# Correction to "Sharp estimates for the convergence of the density of the Euler scheme in small time",

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We thank A. Kohatsu-Higa for pointing us an error in the proof of [Theorem 2.5, GL08]. The aim of this note is to correct it, the result statement is unchanged. Actually, our initial proof gives rise to an extra $\log(N)$-factor in one of the upper bounds about the discretization error, a factor which we have omitted: we provide here a slightly modified proof which gives the announced statement. For the notation, we refer to [GL08]. The result concerned by the correction is the following one.

**Hypothesis 1.** *$\sigma$ is uniformly elliptic, $b$ and $\sigma$ are in $C_b^{1,3}$ and $\partial_t \sigma$ is in $C_b^{0,1}$.*

**Theorem 1.** *Assume Hypothesis 1. For any function $f$ such that $|f(x)| \leq c_1 e^{c_2|x|}$, it holds*

$$\left|\mathbb{E}[f(X_T^N) - f(X_T)]\right| \leq c_1 e^{c_2|x|} K(T) \frac{\sqrt{T}}{N},$$
$$\left|\mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)}) ds\right] - \mathbb{E}\left[\int_0^T f(X_s) ds\right]\right| \leq c_1 e^{c_2|x|} K(T) \frac{T}{N}.$$

*Proof.* The correction concerns the second result. The term $\mathbb{E}\left[\int_0^T \left(f(X^N_{\varphi(s)}) - f(X_s)\right) ds\right]$ is split in two terms $\mathbb{E}\left[\int_0^T \left(f(X^N_{\varphi(s)}) - f(X_{\varphi(s)})\right) ds\right]$ and $\mathbb{E}\left[\int_0^T \left(f(X_{\varphi(s)}) - f(X_s)\right) ds\right]$. Theorem 2.3 in [GL08] enables to bound the first term

$$\left|\mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)}) ds - \int_0^T f(X_{\varphi(s)}) ds\right]\right| = \left|\int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds f(y)(p^N(0,x;\varphi(s),y) - p(0,x;\varphi(s),y))\right|$$
$$\leq \frac{K(T)T}{N} c_1 e^{c_2|x|} \int_{\frac{T}{N}}^T \frac{ds}{\sqrt{\varphi(s)}},$$

which readily leads to an upper bound as advertised.
To bound the second term, it remains to prove that

$$I := \mathbb{E}\int_0^T f(X_s) ds - \frac{T}{N}\mathbb{E}\sum_{i=0}^{N-1} f(X_{t_i}) \qquad \text{and} \qquad |I| \leq c_1 e^{c_2|x|} K(T) \frac{T}{N}.$$

Let us introduce $\tilde{I} := \mathbb{E}\int_{\frac{T}{N}}^{T} f(X_s)ds - \frac{T}{N}\mathbb{E}\sum_{i=1}^{N-1} f(X_{t_i})$. Since $|I - \tilde{I}| \leq c_1 e^{c_2|x|}K(T)\frac{T}{N}$, it remains to bound $|\tilde{I}|$. To do so, we introduce $u : s \longmapsto \mathbb{E}[f(X_s)]$. We have

$$\begin{aligned}
\tilde{I} &= \int_{\frac{T}{N}}^{T} u(s)ds - \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} u(t_i)ds = \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} (u(s) - u(t_i))\, ds, \\
&= \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} \left( u'(t_i)(s-t_i) + u''(\theta_i(s))\frac{(s-t_i)^2}{2} \right) ds := \tilde{I}_1 + \tilde{I}_2,
\end{aligned}$$

where $t_i := \frac{iT}{N}$ for all $i \in \{0, \cdots, N\}$ and $\theta_i(s)$ belongs to $[t_i, s]$.
Before bounding $\tilde{I}_1$ and $\tilde{I}_2$, we recall (see [Proposition A.2, GL08]) for any $s \in ]0, T]$

$$|u(s)| \leq K(T)c_1 e^{c_2|x|}, \qquad |u'(s)| \leq K(T)\frac{c_1}{s}e^{c_2|x|}, \qquad |u''(s)| \leq K(T)\frac{c_1}{s^2}e^{c_2|x|}. \tag{1}$$

**Bound for $\tilde{I}_1 := \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} u'(t_i)(s-t_i)ds$.** We have

$$\tilde{I}_1 = \frac{T^2}{2N^2}\sum_{i=1}^{N-1} u'(t_i) = \frac{T}{2N}\left( \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} (u'(t_i) - u'(s))ds + \underbrace{\int_{t_1}^{T} u'(s)ds}_{=u(T)-u(t_1)} \right).$$

Using the first and third inequalities of (1) gives

$$\begin{aligned}
|\tilde{I}_1| &\leq \frac{T}{2N}\left( \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} \frac{c_1 e^{c_2|x|}K(T)}{t_i^2}\frac{T}{N}ds + c_1 e^{c_2|x|}K(T) \right), \\
&\leq \frac{T}{2N}\left( \sum_{i=1}^{N-1} \frac{c_1 e^{c_2|x|}K(T)}{t_i^2}\left(\frac{T}{N}\right)^2 + c_1 e^{c_2|x|}K(T) \right).
\end{aligned}$$

Then,

$$|\tilde{I}_1| \leq \frac{T}{2N}c_1 e^{c_2|x|}K(T)\underbrace{\left( \sum_{i=1}^{N-1}\frac{1}{t_i^2}\left(\frac{T}{N}\right)^2 + 1 \right)}_{\leq \sum_{i\geq 1}\frac{1}{i^2} + 1 < +\infty}.$$

**Bound for $\tilde{I}_2 := \sum_{i=1}^{N-1}\int_{t_i}^{t_{i+1}} u''(\theta_i(s))\frac{(s-t_i)^2}{2}ds$.** Using the last inequality of (1) yields

$$|\tilde{I}_2| \leq c_1 e^{c_2|x|}K(T)\left(\frac{T}{N}\right)^3\sum_{i=1}^{N-1}\frac{1}{t_i^2} \leq c_1 e^{c_2|x|}K(T)\frac{T}{N}\sum_{i\geq 1}\frac{1}{i^2}.$$

■

## References.


[GL08] E. Gobet and C. Labart. Sharp estimates for the convergence of the density of the Euler scheme in small time. *Electronic Communications in Probability*, 13:311–322, 2008.