\begin{aligned}
X_t^\delta &= X_0 + \int_0^t \beta(X_{\phi(s)}^\delta, V_{\phi(s)}^\delta)ds + \int_0^t \sigma(X_{\phi(s)}^\delta, V_{\phi(s)}^\delta)dB_s + \int_0^t \gamma(X_{\phi(s)}^\delta, V_{\phi(s)}^\delta)dW_s, \\
Z_t^\delta &= \int_0^t h(X_{\phi(s)}^\delta, V_{\phi(s)}^\delta).dW_s - \frac{1}{2}\int_0^t |h(X_{\phi(s)}^\delta, V_{\phi(s)}^\delta)|^2 ds,
\end{aligned}$$

$$< V_t^\delta, f > \quad = \quad E_W\left[f(X_t^\delta)\exp(Z_t^\delta)\right].$$

By denoting $\bar{X}_k = X^\delta_{t_k}$, $\bar{V}_k = V^\delta_{t_k}$, $\Delta\bar{B}_k = B_{t_k} - B_{t_{k-1}}$, $\Delta\bar{W}_k = W_{t_k} - W_{t_{k-1}}$, the Euler scheme reads at the discretization times t_k, $k = 0, \ldots, n$:

$$\bar{X}_{k+1} = \bar{X}_k + \beta(\bar{X}_k, \bar{V}_k)\delta + \sigma(\bar{X}_k, \bar{V}_k)\Delta\bar{B}_{k+1} + \gamma(\bar{X}_k, \bar{V}_k)\Delta\bar{W}_{k+1}, \tag{2.1}$$

$$\bar{X}_0 = X_0 \rightsquigarrow \mu_0, \tag{2.2}$$

$$< \bar{V}_k, f > = E_W\left[f(\bar{X}_k)\exp\left(\sum_{j=0}^{k-1} g(\bar{X}_j, \bar{V}_j, \Delta\bar{W}_{j+1})\right)\right], \tag{2.3}$$

where

$$g(x, V, \Delta\bar{W}) \quad = \quad h(x, V).\Delta\bar{W} - \frac{1}{2}|h(x, V)|^2\delta.$$

Denote by $\bar{P}_{k,W}(x, v, dx')$ the conditional probability of $\bar{X}_k$ given W, $\bar{X}_{k-1} = x$ and $\bar{V}_{k-1} = v$. From (2.1), we have:

$$\bar{P}_{k,W}(x, v, dx') \quad \rightsquigarrow \quad \mathcal{N}\left(x + \beta(x,v)\delta + \gamma(x,v)\Delta\bar{W}_k, \delta\sigma(x,v)\sigma^\intercal(x,v)\right).$$

As usual, we set for any $f \in \mathcal{B}(\mathbb{R}^d)$, set of bounded measurable functions on $\mathbb{R}^d$:

$$\begin{aligned}\bar{P}_{k,W} f(x, v) &= E_W\left[f(\bar{X}_k)\middle|\, \bar{X}_k = x, \bar{V}_k = v\right] \\ &= \int f(x')\bar{P}_{k,W}(x, v, dx'),\end{aligned}$$

for any $x \in \mathbb{R}^d$ and $v \in \mathcal{M}(\mathbb{R}^d)$. Hence, by the distribution of iterated conditional expectations, we have the following inductive formula for $\bar{V}_k$, $k = 0, \ldots, n$:

$$< \bar{V}_{k+1}, f > \; = \; < \bar{V}_k, \exp\left(g(., \bar{V}_k, \Delta\bar{W}_{k+1})\right)\bar{P}_{k+1,W} f(., \bar{V}_k) >, \tag{2.4}$$

$$\bar{V}_0 = \mu_0. \tag{2.5}$$

We denote by $BL_1(\mathbb{R}^d)$ the unit ball of bounded Lipschitz functions on $\mathbb{R}^d$:

$$BL_1(\mathbb{R}^d) \quad = \quad \{f : \mathbb{R}^d \mapsto \mathbb{R} \text{ satisfying } |f(x)| \leq 1 \text{ and } |f(x) - f(y)| \leq |x - y| \text{ for all } x, y\}$$

and we consider the following metric on $\mathcal{M}(\mathbb{R}^d)$:

$$\rho(V_1, V_2) \quad = \quad \sup\left\{|< V_1, f > - < V_2, f >|, \quad f \in BL_1(\mathbb{R}^d)\right\},$$

for any $V_1, V_2 \in \mathcal{M}(\mathbb{R}^d)$.

2.1 Zakai equation

To simplify the following convergence analysis, we assume that coefficients are very smooth and in addition, that they satisfy a uniform ellipticity condition.

(H1) (i) The functions β, σ and γ are of class C^∞ with bounded derivatives.

(*ii*) The function h is of class C^∞, is bounded and its derivatives as well.
(*iii*) For some $\epsilon_0 > 0$, one has $\sigma\sigma^\intercal(x) \geq \epsilon_0$ Id uniformly in x.

We recall some notation from [14]. We set $X_t^{\delta,\lambda} = X_t^\delta + \lambda(X_t - X_t^\delta)$, $a'(t) = \int_0^1 a'(X_t^{\delta,\lambda})d\lambda$ for a smooth function a (with derivative a') and $e^{\bar{Z}_T^\delta} = \int_0^1 e^{Z_T^\delta + \lambda(Z_T - Z_T^\delta)}d\lambda$. Now, consider the unique solution of the linear equation $\mathcal{E}_t = \mathrm{Id} + \int_0^t \beta'(s)\mathcal{E}_s\, ds + \sum_{j=1}^d \int_0^t \sigma_j'(s)\mathcal{E}_s\, dB_s^j + \sum_{j=1}^q \int_0^t \gamma_j'(s)\mathcal{E}_s\, dW_s^j$. Then, Lemma 4.3 in [14] gives

$$\begin{aligned}
X_t - X_t^\delta = \mathcal{E}_t \int_0^t & \mathcal{E}_s^{-1}\{[\beta(X_s^\delta) - \beta(X_{\phi(s)}^\delta)] \\
& - \sum_{j=1}^d \sigma_j'(s)[\sigma_j(X_s^\delta) - \sigma_j(X_{\phi(s)}^\delta)] - \sum_{j=1}^q \gamma_j'(s)[\gamma_j(X_s^\delta) - \gamma_j(X_{\phi(s)}^\delta)]\}\, ds \\
+ \sum_{j=1}^d \mathcal{E}_t \int_0^t & \mathcal{E}_s^{-1}[\sigma_j(X_s^\delta) - \sigma_j(X_{\phi(s)}^\delta)]\, dB_s^j + \sum_{j=1}^q \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1}[\gamma_j(X_s^\delta) - \gamma_j(X_{\phi(s)}^\delta)]\, dW_s^j.
\end{aligned} \tag{2.6}$$

For any $f \in BL_1(\mathbb{R}^d)$, we put $f_\delta(x) = E(f(x + \delta \bar{B}_T))$ where $\bar{B}$ is an extra d-dimensional Brownian motion independent on B and W. Clearly, f_δ is of class C_b^∞, $\|f_\delta\|_\infty + \sup_{x \neq y} \frac{|f_\delta(x) - f_\delta(y)|}{|x-y|} \leq C$, $\|f_\delta - f\|_\infty \leq C\delta$, both estimates being uniform in $BL_1(\mathbb{R}^d)$.

The main result of this section is the following.

Theorem 2.1 *(Case A: Zakai equation)*
Assume **(H1)**. *For $f \in BL_1(\mathbb{R}^d)$, set*

$$\begin{aligned}
A_1(f) &= -e^{Z_T^\delta} f_\delta'(T)\mathcal{E}_T[\sum_{j=1}^q \int_0^T (\mathcal{E}_s^{-1} \int_{\phi(s)}^s \gamma_j'(X_r^\delta)\gamma(X_{\phi(r)}^\delta)dW_r)dW_s^j], \\
A_2(f) &= -e^{\bar{Z}_T^\delta} f(X_T)(\sum_{i=1}^q \int_0^T [\int_{\phi(s)}^s h_i'(X_r^\delta)\gamma(X_{\phi(r)}^\delta)dW_r]dW_s^i), \\
A_3(f) &= -\sum_{i,j=1}^q f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T h_i'(s)\mathcal{E}_s(\int_0^s \mathcal{E}_r^{-1}[\int_{\phi(r)}^r \gamma_j'(X_u^\delta)\gamma(X_{\phi(u)}^\delta)dW_u]dW_r^j)dW_s^i), \\
A_4(f) &= \frac{1}{2}e^{\bar{Z}_T^\delta} f(X_T) \int_0^T [(\|h\|^2)'(s)\mathcal{E}_s(\sum_{j=1}^q \int_0^s \mathcal{E}_r^{-1}(\int_{\phi(r)}^r \gamma_j'(X_u^\delta)\gamma(X_{\phi(u)}^\delta)dW_u)dW_r^j)]ds.
\end{aligned}$$

Then, one has

$$\left\|\rho(V_T, V_T^\delta)\right\|_2 \leq C\delta + \sup_{f \in BL_1(\mathbb{R}^d)} \|E_W\,[A_1(f) + A_2(f) + A_3(f) + A_4(f)]\|_2\,,$$

with

$$\sup_{f \in BL_1(\mathbb{R}^d)} \|E_W(A_1(f) + A_2(f) + A_3(f) + A_4(f))\|_2 \leq C\sqrt{\delta},$$

for some constant C.

Remark 2.1 The fact that $\sqrt{\delta}$ is an upper bound for the error is clear, if we use classic L^p-estimates between X and X^δ, see e.g. [19]. But we know that this argument involving pathwise errors is not optimal when errors on laws are considered [3]. The result above makes clear the role of the correlation in the error on conditional expectations.
1. When there is no correlation between signal and observation, i.e. $\gamma = 0$, the four terms $A_i(f)$, $i = 1, \ldots, 4$, vanish and the rate of convergence for the approximation of V_T is of order δ, the time discretization step.
2. For constant function γ, the three contributions $A_1(f), A_3(f), A_4(f)$ vanish and it remains $A_2(f)$ of order $\sqrt{\delta}$ coming from the approximation of e^{Z_T}.
3. In the general case, the error will be inexorably of order $\sqrt{\delta}$. Indeed, main contributions in the error essentially behave like $\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (W_s - W_{t_i}) dW_s = \frac{1}{2} \sum_{i=0}^{n-1} ([W_{t_{i+1}} - W_{t_i}]^2 - [t_{i+1} - t_i])$, which L^2-norm equals $C\sqrt{\delta}$.

2.2 Proof of Theorem 2.1

The proof relies on Malliavin calculus techniques: we refer the reader to [22], from which we borrow our notations. For technical reasons, it will be useful to work with the Wiener process $\mathcal{W} = \begin{pmatrix} B \\ \bar{B} \\ W \end{pmatrix}$: all the further Malliavin calculus computations are made relatively to $\mathcal{W}$. Set $H = L^2([0,T], \mathbb{R}^d)$ and denote $\bar{X}_t^{\delta,\lambda} = X_t^{\delta,\lambda} + \frac{\delta}{\sqrt{2}} \bar{B}_t$. For $F \in \mathbb{D}^{1,p}$, we write $\mathcal{D}F = (\mathcal{D}^B F, \mathcal{D}^{\bar{B}} F, \mathcal{D}^W F)$ for the components relatively to the three Brownian motions B, $\bar{B}$ and W; the partial Malliavin covariance matrix of F is denoted by $\gamma^F = \int_0^T [\mathcal{D}_t^B F, \mathcal{D}_t^{\bar{B}} F, 0][\mathcal{D}_t^B F, \mathcal{D}_t^{\bar{B}} F, 0]^* dt = \int_0^T \mathcal{D}_t^B F [\mathcal{D}_t^B F]^* dt + \int_0^T \mathcal{D}_t^{\bar{B}} F [\mathcal{D}_t^{\bar{B}} F]^* dt$.

As in section 4.5.2 of [14], a localization factor $\psi_T^\delta \in [0,1]$ will be needed in the control of residual terms to justify integration by parts formulas: it satisfies the following properties

a) $\psi_T^\delta \in \mathbb{D}_{k,p}$ and $\sup_\delta \|\psi_T^\delta\|_{\mathbb{D}^{k,p}} \leq \frac{C}{T^q}$ for any integers k, p;

b) $P(\psi_T^\delta \neq 1) \leq \frac{C}{T^q} \delta^k$ for any $k \geq 1$;

c) $\{\psi_T^\delta \neq 0\} \subset \{\forall \lambda \in [0,1] : \det(\gamma^{\bar{X}_T^{\delta,\lambda}}) \geq \frac{1}{2} \det(\gamma^{X_T})\}$.

We omit the details of its tedious construction and we simply refer to [14]. To prepare the proof, we now state a series of technical results (justified later), which will help to derive a suitable stochastic analysis conditionally on W.

Lemma 2.1 *In the following, $\Phi(W)$ stands for a functional measurable w.r.t. W, which belongs to $\mathbb{D}^\infty$.*

i) For any r.v. $Y \in L^2$, $E_W(Y)$ is the unique r.v. satisfying the equality $E(Y\Phi(W)) = E(E_W(Y)\Phi(W))$ for any functional $\Phi(W) \in \mathbb{D}^\infty$.

ii) For any $\Phi(W) \in \mathbb{D}^\infty$ and $F \in \mathbb{D}^{1,2}$, one has $\Phi(W)F \in \mathbb{D}^{1,1}$, with $\mathcal{D}^B(\Phi(W)F) = \Phi(W)\mathcal{D}^B F$ and $\mathcal{D}^{\bar{B}}(\Phi(W)F) = \Phi(W)\mathcal{D}^{\bar{B}} F$.

iii) For $\Phi(W)$ and G in $\mathbb{D}^\infty$, $g \in C_b^\infty$ and any multi-index α, one has

$$\left\{ \begin{array}{l} E\left(\Phi(W)\partial^\alpha g(X_T)G\right) = E\left(\Phi(W)g(X_T)H_\alpha(X_T,G)\right), \\ \|H_\alpha(X_T,G)\|_2 \leq C\frac{\|G\|_{\mathbb{D}^{k,p}}}{T^q} \end{array}\right. \tag{2.7}$$

for some integers k, p, q. Furthermore, if $G = 0$ on $\{\psi_T^\delta = 0\}$, then for any $\lambda \in [0,1]$, one has

$$\left\{ \begin{array}{l} E\left(\Phi(W)\partial^\alpha g(\bar{X}_T^{\delta,\lambda})G\right) = E\left(\Phi(W)g(\bar{X}_T^{\delta,\lambda})H_\alpha(\bar{X}_T^{\delta,\lambda},G)\right), \\ \|H_\alpha(\bar{X}_T^{\delta,\lambda},G)\|_2 \leq C\frac{\|G\|_{\mathbb{D}^{k,p}}}{T^q} \end{array}\right. \tag{2.8}$$

with some constants C, k, p, q uniform in δ and $\lambda \in [0,1]$.

The result below is more surprising, in particular the estimates of order δ. Indeed, at the first glance, each stochastic integral (for fixed r) at the left hand side of (2.9) is of order $\sqrt{\delta}$, but the mean over r helps in improving this estimate to get δ, provided that the processes g and h satisfy some suitable controls. Its proof is postponed to the end of this section.

Proposition 2.1 *For $g \in \mathbb{D}^\infty(H)$ and $h \in \mathbb{D}^\infty(H)$, one has*

$$\begin{aligned} \int_0^T g_r\left(\int_{\phi(r)}^r h_u \delta\mathcal{W}_u\right)dr &= \int_0^T \left(\int_0^T g_r h_u \mathbf{1}_{\phi(r)\leq u\leq r} dr\right)\delta\mathcal{W}_u \\ &+ \int_0^T \left(\int_0^T \mathcal{D}_u g_r \cdot h_u \mathbf{1}_{\phi(r)\leq u\leq r} dr\right)du, \end{aligned} \tag{2.9}$$

and the above random variable belongs to $\mathbb{D}^\infty$. Under extra assumptions, both terms in the r.h.s. above are of order δ.

i) Assume that $N_{k,p}(g) = \sum_{j=0}^k [E(\int_0^T \|\mathcal{D}^j g_r\|^p_{L^p([0,T]^j)} dr)]^{1/p} < +\infty$ and $N_{k,p}(h) < +\infty$ for any k and p. Then, the first term of r.h.s. of (2.9) is of order δ in $\mathbb{D}^{k,p}$, for any $k \in \mathbb{N}$ and $p > 1$:

$$\left\|\int_0^T \left(\int_0^T g_r h_u \mathbf{1}_{\phi(r)\leq u\leq r} dr\right)\delta\mathcal{W}_u\right\|_{\mathbb{D}^{k,p}} \leq C\ N_{k+1,q}(g)N_{k+1,q}(h)\ \delta \tag{2.10}$$

for some constants C and q depending only on k and p.

ii) Assume that $M_{k,p}(g) = \sum_{j=1}^k \sup_{0\leq r\leq T}[E\|\mathcal{D}^j g_r\|^p_{L^p([0,T]^j)}]^{1/p} < +\infty$ and $N_{k,p}(h) < +\infty$ for any k and p. Then, the second term of r.h.s. of (2.9) is of order δ in $\mathbb{D}^{k,p}$, for any $k \in \mathbb{N}$ and $p \geq 1$:

$$\left\|\int_0^T \left(\int_0^T \mathcal{D}_u g_r \cdot h_u \mathbf{1}_{\phi(r)\leq u\leq r} dr\right)du\right\|_{\mathbb{D}^{k,p}} \leq C\ M_{k+1,q}(g)\ N_{k,q}(h)\ \delta \tag{2.11}$$

for some constants C and q depending only on k and p.

Let us turn to the proof of Theorem 2.1. It consists in proving

$$\begin{aligned} E(\Phi(W)[f(X_T^\delta)e^{Z_T^\delta} - f(X_T)e^{Z_T}]) &= E(\Phi(W)e^{Z_T^\delta}[(f-f_\delta)(X_T^\delta) - (f-f_\delta)(X_T)]) && (2.12) \\ &+ E(\Phi(W)e^{Z_T^\delta}[f_\delta(X_T^\delta) - f_\delta(X_T)]) && (2.13) \\ &+ E(\Phi(W)f(X_T)[e^{Z_T^\delta} - e^{Z_T}]) && (2.14) \\ &= E(\Phi(W)[A_1(f) + A_2(f) + A_3(f) + A_4(f) + R]) \end{aligned}$$

for any functional $\Phi(W) \in \mathbb{D}^\infty$, with $\|R\|_2 = O(\delta)$ uniformly w.r.t. $f \in BL_1(\mathbb{R}^d)$. Since $\|f - f_\delta\|_\infty \leq C\delta$ for $f \in BL_1(\mathbb{R}^d)$, the term (2.12) can be neglected in our expansion.

In the following computations, we simply write Φ instead of $\Phi(W)$.

2.2.1 Contribution (2.13)

A Taylor's formula combined with (2.6) and Ito's formula between $\phi(s)$ and s gives

$$E(\Phi e^{Z_T^\delta}[f_\delta(X_T^\delta) - f_\delta(X_T)]) = E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha^{0,0}(u)du]ds) \tag{2.15}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha^{0,1}(u)dB_u]ds) \tag{2.16}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha^{0,2}(u)dW_u]ds) \tag{2.17}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha^{1,0}(u)du]dB_s) \tag{2.18}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\sum_{i=1}^d \int_{\phi(s)}^s \alpha_i^{1,1}(u)dB_u]dB_s^i) \tag{2.19}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\sum_{i=1}^d \int_{\phi(s)}^s \alpha_i^{1,2}(u)dW_u]dB_s^i) \tag{2.20}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha^{2,0}(u)du]dW_s) \tag{2.21}$$

$$+E(\Phi e^{Z_T^\delta} f'_\delta(T)\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\sum_{i=1}^q \sum_{j=1}^d \int_{\phi(s)}^s \alpha_{i,j}^{2,1}(u)dB_u^j]dW_s^i) \tag{2.22}$$

$$+E(\Phi A_1(f)), \tag{2.23}$$

where coefficients $\alpha_\cdot^\cdot \in \mathbb{D}^\infty(H)$ with $N_{k,p}(\alpha_\cdot^\cdot) + M_{k,p}(\alpha_\cdot^\cdot) < +\infty$ for any k, p, uniformly w.r.t. δ (actually, this is a consequence of the stronger estimate $\sup_{r\in[0,T]} \|\mathcal{D}^k_{s_1,\cdots,s_k}\alpha_\cdot^\cdot(r)\|_p < \infty$, see [14] for instance).

Terms in factor of Φ in (2.15)(2.18)(2.21) clearly satisfy $\|R\|_2 = O(\delta)$ (remind that $\|f'\|_\infty \leq C$ uniformly in $f \in BL_1(\mathbb{R}^d)$).

The contributions (2.16) and (2.17) give a contribution of order δ in L^p-norm by an application of estimates (2.10-2.11).

Terms (2.19) contain most of the difficulties that we have to face in this error analysis: we may here give detailed arguments ((2.20) is handled in the same way). Note that $f_\delta(x) = E(f_{\delta/\sqrt{2}}(x + \frac{\delta}{\sqrt{2}}\bar{B}_T))$ as well for the derivatives: thus, each term of the sum in (2.19) equals

$$\int_0^1 d\lambda E(\Phi\psi_T^\delta e^{Z_T^\delta} f'_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha_i^{1,1}(u)dB_u]dB_s^i) \tag{2.24}$$

$$+\int_0^1 d\lambda E(\Phi(1-\psi_T^\delta)e^{Z_T^\delta} f'_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha_i^{1,1}(u)dB_u]dB_s^i). \tag{2.25}$$

Since $P(\psi_T^\delta \neq 1) \leq C\frac{\delta^2}{T^q}$, (2.25) provides a negligible contribution. Besides, if we transform the Itô integral w.r.t. B into a Lebesgue integral, using the duality relationship and Property ii) of Lemma 2.1, we obtain that (2.24) is equal to

$$\begin{aligned}
&\int_0^1 d\lambda E(\Phi \int_0^T \mathcal{D}_s^{B^i}[\psi_T^\delta e^{Z_T^\delta} f'_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})\mathcal{E}_T]\mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha_i^{1,1}(u)dB_u]ds) \\
&= \sum_{\kappa:|\kappa|=1,2} \int_0^1 d\lambda E(\Phi f^{(\kappa)}_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda}) \int_0^T \alpha_{\kappa,i}^{1,1}(s)[\int_{\phi(s)}^s \alpha_i^{1,1}(u)dB_u]ds)
\end{aligned}$$

with $N_{k,p}(\alpha_{\kappa,i}^{1,1})+M_{k,p}(\alpha_{\kappa,i}^{1,1}) < +\infty$ for any k and p. If we put $G = \int_0^T \alpha_{\kappa,i}^{1,1}(s)[\int_{\phi(s)}^s \alpha_i^{1,1}(u)dB_u]ds$, we remark that $G \in \mathbb{D}^\infty$, that $G = 0$ if $\psi_T^\delta = 0$ because of the local property of the derivative operator and that $\|G\|_{\mathbb{D}^{k,p}} \leq C\delta$ applying Proposition 2.1. Thus, Lemma 2.1 completes the estimate, and the factor of Φ in (2.24) is of order δ in L^2-norm, uniformly w.r.t. $f \in BL_1(\mathbb{R}^d)$.

We now consider (2.22). As for (2.19), we introduce ψ_T^δ: the term with $1-\psi_T^\delta$ can be neglected as before. Using analogous computations as above, it is straightforward to see that we have to control

$$\begin{aligned}
&\int_0^1 d\lambda E(\Phi\psi_T^\delta e^{Z_T^\delta} f'_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})\mathcal{E}_T \int_0^T \mathcal{E}_s^{-1}[\int_{\phi(s)}^s \alpha_{i,j}^{2,1}(u)dB_u^j]dW_s^i) \\
&= \int_0^1 d\lambda \int_0^T \int_0^T E(\mathcal{D}_u^{B^j}[\mathcal{D}_s^{W^i}[\Phi\psi_T^\delta e^{Z_T^\delta} f'_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})\mathcal{E}_T]\mathcal{E}_s^{-1}]\mathbf{1}_{\phi(s)\leq u\leq s}\alpha_{i,j}^{2,1}(u))du\ ds \\
&= \sum_{\kappa:|\kappa|=1,2} \int_0^1 d\lambda E(\Phi f^{(\kappa)}_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda}) \int_0^T \int_0^T \hat{\alpha}_{i,j}^{\kappa,2,1}(s)\mathbf{1}_{\phi(s)\leq u\leq s}\alpha_{i,j}^{2,1}(u)du\ ds) \quad (2.26) \\
&+ \sum_{\kappa:|\kappa|=1,2} \int_0^1 d\lambda E(\int_0^T \mathcal{D}_s^{W^i}[\Phi f^{(\kappa)}_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})](\int_0^T \alpha_{i,j}^{\kappa,2,1}(s)\mathbf{1}_{\phi(s)\leq u\leq s}\alpha_{i,j}^{2,1}(u)du)ds). \quad (2.27)
\end{aligned}$$

For (2.26), it is enough to apply (2.8) with $G = \int_0^T \int_0^T \hat{\alpha}_{i,j}^{\kappa,2,1}(s)\mathbf{1}_{\phi(s)\leq u\leq s}\alpha_{i,j}^{2,1}(u)du\ ds$ that clearly satisfies $\|G\|_{\mathbb{D}^{k,p}} \leq C\delta$: this proves the expected estimate of order δ. The same conclusion holds for each term in (2.27): indeed, they can be transformed in $\int_0^1 d\lambda E(\Phi f^{(\kappa)}_{\delta/\sqrt{2}}(\bar{X}_T^{\delta,\lambda})$ $\int_0^T (\int_0^T \alpha_{i,j}^{\kappa,2,1}(s)\mathbf{1}_{\phi(s)\leq u\leq s}\alpha_{i,j}^{2,1}(u)du)\delta W_s^i)$ and we conclude with Lemma 2.1.

2.2.2 Contribution (2.14)

It can be decomposed as $E(\Phi f(X_T)[e^{Z_T^\delta} - e^{Z_T}]) = E(\Phi f(X_T)e^{\bar{Z}_T^\delta}[Z_T^\delta - Z_T])$, that is

$$\begin{aligned}
&E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T [h(X_{\phi(s)}^\delta) - h(X_s^\delta)].dW_s)) \quad (2.28) \\
&+E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T [h(X_s^\delta) - h(X_s)].dW_s)) \quad (2.29) \\
&-\frac{1}{2}E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T [\|h\|^2(X_{\phi(s)}^\delta) - \|h\|^2(X_s^\delta)]ds)) \quad (2.30) \\
&-\frac{1}{2}E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T [\|h\|^2(X_s^\delta) - \|h\|^2(X_s)]ds)). \quad (2.31)
\end{aligned}$$

Since (2.28) can be rewritten as $E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\sum_{i=1}^q \int_0^T [h_i(X_{\phi(s)}^\delta)-h_i(X_s^\delta)]dW_s^i))$, it equals

$$-E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\sum_{i=1}^q \int_0^T [\int_{\phi(s)}^s h_i'(X_r^\delta)\beta(X_{\phi(r)}^\delta)dr]dW_s^i)) \tag{2.32}$$

$$-E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\sum_{i=1}^q \int_0^T [\int_{\phi(s)}^s h_i'(X_r^\delta)\sigma(X_{\phi(r)}^\delta)dB_r]dW_s^i)) \tag{2.33}$$

$$-E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\sum_{i=1}^q \int_0^T [\int_{\phi(s)}^s h_i'(X_r^\delta)\gamma(X_{\phi(r)}^\delta)dW_r]dW_s^i)). \tag{2.34}$$

The factor of Φ in (2.32) clearly satisfies the required estimate and can be neglected. The term (2.33) can also be discarded from the main part of the error using the same arguments as for (2.22). Finally, the term (2.34) gives $A_2(f)$.

Term (2.29). Owing to (2.6), it writes $\sum_{i=1}^q E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T [h_i(X_s^\delta) - h_i(X_s)]dW_s^i))$, equals

$$-\sum_{i=1}^q\sum_{j=1}^d E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T h_i'(s)\mathcal{E}_s(\int_0^s \mathcal{E}_r^{-1}[\int_{\phi(r)}^r \sigma_j'(X_u^\delta)\sigma(X_{\phi(u)}^\delta)dB_u]dB_r^j)dW_s^i))$$

$$-\sum_{i=1}^q\sum_{j=1}^d E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T h_i'(s)\mathcal{E}_s(\int_0^s \mathcal{E}_r^{-1}[\int_{\phi(r)}^r \sigma_j'(X_u^\delta)\gamma(X_{\phi(u)}^\delta)dW_u]dB_r^j)dW_s^i))$$

$$-\sum_{i,j=1}^q E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T h_i'(s)\mathcal{E}_s(\int_0^s \mathcal{E}_r^{-1}[\int_{\phi(r)}^r \gamma_j'(X_u^\delta)\sigma(X_{\phi(u)}^\delta)dB_u]dW_r^j)dW_s^i)) \tag{2.35}$$

$$-\sum_{i,j=1}^q E(\Phi f(X_T)e^{\bar{Z}_T^\delta}(\int_0^T h_i'(s)\mathcal{E}_s(\int_0^s \mathcal{E}_r^{-1}[\int_{\phi(r)}^r \gamma_j'(X_u^\delta)\gamma(X_{\phi(u)}^\delta)dW_u]dW_r^j)dW_s^i)) \tag{2.36}$$

$$+E(\Phi R)$$

with $\|R\|_2 = O(\delta)$ by estimates (2.10-2.11). The term (2.36) gives $A_3(f)$, while the other contributions can be neglected. To justify this assertion, let us consider for instance (2.35), techniques being the same for the other ones. First, we can replace f by f_δ since $\|f-f_\delta\|_\infty \leq C\delta$. Then, three applications of duality relationship yield:

$$E(\Phi f_\delta(X_T)e^{\bar{Z}_T^\delta}(\int_0^T h_i'(s)\mathcal{E}_s(\int_0^s \mathcal{E}_r^{-1}[\int_{\phi(r)}^r \gamma_j'(X_u^\delta)\sigma(X_{\phi(u)}^\delta)dB_u]dW_r^j)dW_s^i))$$

$$= \int_0^T\int_0^T\int_0^T E(\mathcal{D}_u^B[\mathcal{D}_r^{W^j}[\mathcal{D}_s^{W^i}[\Phi f_\delta(X_T)e^{\bar{Z}_T^\delta}]h_i'(s)\mathcal{E}_s]\mathcal{E}_r^{-1}]\cdot\gamma_j'(X_u^\delta)\sigma(X_{\phi(u)}^\delta)\mathbf{1}_{\phi(r)\leq u\leq r})du\ dr\ ds.$$

The term inside the expectation can be split into a sum involving the derivative of Φ and of f. Presumably, the more difficult term to estimate is of the form

$$\int_0^T\int_0^T\int_0^T E(\mathcal{D}_r^{W^j}[\mathcal{D}_s^{W^i}[\Phi f_\delta^{(\kappa)}(X_T)]]\alpha(u,r,s)\mathbf{1}_{\phi(r)\leq u\leq r})du\ dr\ ds.$$

We omit the details for the other ones which are easier to handle. Two integration by parts with fixed W (see iii) in Lemma 2.1) show that it equals

$$E(\Phi f_\delta^{(\kappa)}(X_T)\int_0^T(\int_0^T(\int_0^T \alpha(u,r,s)\mathbf{1}_{\phi(r)\leq u\leq r}du)\delta W_r^j)\delta W_s^i).$$

Then, we conclude using (2.7) with $\| \int_0^T(\int_0^T(\int_0^T \alpha(u,r,s)\mathbf{1}_{\phi(r)\leq u\leq r}du)\delta W_r^j)\delta W_s^i\|_{\mathbb{D}^{k,p}} \leq C\delta$.

Term (2.30). It yields a contribution of order δ, by an application of Ito's formula and inequalities (2.10-2.11). At last, the term (2.31) is equal to $-\frac{1}{2}\int_0^T E(\Phi f(X_T)e^{\bar{Z}_T^\delta}[\|h\|^2(X_s^\delta)-\|h\|^2(X_s)])ds$: in this form, the analysis is analogous to that of (2.13) and we omit the details. It gives the contribution $A_4(f)$ and some residual terms of order δ.

2.2.3 Proof of Lemma 2.1

The two first statements are straightforward. Statement i) immediately follows from the fact that any $\Phi(W) \in L^2$ can be approximated in L^2 by a sequence of $\mathbb{D}^\infty$-r.v. using the chaos expansion (see Th. 1.1.1 p.6 in [22]). Statement ii) is clear from the definition of $\mathbb{D}^{1,p}$, $\mathcal{D}^B$ and $\mathcal{D}^{\bar{B}}$.

Statement iii) is an integration by parts formula, that puts the differentiation/integration only on B and $\bar{B}$, but not on W. Its proof is an easy adaptation of Proposition 3.2.1. in [23]. The estimate (2.7) is standard using in particular $\|[\gamma^{X_T}]^{-1}\|_p \leq \frac{C}{T^q}$. We only prove (2.8) which is less usual because of the localization factor G. Using ii), one obtains the following equalities:

$$[\mathcal{D}^B(\Phi(W)g(\bar{X}_T^{\delta,\lambda})), \mathcal{D}^{\bar{B}}(\Phi(W)g(\bar{X}_T^{\delta,\lambda}))] = \Phi(W)g'(\bar{X}_T^{\delta,\lambda})[\mathcal{D}^B\bar{X}_T^{\delta,\lambda}, \mathcal{D}^{\bar{B}}\bar{X}_T^{\delta,\lambda}],$$
$$\int_0^T \mathcal{D}_t(\Phi(W)g(\bar{X}_T^{\delta,\lambda}))[\mathcal{D}_t^B\bar{X}_T^{\delta,\lambda}, \mathcal{D}_t^{\bar{B}}\bar{X}_T^{\delta,\lambda}, 0]^*dt = \Phi(W)g'(\bar{X}_T^{\delta,\lambda})\gamma^{\bar{X}_T^{\delta,\lambda}}.$$

Note that $\gamma^{\bar{X}_T^{\delta,\lambda}} \geq \frac{\delta^2}{2}\mathrm{Id}$ and thus $\gamma^{\bar{X}_T^{\delta,\lambda}}$ is invertible. Then, the duality relationship leads to

$$\begin{aligned} &E(\Phi(W)\partial_{x_i}g(\bar{X}_T^{\delta,\lambda})G) \\ =& E(\int_0^T \mathcal{D}_t(\Phi(W)g(\bar{X}_T^{\delta,\lambda}))[Ge^i\cdot[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathcal{D}_t^B\bar{X}_T^{\delta,\lambda}, Ge^i\cdot[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathcal{D}_t^{\bar{B}}\bar{X}_T^{\delta,\lambda}, 0]^*dt) \\ =& E(\Phi(W)g(\bar{X}_T^{\delta,\lambda})\int_0^T [Ge^i\cdot[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathcal{D}_t^B\bar{X}_T^{\delta,\lambda}, Ge^i\cdot[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathcal{D}_t^{\bar{B}}\bar{X}_T^{\delta,\lambda}, 0]\delta\mathcal{W}_t). \end{aligned}$$

For longer multi-index α, we iterate the procedure and construct $H_\alpha(\bar{X}_T^{\delta,\lambda}, G)$ by the recurrence formula $H_{\alpha'+[e^i]^*}(\bar{X}_T^{\delta,\lambda}, G) = \int_0^T[H_{\alpha'}(\bar{X}_T^{\delta,\lambda}, G)e^i\cdot[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathcal{D}_t^B\bar{X}_T^{\delta,\lambda}, H_{\alpha'}(\bar{X}_T^{\delta,\lambda}, G)e^i\cdot[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathcal{D}_t^{\bar{B}}\bar{X}_T^{\delta,\lambda}, 0]\delta\mathcal{W}_t$. Concerning the estimation on $\|H_\alpha(\bar{X}_T^{\delta,\lambda}, G)\|_2$, remark first that since the derivative operator and the Skorohod integral are local (see Propositions 1.3.6 and 1.3.7 in [22]), one has $H_\alpha(\bar{X}_T^{\delta,\lambda}, G) = H_\alpha(\bar{X}_T^{\delta,\lambda}, G)\mathbf{1}_{\psi_T^\delta>0}$ owing to the property on G. Using the standard inequality $\|H_\alpha(\bar{X}_T^{\delta,\lambda}, G)\mathbf{1}_A\|_p \leq C\|[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathbf{1}_A\|_{q_1}^{p_1}\|\bar{X}_T^{\delta,\lambda}\|_{k_2,q_2}^{p_2}\|G\|_{\mathbb{D}^{k_3,q_3}}$ (Proposition 2.4 in [3]) combined with $\|[\gamma^{\bar{X}_T^{\delta,\lambda}}]^{-1}\mathbf{1}_{\psi_T^\delta>0}\|_p \leq \frac{C}{T^q}$ (take into account c) of property on ψ_T^δ), we easily complete the expected estimation.

2.2.4 Proof of Proposition 2.1

To prove (2.9), take $\Psi \in \mathbb{D}^\infty$ and write using twice Fubini's theorem and the duality relationship alternatively:

$$E(\Psi\int_0^T g_r(\int_{\phi(r)}^r h_u\delta\mathcal{W}_u)dr) = \int_0^T E(\Psi g_r(\int_{\phi(r)}^r h_u\delta\mathcal{W}_u))dr$$

$$= \int_0^T \int_0^T E(\mathcal{D}_u[\Psi g_r]\mathbf{1}_{\phi(r)\le u\le r}\cdot h_u)du\, dr$$
$$= \int_0^T E(\mathcal{D}_u\Psi \cdot \int_0^T g_r h_u \mathbf{1}_{\phi(r)\le u\le r}dr)du + \int_0^T E(\Psi \int_0^T \mathcal{D}_u g_r \cdot h_u \mathbf{1}_{\phi(r)\le u\le r}dr)du$$
$$= E(\Psi \int_0^T (\int_0^T g_r h_u \mathbf{1}_{\phi(r)\le u\le r}dr)\delta\mathcal{W}_u) + E(\Psi \int_0^T (\int_0^T \mathcal{D}_u g_r \cdot h_u \mathbf{1}_{\phi(r)\le u\le r}dr)du).$$

Is is standard to check that $\int_0^T g_r(\int_{\phi(r)}^r h_u\delta\mathcal{W}_u)dr$ belongs to $\mathbb{D}^\infty$ (see Lemma 1.3.4 in [22]). The original feature of our result is specifically related to (2.10) and (2.11). Our key estimates are the following ones: for appropriately defined random variables $(g_{r,s}, h_{u,s}, g_{r,s,u})_{r,s,u}$, we have

$$\left[E\left(\int_{[0,T]^j} ds \int_0^T du |\int_0^T g_{r,s}h_{u,s}\mathbf{1}_{\phi(r)\le u\le r}dr|^2\right)^{p/2}\right]^{1/p} \tag{2.37}$$
$$\le C_{p,q}(T)\ \delta \left[E\left(\int_{[0,T]^{j+1}} |h_{u,s}|^q\ duds\right)\right]^{1/q}\left[E\left(\int_{[0,T]^{j+1}} |g_{r,s}|^q\ drds\right)\right]^{1/q},$$
$$\left[E\left(\int_{[0,T]^j} ds \int_0^T du |\int_0^T g_{r,s,u}h_{u,s}\mathbf{1}_{\phi(r)\le u\le r}dr|^2\right)^{p/2}\right]^{1/p} \tag{2.38}$$
$$\le C_{p,q}(T)\ \delta \left[E\left(\int_{[0,T]^{j+1}} |h_{u,s}|^q\ duds\right)\right]^{1/q} \sup_{0\le r\le T}\left[E\left(\int_{[0,T]^{j+1}} |g_{r,s,u}|^q\ dsdu\right)\right]^{1/q},$$

for q large enough. Indeed, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_0^T du\ |\int_0^T g_{r,s,u}h_{u,s}\mathbf{1}_{\phi(r)\le u\le r}dr|^2 &\le \int_0^T du|h_{u,s}|^2(\int_u^{\phi(u)+\delta}|g_{r,s,u}|\ dr)^2\\ &\le [\int_0^T du|h_{u,s}|^4]^{1/2}[\int_0^T du(\int_u^{\phi(u)+\delta}|g_{r,s,u}|\ dr)^4]^{1/2}\\ &\le \delta^{3/2}[\int_0^T du|h_{u,s}|^4]^{1/2}[\int_0^T du\int_u^{\phi(u)+\delta}|g_{r,s,u}|^4\ dr]^{1/2}.\end{aligned}$$

If g does not depend on u, the last term above is bounded by $\delta^{1/2}[\int_0^T |g_{r,s}|^4\ dr]^{1/2}$. Then, the derivation of (2.37) is easy, using Hölder's inequalities. To obtain (2.38), i.e. when g depends on u, the previous computation to get the missing factor $\delta^{1/2}$ does not directly work: before, one has to integrate over s and ω, the other arguments remaining unchanged.

We are now in a position to derive (2.10). Consider first $k=0$. To control the L^p-norms of the first term in the r.h.s. of (2.9), we invoke the continuity of the Skorohod integral (Proposition 2.4.3 in [23]) to get

$$\|\int_0^T (\int_0^T g_r h_u \mathbf{1}_{\phi(r)\le u\le r}dr)\delta\mathcal{W}_u\|_p$$
$$\le C\Big(\|\int_0^T g_r h_\cdot \mathbf{1}_{\phi(r)\le \cdot\le r}dr\|_{L^p(\Omega,H)} + \|\int_0^T \mathcal{D}(g_r h_\cdot)\mathbf{1}_{\phi(r)\le\cdot\le r}dr\|_{L^p(\Omega,H^{\otimes 2})}\Big). \tag{2.39}$$

From (2.37), we easily get that the first term above is bounded by $N_{0,q}(h)N_{0,q}(h)\delta$, for q large enough. With analogous computations, the second term in the r.h.s. of (2.39) is bounded by $CN_{1,q}(h)N_{1,q}(h)\delta$. Estimates (2.10) have been proved when $k = 0$. For $k \geq 1$, the successive derivative of the r.h.s of (2.9) are standard to compute and can be expressed in a similar form than before: then, analogous computations can be performed and this proves (2.10) for any k. The derivation of (2.11) is analogous, using in addition (2.38).

2.3 Stochastic McKean-Vlasov equation

The detailed analysis of the time discretization error for the Zakai equation illustrates that in general, due to the correlation factor γ, the error will be exactly of order $\sqrt{\delta}$. The situation with the McKean-Vlasov equation is analogous, as we briefly discuss it now.

1. Firstly, the derivation of the order $\sqrt{\delta}$ is standard, using usual techniques. In the case of deterministic McKean-Vlasov equation, see [18] Lemma 3.1. But in view of their proofs, it is easy to see that considering conditional laws do not modify the final estimates.

2. Secondly, without correlation (i.e. $\gamma = 0$), it is proved in [2] that the error is of order δ under a non-degeneracy condition. The result is proved in the case $d = 1$ but an extension to higher dimensions is possible.

3. Thirdly, it remains to justify why, in general ($\gamma \neq 0$), we can not expect the error to be of order δ but only $\sqrt{\delta}$. A simple example in dimension 1 may be $\beta \equiv 0$, $\sigma \equiv 0$ and $\gamma(x, V) = x$. In that case, the error coincides with that of the Zakai equation (with $h \equiv 0$). Only the contribution $A_1(f)$ remains (see Theorem 2.1), which is clearly of order $\sqrt{\delta}$.

3 Simulation of Zakai equation and quantization error

3.1 The quantization algorithm

In this section, we propose a quantization approach for the numerical implementation of formulae in (2.1), (2.3) and (2.5) in Case **A** of Zakai equation. Here, those formulae are written as:

$$\begin{aligned} \bar{X}_{k+1} &= \bar{X}_k + \beta(\bar{X}_k)\delta + \sigma(\bar{X}_k)\Delta\bar{B}_{k+1} + \gamma(\bar{X}_k)\Delta\bar{W}_{k+1} \\ &=: F_\delta(\bar{X}_k, \Delta\bar{B}_{k+1}, \Delta\bar{W}_{k+1}), \qquad (3.1) \\ < \bar{V}_{k+1}, f > &= < \bar{V}_k, \exp\left(g(., \Delta\bar{W}_{k+1})\right) \bar{P}_{k+1,W} f > \qquad (3.2) \end{aligned}$$

for $k = 0, \ldots, n-1$, with

$$g(x, \Delta W) = h(x).\Delta W - \frac{1}{2}|h(x)|^2\delta, \qquad (3.3)$$

and $\bar{P}_{k+1,W}(x, dx')$ is a normal distribution with mean $x + \beta(x)\delta + \gamma(x)\Delta\bar{W}_{k+1}$ and variance $\sigma(x)\sigma^\intercal(x)\delta$.

We construct an approximation of $\bar{V}_k$ as follows. At each time t_k, $k = 0, \ldots, n$, we are given a grid $\Gamma_k = \{x_k^1, \ldots, x_k^{N_k}\}$ of N_k points in $\mathbb{R}^d$, associated to Voronoi tessellations $C_i(\Gamma_k)$, $i = 1, \ldots, N_k$:

$$C_i(\Gamma_k) \quad = \quad \left\{ u \in \mathbb{R}^d : |u - x_k^i| = \min_j |u - x_k^j| \right\}.$$

We then approximate the process $(\bar{X}_k)$ by the marginal quantized process $(\hat{X}_k)$ defined as:

$$\hat{X}_k \quad = \quad \mathrm{Proj}_{\Gamma_k}(\bar{X}_k) \; := \; \sum_{i=1}^{N_k} x_k^i 1_{\{\bar{X}_k \in C_i(\Gamma_k)\}}.$$

We thus define the conditional probability $\hat{P}_{k,W}$ of $\hat{X}_k$ given $\hat{X}_{k-1}$, and W. In other words, $\hat{P}_{k,W}$ is a (random) probability transition matrix $\{\hat{p}^{ij}_{k,W},\ i = 1, \ldots, N_{k-1}, j = 1, \ldots, N_k\}$ characterized by:

$$\hat{p}^{ij}_{k,W} \quad = \quad P_W \left[\hat{X}_k = x_k^j \middle| \hat{X}_{k-1} = x_k^i \right].$$

Finally, the random measure-valued process $(\bar{V}_k)$ is approximated by the discrete random measure process $(\hat{V}_k)$ defined by:

$$\begin{aligned} \hat{V}_0 \quad &= \quad \text{law of } \hat{X}_0, \\ < \hat{V}_{k+1}, f > \quad &= \quad < \hat{V}_k, \exp\left(g(., \Delta\bar{W}_{k+1})\right) \hat{P}_{k+1,W} f > . \end{aligned} \tag{3.4}$$

From an algorithmic viewpoint, this reads as:

$$\hat{V}_k \quad = \quad \sum_{i=1}^{N_k} \hat{v}_k^i \delta_{x_k^i}, \quad (\delta_x \text{ is the Dirac measure in } x)$$

for $k = 0, \ldots, n$, where the weights $\hat{v}_k^i$ are computed in a forward induction as:

$$\begin{aligned} \hat{v}_0^i \quad &= \quad \hat{p}_0^i \; := \; P[\hat{X}_0 = x_0^i] \; = \; P[\bar{X}_0 \in C_i(\Gamma_0)], \quad i = 1, \ldots, N_0, \\ \hat{v}_{k+1}^j \quad &= \quad \sum_{i=1}^{N_k} \hat{v}_k^i \hat{p}^{ij}_{k+1,W} \exp\left(g(x_k^i, \Delta\bar{W}_{k+1})\right), \quad j = 1, \ldots, N_{k+1}. \end{aligned}$$

The implementation of the above method requires optimally for each $k = 0, \ldots, n$:

- a grid Γ_k which minimizes the L^p-quantization error

$$\|\Delta_k\|_p \quad = \quad \|\bar{X}_k - \hat{X}_k\|_p$$

as well as an estimation of this error,

- the weights of the joint distribution $(\hat{X}_{k-1}, \hat{X}_k)$ and marginal distribution $\hat{X}_{k-1}$:

$$\begin{aligned} \hat{r}^{ij}_{k,W} \quad &= \quad P_W \left[\hat{X}_k = x_k^j,\ \hat{X}_{k-1} = x_{k-1}^i \right] \; = \; P_W \left[\bar{X}_k \in C_j(\Gamma_k),\ \bar{X}_{k-1} \in C_i(\Gamma_{k-1}) \right], \\ \hat{q}^i_{k-1,W} \quad &= \quad P_W \left[\hat{X}_{k-1} = x_k^i \right] \; = \; P_W \left[\bar{X}_{k-1} \in C_i(\Gamma_{k-1}) \right], \end{aligned}$$

for $i = 1, \ldots, N_{k-1}$, $j = 1, \ldots, N_k$, so that

$$\hat{p}^{ij}_{k,W} \;=\; \frac{\hat{r}^{ij}_{k,W}}{\hat{q}^{i}_{k-1,W}}.$$

This program is achieved as follows:

- Monte-Carlo simulation of M independent copies $(\bar{X}_0^{(m)}, \ldots, \bar{X}_n^{(m)})$ distributed according to $(\bar{X}_0, \ldots, \bar{X}_n)$.

- Recursive optimization of the grids $\Gamma_0, \ldots, \Gamma_n$ by a *Competitive Learning Vector Quantization* procedure and computation of the probability weights $\hat{r}^{ij}_{k,W}$ and $\hat{q}^{i}_{k-1,W}$, $k = 1, \ldots, n$. As a byproduct, we also have an estimation of the L^2 quantization errors $\|\Delta_k\|_2$, $k = 0, \ldots, n$.

3.2 Analysis of quantization error

The next theorem states an error estimation for the approximation of $\bar{V}_n$ under the following condition on the coefficients of the s.d.e X:

(H2) (i) The functions β, σ and γ are Lipschitz in x.
(ii) The function h is bounded and Lipschitz.

Theorem 3.1 *Under* **(H2)**, *for all* $p \in [1, +\infty)$ *and* $p' > p$, *there exists a positive constant* $C_{p,p'}$ *such that:*

$$\left\| \rho(\bar{V}_n, \hat{V}_n) \right\|_p \;\leq\; C_{p,p'} \frac{1}{\sqrt{\delta}} \sum_{k=0}^{n} \|\Delta_k\|_{p'}.$$

Remark 3.1 In view of Zador's theorem for the rate of convergence of minimal quantization error, the last theorem formally says that given a total number of N points to be dispatched among the n grids in time, we have a rate of convergence for $\left\| \rho(\bar{V}_n, \hat{V}_n) \right\|_p$ of order:

$$\frac{n^{\frac{1}{d}+\frac{3}{2}}}{N^{\frac{1}{d}}}.$$

We first need the following classic result about L^p-Lipschitz property of Euler schemes.

Lemma 3.1 *Let* G_δ *be a functional in the form:*

$$G_\delta(x, \varepsilon) \;=\; x + \delta B(x) + \sqrt{\delta} \Sigma(x) \varepsilon,$$

where B *and* Σ *are Lipschitz functions on* $\mathbb{R}^d$, *and* ε *is a Gaussian white noise. Then, for all* $p \in [1, \infty)$, *there exists a constant* C_p *such that for all* $x, x' \in \mathbb{R}^d$:

$$\left\| G_\delta(x, \varepsilon) - G_\delta(x', \varepsilon) \right\|_p \;\leq\; C_p (1 + \delta) |x - x'|.$$

We refer *e.g.* [26] for a detailed proof in a slightly more general setting where ε is only symmetric and lies in L^p.

One defines for every $k = 1, \dots, n$ the operator $\bar{H}_{k,W}$ by

$$\bar{H}_{k,W}(f)(x) = \exp g(x, \Delta \bar{W}_k)\bar{P}_{k,W}(f)(x), \quad \forall f \in BL_1(\mathbb{R}^d),\ \forall x \in \mathbb{R}^d,$$

where g is defined by (3.3). One defines

$$\bar{H}_{0,W}(f) \ = < \mu_0, f > .$$

One easily checks that (with the former notations)

$$< \bar{V}_k, f > = \ E_W(\bar{H}_{k,W}(f)(\bar{X}_{k-1})) \ = < \bar{V}_{k-1}, \bar{H}_{k,W}(f) >$$

so that, for every $k = 0, \dots, n$,

$$< \bar{V}_k, f > = \ (\bar{H}_{0,W} \circ \bar{H}_{1,W} \circ \cdots \circ \bar{H}_{k,W})(f).$$

This equality can be written either in forward or backward recursive form. The backward form will be an important tool for proofs:

$$\begin{aligned} \bar{U}_{n,W} f &:= f, \\ \bar{U}_{k-1,W} f &:= \bar{H}_{k,W}(\bar{U}_{k,W} f), \qquad k = 1, \dots n. \end{aligned} \tag{3.5}$$

then, one checks *using the Markov property and the iterated conditional expectation rule* that

$$\bar{U}_{0,W} f \ = < \bar{V}_n, f > .$$

For every $k = 1, \dots, n$, one approximates the operator $\bar{H}_{k,W}$ by its natural quantized counterpart $\widehat{H}_{k,W}$ defined on the grid $\Gamma_{k-1} = \{x^1_{k-1}, \dots, x^i_{k-1}, \dots, x^{N_{k-1}}_{k-1}\}$ by

$$\widehat{H}_{k,W}(f)(x^i_{k-1}) := \exp g(x^i_{k-1}, \Delta \bar{W}_k) \sum_j f(x^j_k) P_W(\widehat{X}_k = x^j_k \,|\, \widehat{X}_{k-1} = x^i_{k-1})$$

so that

$$\widehat{H}_{k,W}(f)(\widehat{X}_{k-1}) = \exp g(\widehat{X}_{k-1}, \Delta \bar{W}_k)\, E_W(f(\widehat{X}_k) \,|\, \widehat{X}_{k-1}).$$

Then, one sets

$$\widehat{H}_{0,W}(f) := \sum_j f(x^j_0) P_W(\widehat{X}_0 = x^j_0).$$

We then notice that the approximation of $\bar{V}_k$ defined in (3.4) satisfies:

$$< \widehat{V}_k, f > = \ (\widehat{H}_{0,W} \circ \widehat{H}_{1,W} \circ \cdots \circ \widehat{H}_{k,W})(f), \quad k = 1, \dots, n. \tag{3.6}$$

Once again, this equality can be read in backward form as follows:

$$\begin{aligned} \widehat{U}_{n,W} f(x^i_n) &:= f(x^i_n),\ i = 1, \dots, N_n, \\ \widehat{U}_{k-1,W} f(x^i_{k-1}) &:= \widehat{H}_{k,W}(\widehat{U}_{k,W} f)(x^i_{k-1}),\ i = 1, \dots, N_{k-1},\ k = 1, \dots n, \end{aligned} \tag{3.7}$$

so that

$$< \widehat{V}_n, f > \ = \ \widehat{U}_{0,W} f. \tag{3.8}$$

The proof is designed as follows: we wish to establish a backward induction between the error terms $\|\bar{U}_{k,W}f(\bar{X}_k) - \widehat{U}_{k,W}f(\widehat{X}_k)\|_p$ at successive times k and $k+1$ involving the quantization error $\|\bar{X}_{k+1} - \hat{X}_{k+1}\|_p$ of the Euler scheme. Unfortunately a naive approach makes the final error explode because of successive use of Holder inequality. So we are led to introduce a process $\bar{Y}_k$ starting at $\bar{X}_0$ but produced by a *biased* dynamics $G_{\delta,p}$ (instead of F_δ) which corresponds to a step-by-step discrete Girsanov (implicit) change of probability. Thus we can simultaneously take advantage of the martingale property of the Doléans exponential and of the independence property of the increments $\Delta\bar{W}_k$: it makes possible not to use Hölder Inequality at a crucial step (see (3.15) below) which would cause an explosion of the constants. Finally we use a revert Girsanov change of probability to come back to the quantization error of the original dynamics $(\bar{X}_k)$.

Proof of Theorem 3.1. We will assume for convenience that $\delta = T/n \in (0,1]$ throughout the proof.

STEP 1: BACKWARD INDUCTION ON THE ERROR $\|\bar{U}_{k,W}f(\bar{Y}_k) - \widehat{U}_{k,W}f(\widehat{Y}_k)\|_p$

Set temporarily

$$\begin{aligned}
G_{\delta,p}(y,v,w) &:= F_\delta(y,v,w+p\delta h(y)) \\
&= y + \delta(\beta(y) + p\gamma(y)h(y)) + \sigma(y)v + \gamma(y)w, \\
\bar{Y}_k &:= G_{\delta,p}(\bar{Y}_{k-1}, \Delta\bar{B}_k, \Delta\bar{W}_k),\ k \geq 1, \\
\bar{Y}_0 &= X_0,
\end{aligned}$$

and

$$\widetilde{Y}_k := F_\delta(\bar{Y}_{k-1}, \Delta\bar{B}_k, \Delta\bar{W}_k),\ k \geq 1.$$

Let $\bar{\mathcal{F}}_k$ denote the σ-field $\sigma(\Delta\bar{B}_\ell, \Delta\bar{W}_\ell, \ell = 1,\ldots,k)$. Set, for every $k = 0,\ldots,n$,

$$\hat{Y}_k := \mathrm{Proj}_{\Gamma_k}(\bar{Y}_k) \quad \text{and} \quad \widehat{\widetilde{Y}}_k := \mathrm{Proj}_{\Gamma_k}(\widetilde{Y}_k).$$

With these notations, one checks that for every $f \in BL_1(\mathbb{R}^d)$,

$$\bar{H}_{k,W}(f)(\bar{Y}_{k-1}) = \exp g(\bar{Y}_{k-1}, \Delta\bar{W}_k) E_W(f(\widetilde{Y}_k) \,|\, \bar{Y}_{k-1}) \tag{3.9}$$

and

$$\widehat{H}_{k,W}(f)(\widehat{Y}_{k-1}) = \exp g(\bar{Y}_{k-1}, \Delta\bar{W}_k) E_W(f(\widehat{\widetilde{Y}}_k) \,|\, \hat{Y}_{k-1}). \tag{3.10}$$

Consequently

$$\begin{aligned}
\bar{U}_{k-1,W}f(\bar{Y}_{k-1}) &- \widehat{U}_{k-1,W}f(\widehat{Y}_{k-1}) \\
&= \bar{H}_{k,W}(\bar{U}_{k,W}f)(\bar{Y}_{k-1}) - \widehat{H}_{k,W}(\widehat{U}_{k,W}f)(\widehat{Y}_{k-1}) \\
&= (\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) - E_W\left((\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) \,|\, \widehat{Y}_{k-1}\right) \\
&\quad + E_W\left(\bar{H}_{k,W}(\bar{U}_{k,W}f)(\bar{Y}_{k-1}) - \widehat{H}_{k,W}(\widehat{U}_{k,W}f)(\widehat{Y}_{k-1}) \,|\, \widehat{Y}_{k-1}\right).
\end{aligned}$$

Let us deal with the two above terms successively. The random vector $\widehat{Y}_{k-1}$ being a function of $\bar{Y}_{k-1}$ and conditional expectation $E(\,.\,|\,W, \widehat{Y}_{k-1})$ being an L^p-contraction, one gets

$$\begin{aligned}\left\|\bar{U}_{k-1,W}f(\bar{Y}_{k-1}) - E_W\left((\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) \;\mid\; \widehat{Y}_{k-1}\right)\right\|_p &\leq \left\|(\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) - (\bar{U}_{k-1,W}f)(\widehat{Y}_{k-1})\right\|_p \\ &+ \left\|E_W\left((\bar{U}_{k-1,W}f)(\widehat{Y}_{k-1}) - (\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) \,|\, \widehat{Y}_{k-1}\right)\right\|_p \\ &\leq 2\left\|(\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) - (\bar{U}_{k-1,W}f)(\widehat{Y}_{k-1})\right\|_p .\end{aligned}$$

Consequently, using the expressions (3.9) and (3.10) and once again the contraction property and the $\sigma(\bar{Y}_{k-1})$-measurability of $\hat{Y}_{k-1}$ yield

$$\begin{aligned}\left\|\bar{U}_{k-1,W}f(\bar{Y}_{k-1}) \right. &- \left. \widehat{U}_{k-1,W}f(\widehat{Y}_{k-1})\right\|_p \\ &\leq 2\left\|(\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) - (\bar{U}_{k-1,W}f)(\widehat{Y}_{k-1})\right\|_p \qquad (3.11) \\ &+ \left\|e^{g(\bar{Y}_{k-1},\Delta\bar{W}_k)}(\bar{U}_{k,W}f)(\widetilde{Y}_k) - e^{g(\widehat{Y}_{k-1},\Delta\bar{W}_k)}(\widehat{U}_{k,W}f)(\widehat{\widetilde{Y}}_k)\right\|_p\end{aligned}$$

(when $p = 2$, the 2 factor can be deleted). Let us deal now with the second term of the sum in the right hand side. The definition of $\bar{H}_{k,W}$ and the contraction property lead to

$$\begin{aligned}&\left\|e^{g(\bar{Y}_{k-1},\Delta\bar{W}_k)}(\bar{U}_{k,W}f)(\widetilde{Y}_k) - e^{g(\widehat{Y}_{k-1},\Delta\bar{W}_k)}(\widehat{U}_{k,W}f)(\widehat{\widetilde{Y}}_k)\right\|_p \\ \leq\; &\left\|\exp g(\bar{Y}_{k-1},\Delta\bar{W}_k)\left(\bar{U}_{k,W}f(\widetilde{Y}_k) - \exp\left(g(\widehat{Y}_{k-1},\Delta\bar{W}_k) - g(\bar{Y}_{k-1},\Delta\bar{W}_k)\right)\widehat{U}_{k,W}f(\widehat{\widetilde{Y}}_k)\right)\right\|_p .\end{aligned}$$

Set $L_p(\delta) := \exp\left((p-1)\|h\|_\infty^2\,\delta/2\right)$. A change of variable "à la Girsanov" yields for every nonnegative Borel function Θ and every $p \in (1,+\infty)$,

$$\begin{aligned}&\left\|\exp\left(g(\bar{Y}_{k-1},\Delta\bar{W}_k)\right)\Theta(\bar{Y}_{k-1},\Delta\bar{B}_k,\Delta\bar{W}_k)\right\|_p^p \\ \leq\; &(L_p(\delta))^p E\left(\exp\left(ph(\bar{Y}_{k-1}).\Delta\bar{W}_k\right) - p^2|h(\bar{Y}_{k-1})|^2\delta/2)\Theta^p(\bar{Y}_{k-1},\Delta\bar{B}_k,\Delta\bar{W}_k)\right) \\ \leq\; &(L_p(\delta))^p E\left(\Theta^p(\bar{Y}_{k-1},\Delta\bar{B}_k,\Delta\bar{W}_k + p\delta h(\bar{Y}_{k-1}))\right)\end{aligned}$$

so that

$$\left\|\exp(g(\bar{Y}_{k-1},\Delta\bar{W}_k))\Theta(\bar{Y}_{k-1},\Delta\bar{B}_k,\Delta\bar{W}_k)\right\|_p \leq L_p(\delta)\|\Theta(\bar{Y}_{k-1},\Delta\bar{B}_k,\Delta\bar{W}_k + p\delta h(\bar{Y}_{k-1})\|_p . \qquad (3.12)$$

Applying the above inequality with $\Theta(y,v,w) = (\bar{U}_{k,W}f)(G_{\delta,p}(y,v,w))$ leads to

$$\begin{aligned}&\left\|e^{g(\bar{Y}_{k-1},\Delta\bar{W}_k)}(\bar{U}_{k,W}f)(\widetilde{Y}_k) - e^{g(\widehat{Y}_{k-1},\Delta\bar{W}_k)}(\widehat{U}_{k,W}f)(\widehat{\widetilde{Y}}_k)\right\|_p \\ \leq\; &L_p(\delta)\left\|\left(\bar{U}_{k,W}f(\bar{Y}_k) - \exp\left(g(\widehat{Y}_{k-1},\Delta\bar{W}_k + p\delta h(\bar{Y}_{k-1})) - g(\bar{Y}_{k-1},\Delta\bar{W}_k + p\delta h(\bar{Y}_{k-1}))\right)\widehat{U}_{k,W}f(\widehat{Y}_k)\right)\right\|_p\end{aligned}$$

$$
\begin{aligned}
&\le L_p(\delta)\left\|\bar U_{k,W}f(\bar Y_k)-\widehat U_{k,W}f(\widehat Y_k)\right\|_p\\
&\quad +L_p(\delta)\left\|\left(1-\exp\left(g(\widehat Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))-g(\bar Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))\right)\right)\widehat U_{k,W}(f)(\widehat Y_k)\right\|_p\\
&\le L_p(\delta)\left\|\bar U_{k,W}f(\bar Y_k)-\widehat U_{k,W}f(\widehat Y_k)\right\|_p\\
&\quad +L_p(\delta)\left\|1-\exp\Big(g(\widehat Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))-g(\bar Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))\Big)\right\|_{rp}\left\|\widehat U_{k,W}f(\widehat Y_k)\right\|_{sp} \quad (3.13)
\end{aligned}
$$

where $r>1$ and $s=\frac{r}{r-1}$ are conjugate Holder exponents. Now

$$
\left\|\widehat U_{k,W}f(\widehat Y_k)\right\|_{sp} = \left\|\exp g(\widehat Y_k,\Delta\bar W_k)\,\widehat U_{k+1,W}f(\widehat Y_k)\right\|_{sp}.
$$

Applying (3.12) (with sp) yields

$$
\left\|\widehat U_{k,W}f(\widehat Y_k)\right\|_{sp}\le L_{sp}(\delta)\left\|\widehat U_{k+1,W}f(\widehat Y^{(sp)}_{k+1})\right\|_{sp}
$$

for some $\bar{\mathcal F}_{k+1}$-measurable random vector $\widehat Y^{(sp)}_{k+1}$ which we have no need to specify (since f is bounded). One derives by induction that

$$
\left\|\widehat U_{k,W}f(\widehat Y_k)\right\|_{sp}\le (L_{sp}(\delta))^{n-k}\|\widehat U_{n,W}f(\widehat Y^{(sp)}_n)\|_{sp}\le (L_{sp}(\delta))^{n-k}\|f\|_\infty\le C_{p,r,\|h\|_\infty,T}\|f\|_\infty \quad (3.14)
$$

with $K_{p,r,\|h\|_\infty,T}=\exp\left((sp-1)\|h\|^2_\infty T/2\right)$.

Let us deal now with the L^{rp}-norm of the exponential term. First temporarily set $\widehat\Delta_k(h):=h(\widehat Y_k)-h(\bar Y_k)$. Then, standard computations show that

$$
\begin{aligned}
&\left\|1-\exp\left(g(\widehat Y_{k-1},\Delta\bar W_k+p\,\delta\,h(\bar Y_{k-1}))-g(\bar Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))\right)\right\|_{rp}\\
&=\left\|1-\exp\left((p-1)\delta h(\bar Y_{k-1}).\widehat\Delta_{k-1}(h)+\widehat\Delta_{k-1}(h)\Delta\bar W_k-|\widehat\Delta_{k-1}(h)|^2\delta/2\right)\right\|_{rp}.
\end{aligned}
$$

Now using the elementary inequality $|e^x-1|\le|x|e^{x_+}$ where $x_+:=\max(x,0)$ and the fact that $x\mapsto x_+$ is non-decreasing yield

$$
\begin{aligned}
\|1 \;-\; &\exp\left(g(\widehat Y_{k-1},\Delta\bar W_k+p\,\delta\,h(\bar Y_{k-1}))-g(\bar Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))\right)\Big\|_{rp}\\
\le\;& \left\||\widehat\Delta_{k-1}(h)|\left|(p-1)\delta h(\bar Y_{k-1})+\Delta\bar W_k-(\widehat\Delta_{k-1}(h))\delta/2\right|\exp\left(2(p-1)\delta\|h\|^2_\infty+2\|h\|_\infty|\Delta\bar W_k|\right)\right\|_{rp}\\
\le\;& L_{4p-3}(\delta)\sqrt{\delta}[h]_{\rm Lip}\left\||\bar Y_{k-1}-\widehat Y_{k-1}|\left((p-1)\sqrt{\delta}\|h\|_\infty)+|Z_k|+\|h\|_\infty\sqrt{\delta}\right)\exp\left(2\|h\|_\infty\sqrt{\delta}|Z_k|\right)\right\|_{rp}
\end{aligned}
$$

where $Z_k:=\frac{\Delta\bar W_k}{\sqrt{\delta}}$ is a $\mathcal N(0;I_d)$ random vector *independent* of $\bar{\mathcal F}_{k-1}$. Finally

$$
\begin{aligned}
&\|1-\exp\left(g(\widehat Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))-g(\bar Y_{k-1},\Delta\bar W_k+p\delta h(\bar Y_{k-1}))\right)\Big\|_{rp}\\
&\qquad\le C_{p,r,\delta,\|h\|_\infty,T}\sqrt{\delta}[h]_{\rm Lip}\|\widehat Y_{k-1}-\bar Y_{k-1}\|_{rp} \quad (3.15)
\end{aligned}
$$

with

$$C_{p,r,\delta,\|h\|_\infty,T} = L_{4p-3}(\delta)\left\|\left((p-1)\sqrt{\delta}\|h\|_\infty) + |Z| + \sqrt{\delta}\|h\|_\infty\right)\exp\left(2\|h\|_\infty\sqrt{\delta}|Z|\right)\right\|_{rp}.$$

(Note that this real constant is increasing as a function of δ.) Plugging the estimates in (3.15) and (3.14) into (3.13) yields for every $k = 1, \ldots, n$,

$$\begin{aligned}\left\|e^{g(\bar{Y}_{k-1},\Delta\bar{W}_k)}(\bar{U}_{k,W}f)(\widetilde{Y}_k) - e^{g(\widehat{Y}_{k-1},\Delta\bar{W}_k)}(\widehat{U}_{k,W}f)(\widehat{\widetilde{Y}}_k)\right\|_p &\le L_p(\delta)\left\|\bar{U}_{k,W}f(\bar{Y}_k) - \widehat{U}_{k,W}f(\widehat{Y}_k)\right\|_p \\ &\quad + B(\delta)\|\bar{Y}_{k-1} - \widehat{Y}_{k-1}\|_{rp} \qquad (3.16)\end{aligned}$$

with $B(\delta) := C_{p,r,\|h\|_\infty,T}\sqrt{\delta}[h]_{Lip}\|f\|_\infty$ (with $C_{p,r,\|h\|_\infty,T} = C_{p,r,1,\|h\|_\infty,T}K_{p,r,\|h\|_\infty,T}L_p(1)$).

Now let us pass to the first term in the right hand side of (3.11). Let $(\bar{Y}_\ell^{k,y})_{\ell=k,\ldots,n}$ be the sequence obtained by iterating $G_{p,\delta}(., \Delta\bar{B}_\ell, \Delta\bar{W}_\ell)$ from y at time $\ell = k$ *i.e.*

$$\forall\, \ell \in \{k+1, \ldots, n\}, \qquad \bar{Y}_\ell^{k,y} = G_{p,\delta}(\bar{Y}_{\ell-1}^{k,y}, \Delta\bar{B}_\ell, \Delta\bar{W}_\ell), \quad \bar{Y}_k^{k,y} := y.$$

The same proof as above shows that, for any couple (Z_{k-1}, Z'_{k-1}) of $\bar{\mathcal{F}}_{k-1}$-measurable L^p-integrable random variables

$$\begin{aligned}\left\|(\bar{U}_{k-1,W}f)(Z_{k-1}) - (\bar{U}_{k-1,W}f)(Z'_{k-1})\right\|_p &\le L_p(\delta)\|\bar{U}_{k,W}(\bar{Y}_k^{k-1,Z_{k-1}}) - \bar{U}_{k,W}(\bar{Y}_k^{k-1,Z'_{k-1}})\|_p \\ &\quad + B(\delta)\|\bar{Y}_{k-1}^{k-1,Z_{k-1}} - \bar{Y}_{k-1}^{k-1,Z'_{k-1}}\|_{rp}\end{aligned}$$

so that by induction,

$$\begin{aligned}\left\|(\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) - (\bar{U}_{k-1,W}f)(\widehat{Y}_{k-1})\right\|_p &\le B(\delta)\sum_{\ell=k}^{n}(L_p(\delta))^{\ell-k}\|\bar{Y}_{\ell-1}^{k-1,\bar{Y}_{k-1}} - \bar{Y}_{\ell-1}^{k-1,\widehat{Y}_{k-1}}\|_{rp} \\ &\quad + (L_p(\delta))^{n+1-k}[f]_{\mathrm{Lip}}\|\bar{Y}_n^{k-1,\bar{Y}_{k-1}} - \bar{Y}_n^{k-1,\widehat{Y}_{k-1}}\|_{rp}.\end{aligned}$$

Now, Lemma 3.1 (applied to $G_{\delta,p}$) implies the existence of a real constant $C_{rp} > 0$ such that

$$\|\bar{Y}_\ell^{k-1,\bar{Y}_{k-1}} - \bar{Y}_\ell^{k-1,\widehat{Y}_{k-1}}\|_{rp} \le (1 + C_{rp}\delta)^{\ell+1-k}\|\bar{Y}_{k-1} - \widehat{Y}_{k-1}\|_{rp}.$$

Setting $L'_{p,r}(\delta) = L_p(\delta)(1 + C_{rp}\delta)$ finally yields for every $k = 1, \ldots, n$,

$$\begin{aligned}\left\|(\bar{U}_{k-1,W}f)(\bar{Y}_{k-1}) - (\bar{U}_{k-1,W}f)(\widehat{Y}_{k-1})\right\|_p &\le C(\delta)\|\bar{Y}_{k-1} - \widehat{Y}_{k-1}\|_{2p} \\ \text{with} \qquad C(\delta) &= L_p(T)e^{C_{rp}}\left(C_{p,r,\|h\|_\infty,T}\frac{[h]_{\mathrm{Lip}}\|f\|_\infty\sqrt{\delta}}{L'_{p,r}(\delta) - 1} + [f]_{\mathrm{Lip}}\right). \qquad (3.17) \\ &\le L_p(T)e^{C_{rp}}\left(C'_{p,r,\|h\|_\infty,T}\frac{[h]_{\mathrm{Lip}}\|f\|_\infty}{\sqrt{\delta}} + [f]_{\mathrm{Lip}}\right). \qquad (3.18)\end{aligned}$$

Plugging (3.16) and (3.17) into (3.11) finally yields the induction

$$\begin{aligned}\left\|\bar{U}_{k-1,W}f(\bar{Y}_{k-1}) - \widehat{U}_{k-1,W}f(\widehat{Y}_{k-1})\right\|_p &\le L_p(\delta)\left\|\bar{U}_{k,W}f(\bar{Y}_k) - \widehat{U}_{k,W}f(\widehat{Y}_k)\right\|_p \\ &\quad + A(\delta)\|\bar{Y}_{k-1} - \widehat{Y}_{k-1}\|_{rp}\end{aligned}$$

with

$$
\begin{aligned}
A(\delta) &= B(\delta) + 2\,C(\delta) \leq C''_{p,r,\|h\|_\infty,T}\left([h]_{\rm Lip}\|f\|_\infty(\sqrt{\delta}+\frac{1}{\sqrt{\delta}}) + [f]_{\rm Lip}\right)\\
&\leq \frac{C_{p,r,\|h\|_\infty,[h]_{\rm Lip},\|f\|_\infty,[f]_{\rm Lip},T}}{\sqrt{\delta}}
\end{aligned}
$$

since $\delta\in(0,1]$. A new induction leads to

$$
\begin{aligned}
\left\| <\bar{V}_n, f> - <\hat{V}_n, f>\right\|_p &= \left\|\bar{U}_{0,W} f(\bar{X}_0) - \widehat{U}_{0,W} f(\widehat{X}_0)\right\|_p\\
&= \left\|\bar{U}_{0,W} f(\bar{Y}_0) - \widehat{U}_{0,W} f(\widehat{Y}_0)\right\|_p\\
&\leq A(\delta)\sum_{k=0}^{n}(L_p(\delta))^k\|\bar{Y}_k - (\widehat{U}_{n,W}f)(\widehat{Y}_n)\|_{rp} + (L_p(\delta))^n\|(\bar{U}_{n,W}f)(\bar{Y}_n) - \widehat{Y}_k\|_p\\
&\leq \frac{C_{p,r,\|h\|_\infty,[h]_{\rm Lip},\|f\|_\infty,[f]_{\rm Lip},T}}{\sqrt{\delta}}\sum_{k=0}^{n}\|\bar{Y}_k - \widehat{Y}_k\|_{rp} + L_p(T)[f]_{\rm Lip}\|\bar{Y}_n - \widehat{Y}_n\|_{rp}. \quad (3.19)
\end{aligned}
$$

STEP 2 (GLOBAL REVERT GIRSANOV TRANSFORM): Now, we aim to come back to $\bar{X}_k$ by introducing a revert Girsanov transform:

$$
\|\bar{Y}_k - \widehat{Y}_k\|_{rp}^{rp} = E\left(Z_k(Z_k)^{-1}|\bar{Y}_k - \widehat{Y}_k|^{rp}\right)
$$

where

$$
Z_k = \exp\left(-\sum_{\ell=1}^{k} ph(\bar{Y}_{\ell-1}).\Delta\bar{W}_\ell - p^2|h(\bar{Y}_{\ell-1})|^2\delta/2\right).
$$

It follows that

$$
E\left(Z_k(Z_k)^{-1}|\bar{Y}_k - \widehat{Y}_k|^{rp}\right) = E\left(\exp\left(\sum_{\ell=1}^{k} ph(\bar{X}_{\ell-1}).\Delta\bar{W}_\ell - p^2|h(\bar{X}_{\ell-1})|^2\delta/2\right)|\bar{X}_k - \widehat{X}_k|^{rp}\right)
$$

so that by the Holder inequality applied with two conjugate exponents r', $s'>1$,

$$
\begin{aligned}
\|\bar{Y}_k - \widehat{Y}_k\|_{rp}^{rp} &\leq \left(E\exp\left(\sum_{\ell=1}^{k} s'ph(\bar{X}_{\ell-1}).\Delta\bar{W}_\ell - s'p^2|h(\bar{X}_{\ell-1})|^2\delta/2\right)\right)^{1/s'}\left(E|\bar{X}_k - \widehat{X}_k|^{rr'p}\right)^{1/r'}\\
&\leq \exp\left(k(s'-1)p^2\|h\|_\infty^2\,\delta/2\right)\|\bar{X}_k - \widehat{X}_k\|_{rr'p}^{rp}.
\end{aligned}
$$

Finally

$$
\|\bar{Y}_k - \widehat{Y}_k\|_{rp} \leq \exp\left(kp\|h\|_\infty^2\,\delta/4\right)\|\bar{X}_k - \widehat{X}_k\|_{4p} \leq C_{p,r,r',\|h\|_\infty,T}\|\bar{X}_k - \widehat{X}_k\|_{rr'p}.
$$

One completes the proof by setting $r = r' = \sqrt{p'/p} > 1$ and plugging this last inequality into (3.19). $\diamond$

4 Simulation of Stochastic McKean-Vlasov equation and quantization error

4.1 The quantization algorithm

In this case, formulae (2.1), (2.3) are written as:

$$\begin{aligned}
\bar{X}_{k+1} &= \bar{X}_k + \int \tilde{\beta}(\bar{X}_k, y)\bar{V}_k(dy)\delta + \int \tilde{\sigma}(\bar{X}_k, y)\bar{V}_k(dy)\Delta\bar{B}_{k+1} \\
&\quad + \int \tilde{\gamma}(\bar{X}_k, y)\bar{V}_k(dy)\Delta\bar{W}_{k+1} \\
&:= F(\bar{X}_k, \bar{V}_k, \Delta\bar{B}_{k+1}, \Delta\bar{W}_{k+1}), \\
< \bar{V}_k, f > &= E_W\left[f(\bar{X}_k)\right].
\end{aligned}$$

The last relation means that $\bar{V}_k$ is the conditional distribution of $\bar{X}_k$ given W. For a fixed trajectory of W, we construct an approximation of $\bar{V}_k$ as follows. At each time t_k, $k = 0, \ldots, n$, we are given a grid $\Gamma_k = \{x_k^1, \ldots, x_k^{N_k}\}$ of N_k points in $\mathbb{R}^d$, associated to Voronoi tessellations $C_i(\Gamma_k)$, $i = 1, \ldots, N_k$. We then approximate $(\bar{X}_k, \bar{V}_k)$ by a quantization algorithm defined by:

$$\begin{aligned}
\tilde{X}_0 &= \bar{X}_0, \\
\hat{V}_0 &= \text{probability distribution of } \hat{X}_0 = \mathrm{Proj}_{\Gamma_0}(\tilde{X}_0),
\end{aligned}$$

and for $k = 1, \ldots, n$:

$$\begin{aligned}
\tilde{X}_k &= F(\tilde{X}_{k-1}, \hat{V}_{k-1}, \Delta\bar{B}_k, \Delta\bar{W}_k), \\
\hat{V}_k &= \text{probability distribution of } \hat{X}_k = \mathrm{Proj}_{\Gamma_k}(\tilde{X}_k).
\end{aligned}$$

The implementation of the above method requires optimally for each $k = 0, \ldots, n$:

- a grid Γ_k which minimizes the L^p-quantization error

$$\|\Delta_k\|_p = \|\tilde{X}_k - \hat{X}_k\|_p$$

as well as an estimation of this error,

- the weights of the discrete probability distribution $\hat{V}_k = \sum_{i=1}^{N_k} v_k^i \delta_{x_k^i}$:

$$v_k^i = P\left[\hat{X}_k = x_k^i\right] = P\left[\tilde{X}_k \in C_i(\Gamma_k)\right].$$

This program is achieved by successive stochastic gradient descent methods, known as *Competitive Learning Vector Quantization* algorithm, based on Monte-Carlo simulation of $(\tilde{X}_k)$:

► $k = 0$:

- simulation (and storing until time $k = 1$) of M independent copies $\tilde{X}_0^{(m)}$ distributed according to $\bar{X}_0$.

- optimization of the grid Γ_0 by a *CLVQ* procedure and computation of the probability weights v_0^i, $i = 1, \ldots, N_0$ of $\hat{X}_0 = \mathrm{Proj}_{\Gamma_0}(\tilde{X}_0)$. As a byproduct, we also have an estimation of the L^2 quantization error $\|\Delta_0\|_2$.

▶ $k = 1, \dots, n$:
- for every $m = 1, \dots, M$, one simulates and stores until next time $k+1$

$$\tilde{X}_k^{(m)} = F(\tilde{X}_{k-1}^{(m)}, \hat{V}_{k-1}, \Delta\bar{B}_k^{(m)}, \Delta\bar{W}_k).$$

- optimization of the grid Γ_k by a *CLVQ* procedure and computation of the probability weights v_k^i, $i = 1, \dots, N_k$ of $\hat{X}_k = \mathrm{Proj}_{\Gamma_k}(\tilde{X}_k)$. As a byproduct, we also have an estimation of the L^2 quantization error $\|\Delta_k\|_2$.

SOME COMMENTS: the usual asset of quantization based algorithms (including the one proposed here to solve Zakai equation) is that a significant part of the computation can be kept off-line. Then the implementation of the procedure for a given function f is almost instantaneous; this is no longer the case here. However it remains an attractive procedure in comparison with particle algorithms because of its lower complexity. This comes once again from the quantization feature. Let us be more specific: At every time step k, the main task of the CLVQ algorithm is to search the closest neighbour of $X_k^{(m)}$ among the N_k points of the grid Γ_k in order to update it. In some way this phase corresponds to the interaction phase in particle algorithms. The complexity of such a procedure when appropriately implemented is $O(\log(N_k))$ *in average* (see [11]) (and $O(N_k)$ in case of a naive search). Then, one has to simulate M independent copies of the Euler scheme at time $k+1$ based on (2.1) with a cost $M \times N_k$ (due to the computation of M integrals with respect to $\widehat{V}_k$). So the global complexity induced by time step k is upper-bounded by $C \times M \times N_k$ where C is a real constant not depending on k. The resulting global complexity behaves like

$$C \times M \times \overline{N} \times n$$

where $\overline{N} = (N_0 + N_1 + \cdots + N_n)/(n+1)$ is the average number of elementary quantizers used per time step for the quantization of the measures $\widehat{V}_k$. This is to be compared to the complexity of an algorithm based on interacting particles like the one implemented in [7] which is proportional to

$$M \times M \times n$$

in full generality.

4.2 Analysis of quantization error

The next theorem states an error estimation for the approximation of $\bar{V}_k$ by $\hat{V}_k$. We make the following condition on the coefficients of the s.d.e X:

(H3) The functions $\tilde{\beta}$, $\tilde{\sigma}$ and $\tilde{\gamma}$ are bounded and Lipschitz in x, uniformly in y: there exists some positive constant C such that

$$\begin{aligned} |\tilde{\beta}(x,y)| + |\tilde{\sigma}(x,y)| + |\tilde{\gamma}(x,y)| &\leq C, \\ |\tilde{\beta}(x,y) - \tilde{\beta}(x',y)| + |\tilde{\sigma}(x,y) - \tilde{\sigma}(x',y)| + |\tilde{\gamma}(x,y) - \tilde{\gamma}(x',y)| &\leq C|x - x'|, \end{aligned}$$

for all $x, x', y \in \mathbb{R}^d$.

Theorem 4.1 *Under* **(H3)**, *for all* $p \in [1, \infty)$, *there exists a positive constant* C_p *such that:*

$$\left\| \rho(\bar{V}_n, \hat{V}_n) \right\|_{2p} \;\leq\; \|\Delta_n\|_{2p} \;+\; C_p \delta^{\frac{1}{2p}} \sum_{k=0}^{n-1} \|\Delta_k\|_{2p} .$$

Remark 4.1 In view of the Zador-Wise theorem for the rate of convergence of minimal quantization error, the last theorem formally says that given a total number of N points to be dispatched among the n grids in time, we have a rate of convergence for $\left\| \rho(\bar{V}_n, \hat{V}_n) \right\|_{2p}$ of order:

$$\frac{n^{1+\frac{1}{d}-\frac{1}{2p}}}{N^{\frac{1}{d}}}.$$

Proof of Theorem 4.1.
Step 1. We set for all $x \in \mathbb{R}^d$ and $v \in \mathcal{M}(\mathbb{R}^d)$

$$\beta(x,v) \;=\; \int \tilde{\beta}(x,y)v(dy), \;\; \sigma(x,v) \;=\; \int \tilde{\sigma}(x,y)v(dy), \;\; \gamma(x,v) \;=\; \int \tilde{\gamma}(x,y)v(dy),$$

and we notice that under **(H3)**, the following Lipschitz condition holds: there exists some positive constant C such that

$$\begin{aligned} & |\beta(x,v) - \beta(x',v')| + |\sigma(x,v) - \sigma(x',v')| + |\gamma(x,v) - \gamma(x',v')| \\ \leq \;\; & C \left[|x - x'| + \rho(v,v') \right], \end{aligned} \tag{4.1}$$

for all $x, x' \in \mathbb{R}^d$, $v, v' \in \mathcal{M}(\mathbb{R}^d)$.

We consider the continuous Euler scheme associated to $(\bar{X}_k)$ and $(\tilde{X}_k)$: It is written for all $t \in [t_k, t_{k+1}]$, $k = 0, \ldots, n-1$, as

$$\begin{aligned} X_t^\delta &= \bar{X}_k + \beta(\bar{X}_k, \bar{V}_k)(t - t_k) + \sigma(\bar{X}_k, \bar{V}_k)(B_t - B_{t_k}) + \gamma(\bar{X}_k, \bar{V}_k)(W_t - W_{t_k}) \\ \tilde{X}_t^\delta &= \tilde{X}_k + \beta(\tilde{X}_k, \hat{V}_k)(t - t_k) + \sigma(\tilde{X}_k, \hat{V}_k)(B_t - B_{t_k}) + \gamma(\tilde{X}_k, \hat{V}_k)(W_t - W_{t_k}). \end{aligned}$$

We denote $D_t = X_t^\delta - \tilde{X}_t^\delta$. Applying Itô's formula to $|D|^{2p}$ between t_k and $t \in [t_k, t_{k+1}]$, standard computations as for the estimation of L^p-moments of s.d.e. show the existence of some positive constant C_p such that:

$$\begin{aligned} E_k|D_t|^{2p} \;\; \leq \;\; & |\bar{X}_k - \tilde{X}_k|^{2p} + C_p \int_{t_k}^t E_k|D_u|^{2p} du \\ & + C_p \int_{t_k}^t E_k \left[|\beta(\bar{X}_k, \bar{V}_k) - \beta(\tilde{X}_k, \hat{V}_k)|^{2p} + |\sigma(\bar{X}_k, \bar{V}_k) - \sigma(\tilde{X}_k, \hat{V}_k)|^{2p} \right] du \\ & + C_p \int_{t_k}^t E_k \left[|\gamma(\bar{X}_k, \bar{V}_k) - \gamma(\tilde{X}_k, \hat{V}_k)|^{2p} \right] du. \end{aligned}$$

Here E_k denotes the conditional expectation given $\mathcal{F}_k$. From the Lipschitz condition (4.1), we then have:

$$E_k|D_t|^{2p} \;\; \leq \;\; (1 + C_p\delta)|\bar{X}_k - \tilde{X}_k|^{2p} + C_p\delta|\rho(\bar{V}_k, \hat{V}_k)|^{2p} + C_p \int_{t_k}^t E_k|D_u|^{2p} du.$$

Here and in the sequel, C_p denotes a generic constant dependent of p (and independent of δ) which may change along the different lines. By Gronwall lemma, and recalling that $D_{t_{k+1}} = \bar{X}_{k+1} - \tilde{X}_{k+1}$, we get:

$$E_k|\bar{X}_{k+1} - \tilde{X}_{k+1}|^{2p} \quad \leq \quad (1 + C_p\delta)|\bar{X}_k - \tilde{X}_k|^{2p} + C_p\delta|\rho(\bar{V}_k, \hat{V}_k)|^{2p}.$$

This clearly implies

$$\left\|\bar{X}_{k+1} - \tilde{X}_{k+1}\right\|_{2p}^{2p} \quad \leq \quad (1 + C_p\delta)\left\|\bar{X}_k - \tilde{X}_k\right\|_{2p}^{2p} + C_p\delta\left\|\rho(\bar{V}_k, \hat{V}_k)\right\|_{2p}^{2p}. \tag{4.2}$$

Step 2. For any $f \in BL_1(\mathbb{R}^d)$, we have

$$\begin{aligned}\left|< \bar{V}_k, f > - < \hat{V}_k, f >\right| &= \left|E_W\left[f(\bar{X}_k) - f(\hat{X}_k)\right]\right| \\ &\leq E_W|\bar{X}_k - \hat{X}_k|,\end{aligned}$$

so that by Jensen's inequality and the law of iterated conditional expectation:

$$\begin{aligned}\left\|\rho(\bar{V}_k, \hat{V}_k)\right\|_{2p} &\leq \left\|\bar{X}_k - \hat{X}_k\right\|_{2p} \\ &\leq \left\|\bar{X}_k - \tilde{X}_k\right\|_{2p} + \|\Delta_k\|_{2p}.\end{aligned} \tag{4.3}$$

Substituting this last inequality into (4.2) and using the elementary relation $(a+b)^{2p} \leq C_p(a^{2p} + b^{2p})$, we obtain:

$$\left\|\bar{X}_{k+1} - \tilde{X}_{k+1}\right\|_{2p}^{2p} \quad \leq \quad (1 + C_p\delta)\left\|\bar{X}_k - \tilde{X}_k\right\|_{2p}^{2p} + C_p\delta\|\Delta_k\|_{2p}^{2p}.$$

By induction and recalling that $\tilde{X}_0 = \bar{X}_0$, we get:

$$\begin{aligned}\left\|\bar{X}_n - \tilde{X}_n\right\|_{2p}^{2p} &\leq C_p\delta\sum_{k=0}^{n-1}(1 + C_p\delta)^{n-1-k}\|\Delta_k\|_{2p}^{2p} \\ &\leq C_p\delta\sum_{k=0}^{n-1}\|\Delta_k\|_{2p}^{2p}.\end{aligned}$$

By using the elementary inequality $(a+b)^{\frac{1}{2p}} \leq a^{\frac{1}{2p}} + b^{\frac{1}{2p}}$, this yields

$$\left\|\bar{X}_n - \tilde{X}_n\right\|_{2p} \quad \leq \quad C_p\delta^{\frac{1}{2p}}\sum_{k=0}^{n-1}\|\Delta_k\|_{2p}.$$

Plugging finally this last inequality into (4.3) proves the required result. □

5 Numerical simulations and estimation of the rates of convergence

5.1 Zakai equation in the linear case

We consider the linear case:

$$\begin{aligned}\beta(x) &= (A - \Gamma H)x, & h(x) &= Hx, \\ \gamma(x) &= \Gamma, & \sigma(x) &= \Sigma,\end{aligned}$$

where A, Γ, Σ and H are constant matrices of appropriate dimensions. We also suppose that μ_0 is a Gaussian law with mean m_0 and covariance matrix R_0. Then it is well-known that the solution to the Zakai equation (1.6) is explicitly given by:

$$< V_t, f > \quad = \quad \left[\int f(\hat{m}_t + R(t)^{\frac{1}{2}} x) \frac{\exp\left(-\frac{1}{2}|x|^2\right)}{(2\pi)^{\frac{d}{2}}} dx \right] < V_t, 1 >, \tag{5.1}$$

where $R(t)$ is the solution to the Riccati equation:

$$\begin{aligned} \frac{dR}{dt} &= AR + RA^\intercal + \Sigma\Sigma^\intercal + \Gamma\Gamma^\intercal - (RH^\intercal + \Gamma)(HR + \Gamma^\intercal), \\ R(0) &= R_0, \end{aligned} \tag{5.2}$$

$\hat{m}_t$ is solution of:

$$\begin{aligned} d\hat{m}_t &= A\hat{m}_t dt + (RH^\intercal + \Gamma)(dW_t - H\hat{m}_t dt), \\ \hat{m}_0 &= m_0, \end{aligned} \tag{5.3}$$

and

$$< V_t, 1 > \quad = \quad \exp\left(\int_0^t H\hat{m}_s.dW_s - \frac{1}{2} \int_0^t |H\hat{m}_s|^2 ds \right). \tag{5.4}$$

In other words, the normalized measure π_t defined by

$$< \pi_t, f > \quad = \quad \frac{< V_t, f >}{< V_t, 1 >},$$

is a Gaussian distribution with mean $\hat{m}_t$ and variance $R(t)$.

We introduce now the quantized normalized filter for a given function $f \in BL_1(\mathbb{R})$ as

$$< \hat{\pi}_k^\delta, f >:= \frac{< \hat{V}_k, f >}{< \hat{V}_k, 1 >}, \quad k = 0, \ldots, n,$$

where we have emphasized the dependence of the filter in $\delta = T/n$ by a superscript. The unnormalized filters $\hat{V}_k$ are computed according to algorithm (3.4).

The exact normalized filter is approximated owing to (5.1) using the following way. Since R is an explicitly known function (solution of (5.2)), it is sufficient to approximate $\hat{m}_t$, solution of the SDE (5.3) with a refined Euler scheme of step

$$\delta_{ref} = \frac{T}{1024} \ll \delta.$$

Indeed, for each path of the observation W, (5.3) and (5.4) are discretized as

$$\bar{m}_{l+1} \quad = \quad \bar{m}_l + \delta_{ref} A\bar{m}_l + (R(l\delta_{ref})H^\intercal + \Gamma)(W_{(l+1)\delta_{ref}} - W_{l\delta_{ref}} - H\bar{m}_l\delta_{ref}), \tag{5.5}$$

$$\bar{Z}_{l+1} \quad = \quad \bar{Z}_l + H\bar{m}_l.(W_{(l+1)\delta_{ref}} - W_{l\delta_{ref}}) - \frac{1}{2}|H\bar{m}_l|^2\delta_{ref}, \quad \bar{\xi}_l = \exp(\bar{Z}_l), \tag{5.6}$$

and so a very close approximation of the exact normalized filter is

$$< \pi_{l\delta_{ref}}^{\delta_{ref}}, f >:= \int f(\bar{m}_l + R(l\delta_{ref})^{\frac{1}{2}} x) \frac{\exp\left(-\frac{1}{2}|x|^2\right)}{(2\pi)^{\frac{d}{2}}} dx,$$

where $R(t)$ is computed owing to an exact quadrature formula.

We now estimate the rate of convergence of the scheme with respect to the spatial and time discretization. In order to remove undesirable time oscillations of the error, we focus on the following temporal mean of the quadratic quantization error for the normalized filter, namely

$$\mathrm{E}rr(\delta, \bar{N}) = \frac{1}{n} E \sum_{k=0}^{n} \left| < \hat{\pi}_k^{\delta}, f > - < \pi_{t_k}^{\delta_{ref}}, f > \right|^2, \tag{5.7}$$

where $t_k = k\delta = l(k)\delta_{ref}$.

We test the error for the following test functions:

$$f_0(x) \; = \; x, \quad f_1(x) = \exp(-x^2), \quad f_2(x) = \exp(-x). \tag{5.8}$$

The expectation in (5.7) is computed by a Monte Carlo method with $M = 100$ trajectories of the observations W.

The parameters of our simulations are

$$\Sigma = 1, \quad B = -0.5, \quad H = 1, \quad T = 1.$$

Such a choice of parameter is motivated by the fact that it provides not too small values for $R(t)$. Otherwise, there would not be enough points around $m_0 = 0$ in order to be able to "capture" the behaviour of the signal around its mean 0.

We will also change a bit the model and consider the following equations:

$$\begin{cases} dX_t = BX_t\, dt + \Sigma\, dB_t + \Gamma dW_t, \\ \\ dW_t = HX_t dt + \varepsilon dU_t. \end{cases} \tag{5.9}$$

The formulæ above need to be changed as follows $\Gamma \rightsquigarrow \varepsilon\Gamma$ and $H \rightsquigarrow H/\varepsilon$. The reason for introducing this new degree of freedom on the noise level may look paradoxical since small ε will provide large errors. But precisely, these large errors make it possible to display the rate of convergence more efficiently than with $\varepsilon = 1$ which produces smaller errors. Indeed, we will see that as the discretization parameters δ (resp. $\bar{N}$) get smaller and smaller the error $\mathrm{Err}(\delta, \bar{N})$ is decreasing as a function of δ (resp. $\bar{N}$) until some threshold depending on $\bar{N}$ (resp. δ) and in the number M of observations (*i.e.* paths of W). Beyond this threshold, the error becomes more or less constant because the difference with the exact solution will be of the same order of the spatial discretization (resp. temporal discretization). Subsequently the sum of the two errors will become indistinguishable from the spatial one (resp. temporal one). Therefore a small ε will provide bigger errors and so we will have more relevant points before reaching this threshold.

• **Estimation of the spatial discretization rate.** We first estimate the spatial rate of convergence in the case $\Gamma = 0$ (no correlation between the signal process X and the observation process W). For four values of $n = 1/\delta \in \{16, 32, 64, 128\}$, we estimate $\bar{N} \mapsto Err(\delta, \bar{N})$ with $\bar{N} = 2^{-\ell}$, $\ell = 1, \ldots, 7$. As a first step, for each value of n and of $\bar{N}$, we compute an optimal quantization $(\hat{X}_k)_k$ of the Euler scheme $(\bar{X}_k)_k$ of (5.9) (which is a version of (3.1)), according to the algorithm described in subsection 3.1. Then, for each test function f in (5.8) and each observation path of W, we compute recursively $< \hat{V}^\delta_k, f >$ and $< \hat{V}^\delta_k, 1 >$ using (3.4) and then $< \hat{\pi}^\delta_k, f >$. On the other hand, we compute the exact solutions using (5.5) and finally we compute $Err(\delta, \bar{N})$ as defined by(5.7) by summing up over the M trajectories sampled from the observation process W.

Note that since $\Gamma = 0$, the quantization optimization procedure of $(X_k)_k$ is a one shot process which does not depend on the observations W.

The results are summarized in Figure 1. In this case $\varepsilon = 0.1$. We have plotted $\log \mathrm{Err}(\delta, \bar{N})$ against $\log(\bar{N})$. It shows that the rate of convergence of the square root of the error (5.7) with respect to $\bar{N}$ toward 0 seems to behave like $O(1/\bar{N})$. This remains true for all the four selected time steps δ as well as for all the test functions (5.8). This is in accordance with the results given by Theorem 3.1.

In Figure 2 are depicted the same curves for $n = 256$. The slope of the lines seems to be closer to 1 than 2. This could suggest a slower rate of convergence $O((\bar{N})^{-1/2})$. In fact, this emphasizes that the scheme needs some stability criterion involving n and $\bar{N}$ in order to converge at the true rate $O(1/\bar{N})$. The quantization step of the algorithm can also be the cause of this rate. Indeed, during the quantization optimization of the signal X, we need to simulate at each time step an Euler increment of X in (5.9). This simulation is used to compute the weights of the “quantization tree" of X (weight of the Voronoi cells and the transition probabilities) and to process the optimization. Here the Euler increment of X, namely $\Sigma \sqrt{\delta}\, \chi$ where χ denotes a real valued normal random variable becomes very small as n grows; and so it is when $n = 256$. This implies that the Euler increment will mainly “hit" the closest cell in the upper time layer (not to speak about the ability of random number generator to simulate the tail of distributions). Consequently the transition probabilities are not computed accurately enough, given the size of the simulation and can explain the downgrading of the rate of convergence in time. One can conclude this experiment by saying that there is a “CFL" involving the mean spatial unit length and the time step parameter and a second “CFL" involving the time discretization parameter and the size of the simulation (this one has been precisely analyzed in [5]).

• **Estimation of the time discretization rate of convergence.** Now we look for the rate of convergence with respect to δ. For that purpose, we use $\bar{N} = 100$ quantization points in each time layer. The rate of convergence in time will be estimated with

$$\Gamma \in \{0, 0.5\}, \quad \varepsilon \in \{0.1, 0.5, 1.0\}, \quad \delta = 2^{-m}, \quad m = 1, \ldots, 8.$$

Let us see now why we used normalized filter instead of non normalized one. In Figure 3 are displayed typical examples of graphs $k \mapsto < \hat{V}^\delta_k, f >$, $t \mapsto < V_t, f >$, $k \mapsto < \hat{\pi}^\delta_k, x >$ and $t \mapsto < \pi_t, x >$ for $\Gamma = 0$, $\varepsilon = 0.1$, $\delta = 1/256$ and $\bar{N} = N_n = 100$. The exact filters are still computed using (5.5) and (5.6). We verify on that example that the normalized filter seems to be better computed than the unnormalized one. It explains why we did not use the unnormalized version of the error. Indeed, for such a level of noise for the observations, ($\varepsilon = 0.1$) the unnormalized filter $< \hat{V}^\delta_k, f >$ has very large values. This is true for all tested functions f and all time discretization $\delta = 1/n$. Furthermore, it is also true on all sampled trajectories of W (not all depicted). Therefore it is difficult for numerical reasons to compute errors based on $< \hat{V}^\delta_k, f >$ for $\varepsilon = 0.1$.

Let us consider first the uncorrelated case ($\Gamma = 0$). Figure 4 shows the error plotted against the time step in a $\log - \log$ scale for f given by (5.8). We can see again that for a fixed ε given, the time error decreases until a threshold and then remains flat. We also see that this threshold grows as the inverse of the noise level ε. Before reaching this threshold, for every ε and every function f, the rate seems to be of order $\delta = 1/n$ as established in Theorem 2.1.

Let emphasize that, once again in this case, the quantization procedure does not depend on the observations. Therefore, it can be carried out *off-line*. This is no longer true in the correlated case. Then (*e.g.* if $\Gamma = 0.5$), we will have to compute $M = 100$ quantizations (one per observation path) of the signal $(X_k)_k$ for every $n \in \{2, 4, 8, 16, 32, 64, 128, 256\}$, i.e. 800 optimal grids. The previous study in the uncorrelated case seems to indicate that we need a small level of noise on the observations in order to display a rate with a significant number of time steps. This is why we have chosen $\varepsilon = 0.1$ for the simulations. Figure 5 shows the errors obtained as a function of n in a log–log scale for the functions (5.8). The rates of convergence are the same in each case. A linear regression seems to indicate a rate of $O(n^{-3/4})$ which is better than $O(n^{-1/2})$ stated in Theorem 2.1. An explanation of this unexpected behavior could be the following one. The constant in factor of the term $n^{-1/2}$ is presumably very small compared to the one associated to n^{-1}: thus, small values of n make an intermediate rate of convergence appear, while the rate $n^{-1/2}$ would be observed for larger n (in the asymptotic regime).

5.2 McKean Vlasov equation

In order to compare the performances of the quantization approach, i.e. the spatial rate of convergence estimated in Theorem 4.1, we implement our procedure in the case of a deterministic one-dimensional McKean-Vlasov equation, closely following the setting tested in [7]. Namely we set

$$\sigma(x, v) = \sigma = \sqrt{0.2}, \quad \gamma(x, v) = 0$$

and

$$\beta(x, v) = \int \widetilde{\beta}(x, y)v(dy) \quad \text{with } \widetilde{\beta}(x, v) = \beta\, H(x - y),$$

where H denotes the Heavyside function and $\beta = \pm 1$. Note that this corresponds to a non-Lipschitz setting. Then, one checks (see [7]) that the distribution function

$$F(t, x) = \int_{-\infty}^{x} V_t(dy)$$

satisfies the following initial value problem equation with its initial condition

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2} - F\frac{\partial F}{\partial x} \quad \text{on } (0, T) \times \mathbb{R} \\ F(0, x) &= \int_{-\infty}^{x} V_0(dy), \end{aligned} \tag{5.10}$$

where $V_0 = H$ if $\beta = 1$ or $V_0 = 1 - H$ if $\beta = -1$.

In this setting, the process (X_t) satisfies the following SDE:

$$dX_t = \beta(X_t, V_t)\, dt + \sigma dB_t, \quad X_0 = 0.$$

Hence, only the drift depends on the conditional distribution of X given W, namely V.

We consider an horizon $T = 1$. The McKean-Vlasov equation is discretized using an Euler scheme with discretization step $\delta = 1/50 = 0.02$. The quantization procedure is carried out according to the following Euler scheme described in the subsection 4.1:

$$\bar{X}_{k+1} = \bar{X}_k + \delta \sum_{i=1}^{N_k} v_k^i \tilde{\beta}(\bar{X}_k, x_k^i) + \sqrt{\delta}\sigma\chi^{k+1},$$

where (χ^k) denotes a sequence of i.i.d. real valued normally distributed random variables and

$$\tilde{\beta}(\bar{X}_k, x_k^i) = \beta H(\bar{X}_k - x_k^i), \quad \beta \in \{-1, 1\}.$$

Let us note that $\beta = 1$ corresponds to an expanding wave whereas $\beta = -1$ corresponds to a shock wave. Note that in this setting the quantization optimization algorithm is processed *on line*, *i.e.* during the evaluation of the quantized solution itself. This was not the case for the (uncorrelated) Zakai equation.

We reproduce here the exact solution of (5.10) (see [7]):

$$F(x,t) = \frac{\int_0^\infty \exp(-(1/2(x-y)^2/t + \beta g(y))/\sigma^2)\, dy}{\int_{-\infty}^\infty \exp(-(1/2(x-y)^2/t + \beta g(y))/\sigma^2)\, dy},$$

where $g(y) = \int_0^y H(z)\, dz$.

Let $(x_k^i, v_k^i)_{1 \le i \le N_k}$ denote the quantization system obtained at time step k. We define the approximate solution by

$$\widehat{F}(x_k^i) := P(\widehat{X}_k \le x_k^i) = \sum_{j \le i} v_k^i$$

We evaluate the induced quadratic error using the closed form for F given in [7] by

$$\|F(t_k, \widehat{X}_k) - \widehat{F}_k(\widehat{X}_k)\|_2 = \left(\sum_{i=1}^{N_k} v_k^i (F(t_k, x_k^i) - \widehat{F}_k(x_k^i))^2 \right)^{\frac{1}{2}}$$

In Figure 6, we plot the quantized solution together with the exact solution at $t = T = 1$ in the case $\beta = 1$ (a) and $\beta = -1$ (b). We can check on that example that both behaviours (expanding and shock waves) are well reproduced by the scheme.

In Table 1, we compute the errors in the expanding wave case ($\beta = 1$) with several values of the number N_n of quantization points on the top time layer and corresponding values of the total number of points $N = N_{total}$. This confirms a spatial rate of convergence of order $1/N$ as stated in Theorem 4.1.

Table 1: L^2-quantization error for the McKean-Vlasov as a function of the space discretization where $\delta = 0.02$, $\beta = 1$, $T = 1$, $\sigma^2/2 = 0.1$ and $n = 50$.

N_n	50	100	200	400
$\|F(T, \widehat{X}_n) - \widehat{F}_n(\widehat{X}_n)\|_2$	4.82(-2)	3.91(-2)	1.75(-2)	6.61(-3)
N	2 020	4 036	8 074	16 153
CPU	575	975	1 876	4 017

References

[1] A. Alabert, J. Gaines, I. Gyöngy (2000): On numerical approximations of stochastic Burgers' equations, Preprint.

[2] F. Antonelli and A. Kohatsu-Higa (2002): Rate of convergence of a particle method to the solution of the McKean-Vlasov equation, *Annals of Applied Probability*, 12, 423-476.

[3] V. Bally, D. Talay (1996): The distribution of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function, *Probab. Theory Related Fields*, 104-1, 43-60 (1996).

[4] V. Bally, G. Pagès (2003): A quantization algorithm for solving discrete time multidimensional optimal stopping problems, *Bernoulli*, **9**, $n^0 6$, pp.1003-1049.

[5] V. Bally, G. Pagès (2003), Error analysis of the optimal quantization algorithm for obstacle problems, *Stoch. Proc. and their Appl.*, **106**, $n^0 6$, pp.1-40.

[6] A. Bensoussan, R. Glowinski, R. Rascanu (1989): Approximation of Zakai equation by the splitting-up method, *Stochastic Systems and Optimization*, Lect. Notes in Cont. Inf. Sci., 136, 257-265, Springer Verlag.

[7] M. Bossy, D. Talay (1997): A stochastic particle method for the McKean-Vlasov equation and the Burgers equation, *Math. Comp.*, 66, 157-192.

[8] A. Budhiraja, G. Kallianpur (1996): Approximations to the solution of the Zakai equation using multiple Wiener and Stratonovitch integral expansions, *Stochastics and Stochastics Reports*, 56, 271-315.

[9] D. Crisan, T. Lyons (1999): A particle approximation of the solution of the Kushner-Stratonovitch equation, *Probability Theory and Related Fields*, 115, 549-578.

[10] P. Del Moral (1995): Nonlinear filtering using random particles, *Theory of Prob. Appl.*, 40, 690-701.

[11] L. Devroye (1986): *Non uniform random variate generation*, Springer Verlag.

[12] G. Di Masi, M. Pratelli, W. Runggaldier (1985): An Approximation for the nonlinear filtering problem with error bounds, *Stochastics and Stochastics Reports*, 14, 247-271.

[13] P. Florchinger, F. Le Gland (1991): Time-discretization of the Zakai equation for diffusion processes observed in correlated noise, *Stochastics and Stochastics Reports*, 35, 233-256.

[14] E. Gobet, R. Munos (2002): Sensitivity analysis using Itô-Malliavin calculus and martingales. Application to stochastic control problem, to appear in *SIAM J. Cont. Optim.*

[15] S. Graf, H. Luschgy, *Foundations of quantization for random vectors*, Lecture Notes in Mathematics n^{0}1730, Springer, 2000, 230p.

[16] I. Gyöngy (2002): Approximations of stochastic partial differential equations, *Stochastic Partial Differential Equations*, Trento, Lect. Notes in Pure and Appl. Math., 227, 287-307, Dekker.

[17] Y. Hu, G. Kallianpur, J. Xiong (2002): An approximation for Zakai equation, *Appl. Math. Optim.*, 45, 23-44.

[18] A. Kohatsu-Higa, S. Ogawa (1997): Weak rate of convergence for an Euler scheme of nonlinear SDE's, *Monte Carlo Methods Appl.*, 3(4), 327-345.

[19] T. Kurtz and J. Xiong (1999): Particle representations for a class of nonlinear SPDEs, *Stochastic Processes and their Applications*, 83, 103-126.

[20] H. Kushner (1979): A robust discrete state approximation of the optimal nonlinear filter for a diffusion, *Stochastics and Stochastics Reports*, 3, 75-83.

[21] S. Lototsky, R. Mikulevicius, B. Rozovskii (1997): Nonlinear filtering revisited: a spectral approach, *SIAM J. Cont. Optim.*, 33, 1716-1730.

[22] D. Nualart (1995): Malliavin calculus and related topics, Springer Verlag.

[23] D. Nualart (1998): Analysis on Wiener space and anticipating stochastic calculus, Lectures on probability theory and statistics (Saint-Flour, 1995), 123-167, Springer Berlin.

[24] G. Pagès, H. Pham (2003): Optimal quantization methods for non linear filtering with discrete-time observations, to appear in *Bernoulli.*

[25] G. Pagès, H. Pham, J. Printems (2004): Optimal quantization methods and applications to numerical problems in finance, *Handbook of numerical methods in finance*, ed. S. Rachev, Birkhauser.

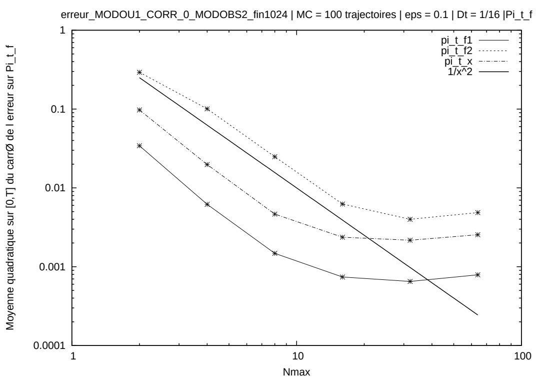
[26] G. Pagès, H. Pham, J. Printems (2004), An Optimal Markovian Quantization Algorithm for Multidimensional Stochastic Control Problems, *Stochastics and Dynamics*, 4, 501-545.

[27] G. Pagès, J. Printems (2003): Optimal quadratic quantization for numerics: the Gaussian case, *Monte Carlo Methods and Applications*, **9**, n^{0}2, pp.135-166.

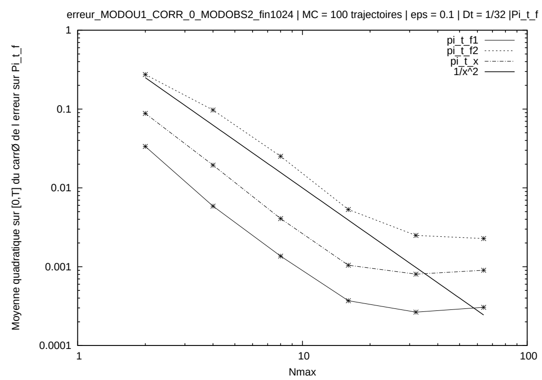
[28] J. Picard (1984): Approximations of nonlinear filtering problems and order of convergence, *Filtering and Control of Random Processes*, Lect. Notes in Cont. Inf. Sci., 61, 219-236, Springer Verlag.

[29] M. Piccioni (1987): Convergence of implicit discretization schemes for linear differential equations with application to filtering, *Stochastic partial differential equations and applications*, Trento, Lect. Notes in Math., 1236, 208-229, Springer.

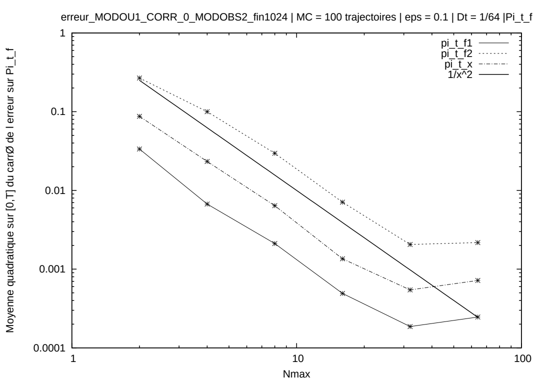
(a) $n = 16$



(b) $n = 32$



(c) $n = 64$



(d) $n = 128$

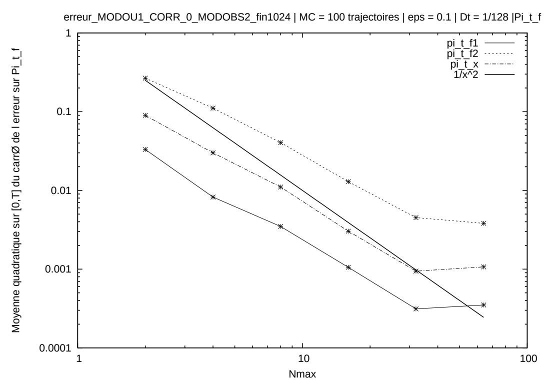


Figure 1: Error $Err(\delta, \bar{N})$ as a function of $\bar{N}$ for several time discretization n: (a) $n = 16$, (b) $n = 32$, (c) $n = 64$ and (d) $n = 128$. The straight line depicts $\bar{N} \mapsto 1/\bar{N}^2$ and the dash lines denotes the errors computed with the different functions (5.8). Here $\varepsilon = 0.1$.

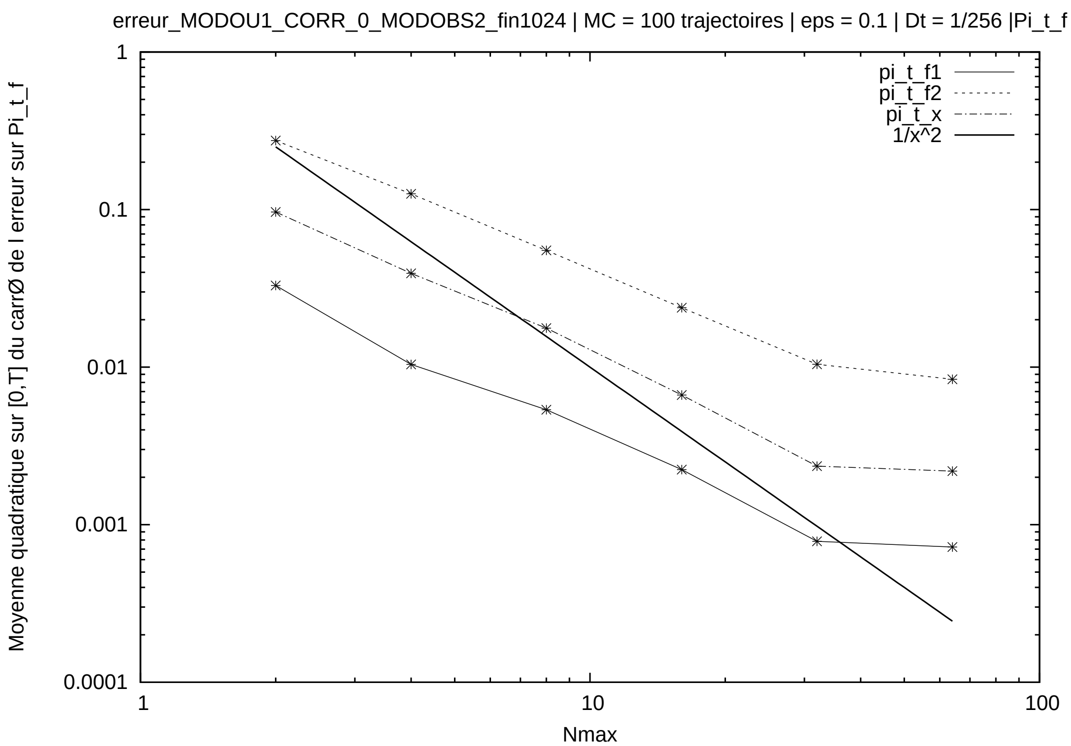
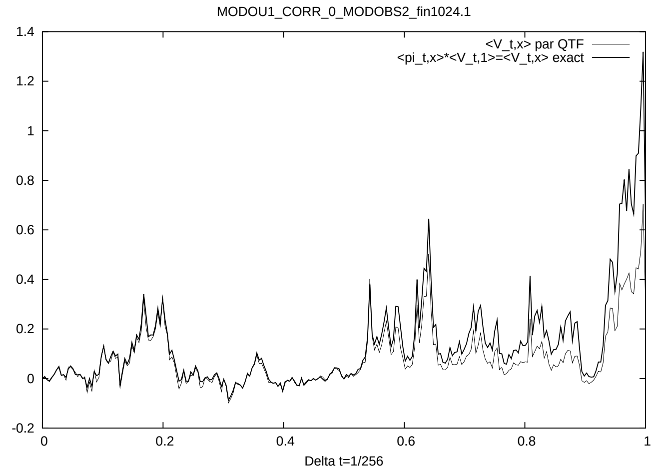
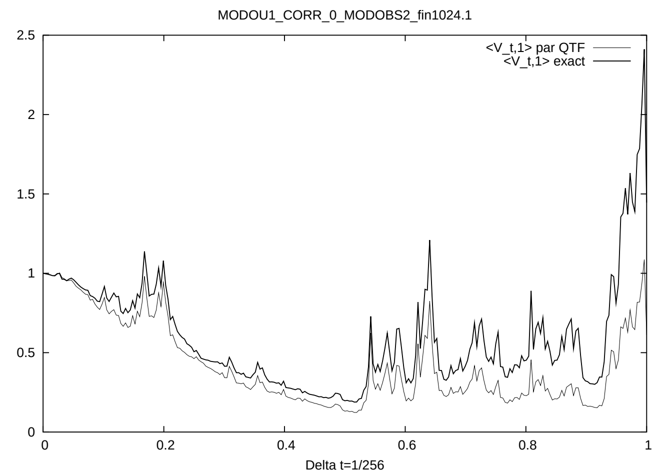


Figure 2: Rate of convergence of (5.7) with $n = 256$. Here again $\varepsilon = 0.1$.

a)



b)



c)

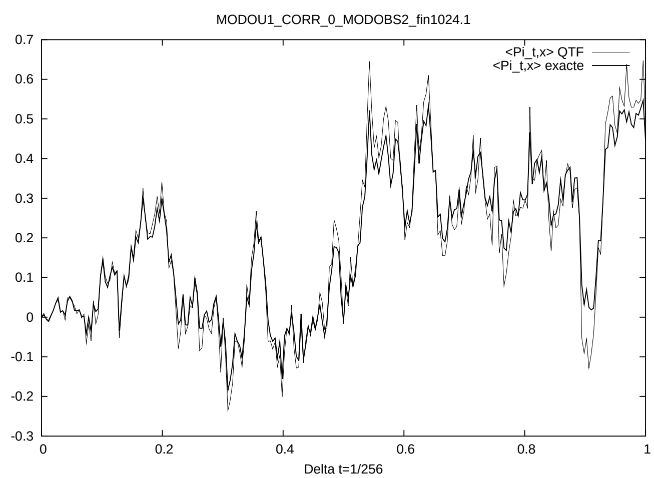
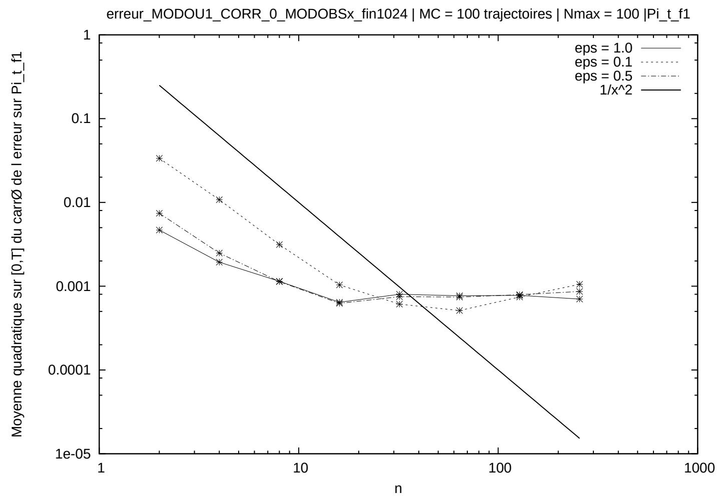
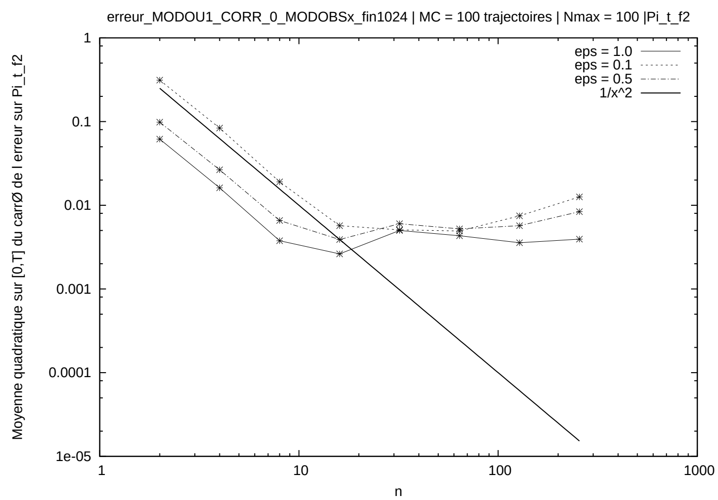


Figure 3: Examples of curves $k \mapsto < \hat{V}_k^\delta, x >$ a), $k \mapsto < \hat{V}_k^\delta, 1 >$ b), $k \mapsto < \hat{\pi}_k^\delta, x >$ c) with $\delta = 1/256$ and $N_n = 100$ computed with the same trajectory of observation. Here $\varepsilon = 0.1$ and $\Gamma = 0$. The thick line depicts the exact filter computed according a time step $\delta_{ref} = 1/1024$ and the thin line the quantized filter.

a)



b)



c)

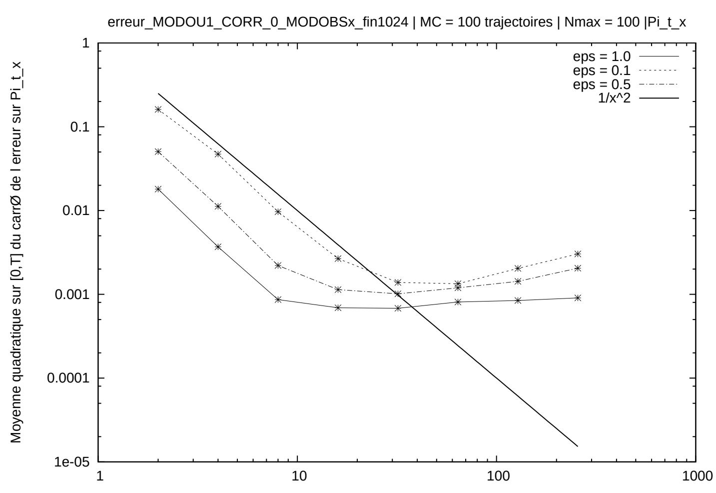


Figure 4: Square of the error (5.7) where $f(x) = \exp(-x^2)$ a), $f(x) = \exp(-x)$ b) and $f(x) = x$ c) as a function of the time step n in a log–log scale. Non correlated case.

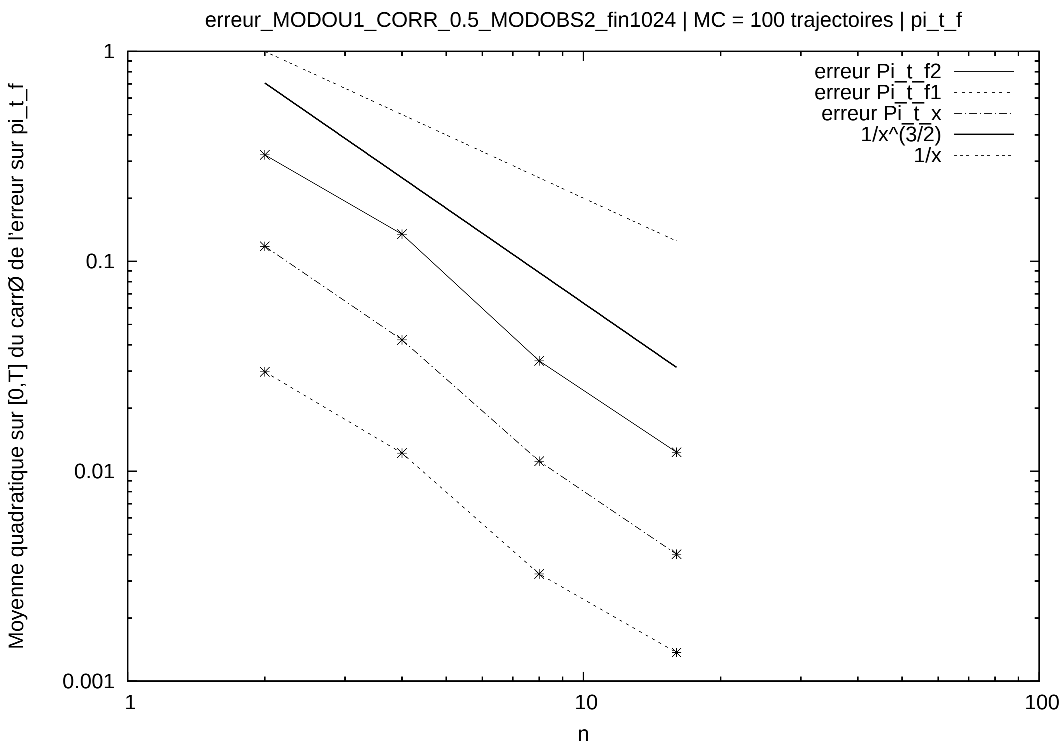
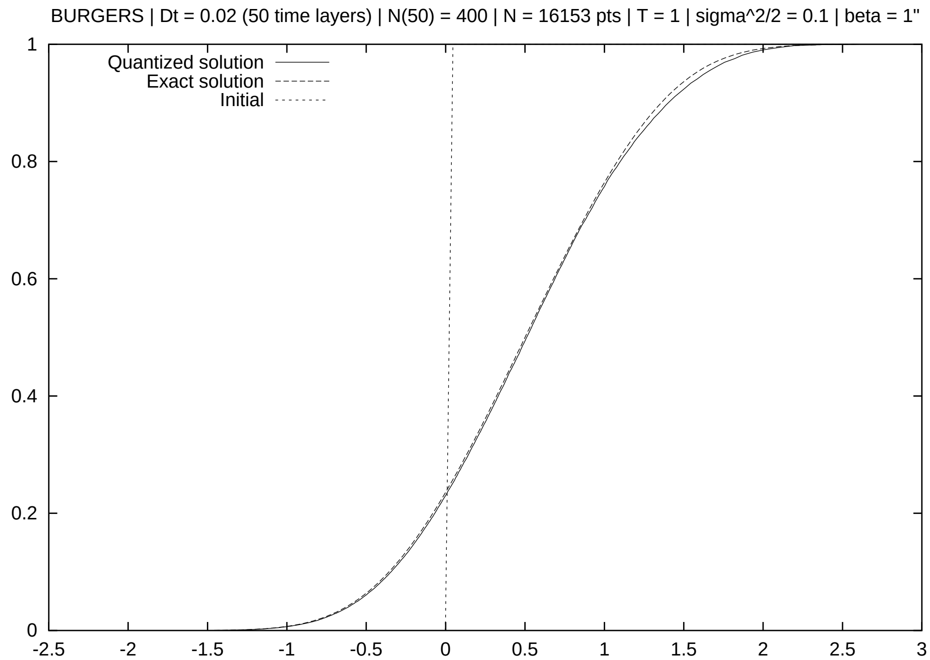


Figure 5: Error (5.7) as a function of the time step n in a log–log scale. Correlated case. The three functions I_d, $f_1(x) = \exp(-x)$ and $f_2(x) = \exp(-x^2))$ are depicted.

(a)



(b)

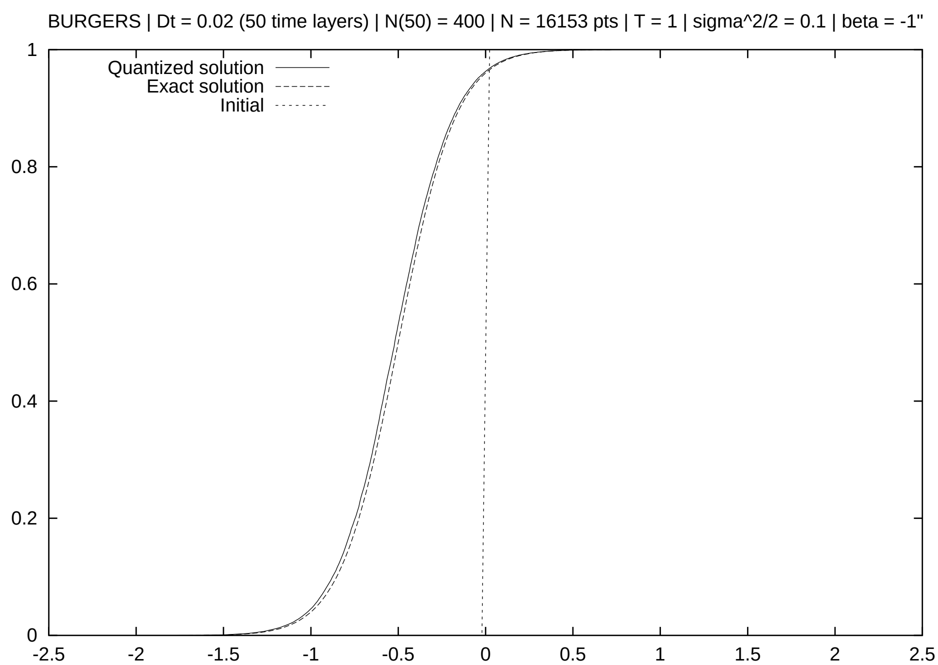


Figure 6: Quantized and exact solutions of (5.10) with $\delta = 0.02$ (i.e. $n = 50$), $N_n = 400$, $N = 16153$, $T = 1$, $\sigma^2/2 = 0.1$. The plain line depicts the quantized solution and the dash line, the exact solution. (a) $\beta = 1$, (b) $\beta = -1$.