Discrete time hedging errors for options with irregular payoffs

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Abstract

In a complete market with a constant interest rate and a risky asset, which is a linear diffusion process, we are interested in the discrete time hedging of an European vanilla option with payoff function f. As regards the perfect continuous hedging, this discrete time strategy induces, for the trader, a risk which we analyze w.r.t. n, the number of discrete times of rebalancing. We prove that the rate of convergence of this risk (when $n \to +\infty$) strongly depends on the regularity properties of f: the results cover the cases of standard options.

KEY WORDS: Discrete time hedging, approximation of stochastic integral, rate of convergence.

1 Introduction

To describe the price of an option at time t, we use a generalized Black and Scholes model with a risky asset (a share of price X_t at time t) and a non risky asset (which price is S_t^0 at time t). The price X_t is given by the following 1-dimensional stochastic differential equation

$$dX_t = \mu(X_t)X_t dt \ + \ \sigma(X_t)X_t dB_t,$$

with $X_0 = x_0 > 0$, and the non risky asset by the ordinary differential equation

$$dS_t^0 = rS_t^0 dt.$$

The coefficients μ, σ fulfill the following assumptions:

(H1) μ and σ are bounded, twice continuously differentiable and the second derivatives satisfy some Hölder conditions. More precisely, if we set $\hat{\mu}(x) = \mu(\exp(x))$ and $\hat{\sigma}(x) = \sigma(\exp(x))$, we assume that there are $\delta \in (0,1)$ and $K > 0$, such that for $(x, x') \in \mathbb{R}^2$, we have

$$\left|\hat{\mu}'(x)\right| + \left|\hat{\mu}''(x)\right| + \frac{|\hat{\mu}''(x) - \hat{\mu}''(x')|}{|x - x'|^\delta} + \left|\hat{\sigma}'(x)\right| + \left|\hat{\sigma}''(x)\right| + \frac{|\hat{\sigma}''(x) - \hat{\sigma}''(x')|}{|x - x'|^\delta} \leq K.$$

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(H2) $\exists\, \sigma_0 > 0$ such that $|\sigma(x)| \geq \sigma_0$ for $x > 0$.

Under these hypotheses, the process $\log(X_t)$ has a smooth transition density $p_t(\cdot,\cdot)$ w.r.t. the Lebesgue measure on $\mathbb{R}$. It implies that the process X_t has also a density $q_t(\cdot,\cdot)$ given by

$$q_t(x,x') = \frac{p_t(\log(x),\log(x'))}{x'} \quad \forall (x,x') \in \mathbb{R}_+^* \times \mathbb{R}_+^*.$$

As usual, we introduce the process $W_t = B_t + \int_0^t \frac{\mu(X_s)-r}{\sigma(X_s)} ds$ which is a Brownian motion under an appropriate probability, called the neutral-risk probability, denoted by $\mathbb{P}$. Thus, the risky asset satisfies a new stochastic differential equation:

$$dX_t = rX_t dt \; + \; \sigma(X_t) X_t dW_t.$$

In the following, we consider European vanilla options with payoff function $f \in L^2(X_T)$. Mathematically, the price of this option is given by

$$h(f) = \mathbb{E}(\exp(-rT) f(X_T) \mid \mathcal{F}_0).$$

If we set

$$u(t,x) \; = \; \mathbb{E}_x \left(e^{-r(T-t)} f(X_{T-t}) \right), \tag{1}$$

note that $h(f)$ is equal to $u(0,x)$ and that u solves the Cauchy problem:

$$-\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\sigma^2(x) x^2 \frac{\partial^2}{\partial x^2} u(t,x) + rx \frac{\partial}{\partial x} u(t,x) - ru(t,x) \quad with\ (t,x) \in [0,T) \times (0,\infty) \tag{2}$$

$$u(T,x) = f(x) \qquad for\ x \in (0,\infty).$$

The well known option valuation formula is

$$e^{-rT} f(X_T) = h(f) + \int_0^T \xi_t \, d\widetilde{X}_t,$$

where $\widetilde{X_t} = e^{-rt} X_t$ is the discounted price of the risky asset. Itô's formula implies that the delta hedging strategy ξ is given by

$$\xi_t = \frac{\partial u}{\partial x}(t, X_t).$$

In other words, to have a perfect hedging, the investor must trade at each time $t \in [0,T]$ and hold ξ_t units of the underlying asset. In practice, this is impossible.
An alternative solution is to hedge only at discrete times. In fact, assume that the investor will trade at n fixed times in the period $[0,T]$. At each trading times defined by $t_k = kT/n$ ($k \in \{0, \dots, n\}$), the trader holds ξ_{t_k} units of the asset X_t. Hence, at maturity the investor will be left with the difference:

$$\begin{aligned} \Delta_n(f) \; &:= \; e^{-rT} f(X_T) - \left(u(0,x) + \int_0^T \frac{\partial u}{\partial x}(\varphi(t), X_{\varphi(t)}) d\widetilde{X_t} \right) \\ &= \; \left(\int_0^T \frac{\partial u}{\partial x}(t, X_t) d\widetilde{X_t} - \int_0^T \frac{\partial u}{\partial x}(\varphi(t), X_{\varphi(t)}) d\widetilde{X_t} \right), \end{aligned}$$

where $\varphi(t) = \sup\{t_i \mid t_i \leq t\}$.
Our purpose is to study some aspects of the convergence of $\Delta_n(f)$ to 0 when n goes to infinity.

2 Results

It is already known that

Proposition 2.1. (Revuz-Yor([2]) Proposition 2.13 p. 135)
With the above notation, $\Delta_n(f)$ converges in probability to 0:

$$\Delta_n(f) \xrightarrow[n \to +\infty]{\mathbb{P}} 0.$$

From now on, we are going to analyze the risk incurred by the trader by evaluating the rate of convergence of the variance of $\Delta_n(f)$. The main contribution of this paper is to prove that the results strongly depend on the regularity property of the payoff function f.

- The case where f is absolutely continuous with polynomial growth (European call or put e.g.) was studied by Zhang [5]: under technical assumptions, the error decreases as $1/n$. One has

$$\mathbb{E}\Delta_n^2(f) = \frac{T}{2n}\mathbb{E}\left(\int_0^T e^{-2rt} X_t^4 \sigma^4(X_t) \left(\frac{\partial^2 u}{\partial x^2}(t, X_t)\right)^2 dt\right) + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

We now focus on more irregular payoffs.

- For an European call digital option with strike $K > 0$ and maturity T (a contingent claim which pays 1 if the price of the underlying risky asset lies above K at maturity and which pays nothing otherwise), the rate of convergence is $1/\sqrt{n}$.

  Theorem 2.1. *Under (H1) and (H2), for the case*

  $$(\mathcal{C}_0) \qquad f(x) = \mathbf{1}_{x \geq K},$$

  one has

  $$\mathbb{E}\Delta_n^2(f) = \sqrt{\frac{T}{n}}\,\frac{C_0}{4\sqrt{\pi}} K \sigma^3(K) e^{-2rT} q_T(x, K) + O\left(\frac{\log(n)}{n}\right)$$

  where C_0 is an universal constant, defined in lemma (3.4) below.

  The same rate of convergence occurs for functions f which can be written as $f(x) = C\,\mathbf{1}_{x \geq K} + g(x)$, for some constants C and K, and for some function g of class C^1_{pol}, e.g. for the digital put $\mathbf{1}_{x \leq K}$.

- Some intermediate rates of convergence (between $1/\sqrt{n}$ and $1/n$) can be achieved with functions f satisfying a Hölder condition i.e. with an intermediate regularity properties between discontinuity and absolute continuity.

Theorem 2.2. *Under (H1) and (H2), for the case*

$$(\mathcal{C}_a) \qquad f(x) = (x-K)_+^a \quad a \in \left(0, \frac{1}{2}\right),$$

one has

$$\mathbb{E}\Delta_n^2(f) = \left(\frac{T}{n}\right)^{\frac{1}{2}+a} \hat{C}_a K^{1+2a} \sigma^{3+2a}(K) e^{-2rT} q_T(x,K) + o\left(\frac{1}{n^{1/2+a}}\right),$$

with

$$\hat{C}_a = C_a \int_{\mathbb{R}} d\delta \left(\int_{-\delta}^{+\infty} \frac{aw\, dw}{\sqrt{2\pi}(\delta+w)^{1-a}} e^{-\frac{w^2}{2}} \right)^2,$$

where C_a is an universal constant, defined in lemma (3.4) below.

Note that these results are still available under the historical probability.
These previous results on the rate of convergence in L^2 norm might be surprising because the weak convergence in the cases $(\mathcal{C}_0)$ and $(\mathcal{C}_a)$ occurs at rate $1/\sqrt{n}$. This can be derived from a general result of Rootzen ([4]) by hard checkings of the assumptions for some particular models.

Theorem 2.3 (Rootzen ([4])). *Let X_t be a diffusion that solves the Black & Scholes equation $dX_t = \sigma X_t dW_t$, then if u is defined as in (1), it follows that*

$$\sqrt{n}\Delta_n(f) \to_d \hat{W}_\tau, \quad n \to \infty, \tag{3}$$

where $\tau = \frac{1}{2}\int_0^T \left(\frac{\partial^2 u}{\partial x^2}(t, X_t)\right)^2 \sigma^4 X_t^4 dt$, and $\hat{W}$ is an extra Brownian motion independent of τ.

Note that this last theorem is not contradictory with the theorems 2.1 and 2.2 since we cannot take the "expectation of the square" in the convergence equation (3). It is even possible to check that for the cases $(\mathcal{C}_0)$ and $(\mathcal{C}_a)$, one has $\mathbb{E}(\tau) = +\infty$.

3 Proof of theorems 2.1 and 2.2

3.1 General decomposition of the error

We are interested in computing

$$\mathbb{E}\left(\Delta_n^2(f)\right) = \mathbb{E}\left[\left(\int_0^T \left(\frac{\partial u}{\partial x}(t, X_t) - \frac{\partial u}{\partial x}(\varphi(t), X_{\varphi(t)})\right) d\widetilde{X}_t\right)^2\right].$$

Since $\widetilde{X}_t$ is a martingale under the neutral-risk probability, one has

$$\mathbb{E}\left(\Delta_n^2(f)\right) = \mathbb{E}\left[\int_0^T \left(\frac{\partial u}{\partial x}(t, X_t) - \frac{\partial u}{\partial x}(\varphi(t), X_{\varphi(t)})\right)^2 X_t^2 e^{-2rt} \sigma^2(X_t) dt\right] = \mathbb{E}\left[\int_0^T M_t^2 dt\right],$$

where we denote

$$M_t = \left(\frac{\partial u}{\partial x}(t, X_t) - \frac{\partial u}{\partial x}(\varphi(t), X_{\varphi(t)})\right) e^{-rt}\sigma(X_t)X_t.$$

Itô's formula for M_t^2, between $\varphi(t)$ and t, yields

$$M_t^2 = 2\int_{\varphi(t)}^t M_s dM_s + \int_{\varphi(t)}^t d\langle M, M\rangle_s.$$

Put $D_u(t) = \frac{\partial u}{\partial x}(t, X_t) - \frac{\partial u}{\partial x}(\varphi(t), X_{\varphi(t)})$ (note that $\varphi(\theta) = \varphi(t)\ \forall\theta \in [\varphi(t), t]$), a straightforward calculation leads to

$$\begin{aligned}
e^{r\theta}dM_\theta = &\left(\frac{\partial^2 u}{\partial x \partial t}(\theta, x)x\sigma(x) - rD_u(\theta)x\sigma(x)\right)\Big|_{x=X_\theta} d\theta \\
&+ \left(rx\left(\frac{\partial^2 u}{\partial x^2}(\theta, x)x\sigma(x) + D_u(\theta)(x\sigma(x))'\right)\right)\Big|_{x=X_\theta} d\theta \\
&+ \left(x\sigma(x)\left(\frac{\partial^2 u}{\partial x^2}(\theta, x)x\sigma(x) + D_u(\theta)(x\sigma(x))'\right)\right)\Big|_{x=X_\theta} dW_\theta \\
&+ \left(\frac{1}{2}x^2\sigma^2(x)\left(\frac{\partial^3 u}{\partial x^3}(\theta, x)x\sigma(x) + 2\frac{\partial^2 u}{\partial x^2}(\theta, x)(x\sigma(x))' + D_u(\theta)(x\sigma(x))''\right)\right)\Big|_{x=X_\theta} d\theta,
\end{aligned}$$

and

$$d\langle M, M\rangle_\theta = \left(\left(\frac{\partial^2 u}{\partial x^2}(\theta, x)x\sigma(x) + D_u(\theta)(x\sigma(x))'\right)^2 e^{-2r\theta}x^2\sigma^2(x)\right)\Big|_{x=X_\theta} d\theta.$$

The derivative of u w.r.t. t can be rewritten using (2) :

$$-\frac{\partial^2 u}{\partial x \partial t}(t, x) = \frac{1}{2}x^2\sigma^2(x)\frac{\partial^3 u}{\partial x^3}(t, x) + x\sigma(x)(x\sigma(x))'\frac{\partial^2 u}{\partial x^2}(t, x) + rx\frac{\partial^2 u}{\partial x^2}(t, x).$$

Consequently, we obtain, after some simplifications,

$$\begin{aligned}
\mathbb{E}\left(\Delta_n^2(f)\right) = \mathbb{E}\int_0^T dt \int_{\varphi(t)}^t d\theta \Bigg[&\Bigg(e^{-2r\theta}D_u(\theta)x\sigma(x)\Big\{-2rD_u(\theta)x\sigma(x) \\
&+ 2rxD_u(\theta)(x\sigma(x))' + x^2\sigma^2(x)D_u(\theta)(x\sigma(x))''\Big\} \\
&+ e^{-2r\theta}x^2\sigma^2(x)D_u^2(\theta)[(x\sigma(x))']^2 + 2e^{-2r\theta}\frac{\partial^2 u}{\partial x^2}(t, x)x^3\sigma^3(x)D_u(\theta)(x\sigma(x))'\Bigg)\Big|_{x=X_\theta} \\
&+ \left(\frac{\partial^2 u}{\partial x^2}(\theta, X_\theta)\right)^2 e^{-2r\theta}\sigma^4(X_\theta)X_\theta^4\Bigg].
\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left(\Delta_n^2(f)\right) = A_1 + A_2 + A_3, \tag{4}$$

with

$$A_1 = \mathbb{E}\int_0^T dt \int_{\varphi(t)}^t d\theta \left(e^{-2r\theta}\frac{\partial^2 u}{\partial x^2}(\theta, X_\theta)\right)^2 \sigma^4(X_\theta)X_\theta^4, \tag{5}$$

$$A_2 = \mathbb{E}\int_0^T dt \int_{\varphi(t)}^t d\theta\, e^{-2r\theta} D_u^2(\theta) X_\theta^2 \sigma(X_\theta) g(X_\theta), \tag{6}$$

$$A_3 = 2\mathbb{E}\int_0^T dt \int_{\varphi(t)}^t d\theta \left(e^{-2r\theta}\frac{\partial^2 u}{\partial x^2}(\theta, x)x^3\sigma^3(x)D_u(\theta)(x\sigma(x))'\right)\Bigg|_{x=X_\theta}, \tag{7}$$

where the function g is defined by

$$\begin{aligned} g(x) &= -2r\sigma(x) + 2r(x\sigma(x))' + (x\sigma(x))''x\sigma^2(x) + [(x\sigma(x))']^2\sigma(x) \\ &= 2\, x\sigma'(x)(r + \sigma^2(x)) + x^2\sigma''(x)\sigma^2(x) + [(x\sigma(x))']^2\sigma(x), \end{aligned}$$

and is bounded under assumption (H1).
The remainder of the proof consists in proving that A_1 gives the main term in the expansion of $\mathbb{E}\left(\Delta_n^2(f)\right)$, whereas A_2 and A_3 are negligible. We first need some estimates to control derivatives of u.

3.2 Preliminary estimates

From now on, $K(T)$ will always stand for a non decreasing, finite, positive map, which can change throughout the calculus, but without numbering in a different way the functions which will appear.

We put $Y_t = \log(X_t)$ and

$$b(y) = r - \frac{\sigma^2(e^y)}{2}; \qquad s(y) = \sigma(e^y); \qquad y_0 = \log(x_0).$$

Hence, obviously, the process Y_t is the solution of the stochastic differential equation:

$$Y_t = y_0 + \int_0^t b(Y_u)du + \int_0^t s(Y_u)dW_u. \tag{8}$$

Note that, under assumption (H1), the coefficients b and s belongs to $\mathcal{C}_b^{2+\delta}(\mathbb{R}, \mathbb{R})$. Hence, we have

Proposition 3.1. (Friedman, [1], Chapter 6)
Under $(H1)$ and $(H2)$, for $t > 0$, the process $Y_t(y)$ has a smooth transition density $p_t(y, \cdot)$ w.r.t. the Lebesgue measure on $\mathbb{R}$, which fulfills:

- $\forall t > 0$, *$p_t(\cdot, \cdot)$ belongs to $\mathcal{C}^4(\mathbb{R}^2, \mathbb{R})$*
- $\forall \alpha, \beta \in \mathbb{N}$ *such as $\alpha + \beta \leq 4$, there exist a function $K(T)$ and a constant $c > 0$, such that:*

$$\forall (t, y, y') \in (0, T] \times \mathbb{R} \times \mathbb{R} \qquad \left|\frac{\partial^{\alpha+\beta} p_t}{\partial y^\alpha \partial y'^\beta}(y, y')\right| \leq \frac{K(T)}{t^{\frac{\alpha+\beta+1}{2}}} e^{-c\frac{(y-y')^2}{t}}. \tag{9}$$

- $p_t(\cdot,\cdot)$ *satisfies the Kolmogorov backward equation:*

$$\frac{\partial p_t}{\partial t}(y,y') = Lp_t(y,y') = \frac{s^2(y)}{2}\frac{\partial^2}{\partial y^2}p_t(y,y') + b(y)\frac{\partial}{\partial y}p_t(y,y'), \tag{10}$$

and the forward equation:

$$\frac{\partial p_t}{\partial t}(y,y') = L^*p_t(y,y') = \frac{\partial^2}{\partial y'^2}\left(\frac{s^2(y')}{2}p_t(y,y')\right) - \frac{\partial}{\partial y'}\left(b(y')p_t(y,y')\right). \tag{11}$$

With the above notation, we define the function v by

$$v(t,y) := u(t,e^y) = \mathbb{E}_y\left(e^{-r(T-t)}f(e^{Y_{T-t}})\right) = e^{-r(T-t)}\int_{\mathbb{R}} f(e^{y'})p_{T-t}(y,y')\,dy',$$

which satisfies the Cauchy problem

$$\begin{aligned} -\frac{\partial}{\partial t}v(t,y) &= \frac{1}{2}s^2(y)\frac{\partial^2}{\partial y^2}v(t,y) + b(y)\frac{\partial}{\partial y}v(t,y) - rv(t,y) \quad && in\ (t,y) \in [0,T)\times\mathbb{R} \\ v(T,y) &= f(e^y) && on\ \mathbb{R}. \end{aligned} \tag{12}$$

Estimates from proposition (3.1) now enable us to establish some specific estimations on the derivatives of v which are not given by standard results on PDE's.

Lemma 3.1. Case $(\mathcal{C}_0)$
Under (H1) and (H2), the function v belongs at least to $\mathcal{C}^{2,4}$ $([0,T)\times\mathbb{R})$ and for $(t,y) \in [0,T)\times\mathbb{R}$ and $\alpha \leq 4$, the following inequalities hold:

$$\left|\frac{\partial^\alpha v}{\partial y^\alpha}(t,y)\right| \leq \frac{K(T)}{(T-t)^{\frac{\alpha}{2}}}, \tag{13}$$

and

$$\mathbb{E}\left(\frac{\partial^2 v}{\partial y^2}(t,Y_t)\right)^2 \leq \frac{K(T)}{T^{\frac{1}{2}}(T-t)^{\frac{3}{2}}}. \tag{14}$$

Lemma 3.2. Case $(\mathcal{C}_a)$
Under (H1) and (H2), the function v belongs at least to $\mathcal{C}^{2,4}$ $([0,T)\times\mathbb{R})$ and for $(t,y) \in [0,T)\times\mathbb{R}$ and $1 \leq \alpha \leq 4$ an integer, the following inequalities hold:

$$\left|\frac{\partial^\alpha v}{\partial y^\alpha}(t,y)\right| \leq \frac{K(T)e^{|y|}}{(T-t)^{\frac{\alpha-a}{2}}}, \tag{15}$$

and

$$\mathbb{E}\left(\frac{\partial^2 v}{\partial y^2}(t,Y_t)\right)^2 \leq \frac{K(T)e^{2|y_0|}}{T^{\frac{1}{2}}(T-t)^{\frac{3}{2}-a}}. \tag{16}$$

Proof of lemma 3.1. Inequality (13) is easy to obtain, using (9), since we have

$$\left|\frac{\partial^\alpha v}{\partial y^\alpha}(t,y)\right| \leq e^{-r(T-t)} \int_{\log(K)}^{+\infty} \left|\frac{\partial^\alpha p_{T-t}}{\partial y^\alpha}(y,y')\right| dy'$$
$$\leq \frac{K(T)}{(T-t)^{\frac{\alpha}{2}}} \int_{\log(K)}^{+\infty} \frac{e^{-c\frac{(y-y')^2}{T-t}}}{\sqrt{T-t}} dy' \leq \frac{K(T)}{(T-t)^{\frac{\alpha}{2}}}.$$

To prove estimate (14), we first use the backward equation (10) to obtain

$$\begin{aligned}\frac{\partial^2 v}{\partial y^2}(t,y) &= e^{-r(T-t)} \int_{\mathbb{R}} f(e^{y'}) \frac{\partial^2 p_{T-t}}{\partial y^2}(y,y') dy' \\ &= e^{-r(T-t)} \int_{\mathbb{R}} f(e^{y'}) \left(-\frac{2b(y)}{s^2(y)} \frac{\partial p_{T-t}}{\partial y}(y,y') - \frac{2}{s^2(y)} \frac{\partial p_{T-t}}{\partial t}(y,y')\right) dy',\end{aligned}$$

and then, the forward one (11) to evaluate the derivative w.r.t. the time, to get

$$\frac{\partial^2 v}{\partial y^2}(t,y) = -\frac{2b(y)}{s^2(y)} \frac{\partial v}{\partial y}(t,y)$$
$$+ e^{-r(T-t)} \int_{\mathbb{R}} f(e^{y'}) \left(\frac{2}{s^2(y)} \left[\frac{\partial^2}{\partial y'^2}\left(\frac{s^2(y')}{2} p_{T-t}(y,y')\right) - \frac{\partial}{\partial y'}\left(b(y') p_{T-t}(y,y')\right)\right]\right) dy' \tag{17}$$

$$= -\frac{2b(y)}{s^2(y)} \frac{\partial v}{\partial y}(t,y) + \frac{2\, e^{-r(T-t)}}{s^2(y)} \left[b(y') p_{T-t}(y,y') - \frac{\partial}{\partial y'}\left(\frac{s^2(y')}{2} p_{T-t}(y,y')\right)\right]\Bigg|_{y'=\log(K)}, \tag{18}$$

where we used an elementary computation of the integral. Using the estimates (9), it readily follows that

$$\left|\frac{\partial^2 v}{\partial y^2}(t,y)\right| \leq \frac{K(T)}{\sqrt{T-t}} \left[1 + \frac{1}{\sqrt{T-t}} \exp\left(-c\frac{(y-\log(K))^2}{T-t}\right)\right].$$

Hence, one has

$$\mathbb{E}\left(\frac{\partial^2 v}{\partial y^2}(t,Y_t)\right)^2 \leq \frac{K(T)}{T-t} \left[1 + \mathbb{E}\left(\frac{1}{T-t} \exp\left(-2c\frac{(Y_t-\log(K))^2}{T-t}\right)\right)\right].$$

To conclude the proof of estimate (14), it remains to show that

$$\mathbb{E}\left(\frac{1}{\sqrt{T-t}} \exp\left(-2c\frac{(Y_t-\log(K))^2}{T-t}\right)\right) \leq \frac{K(T)}{\sqrt{T}},$$

which is easily obtained using an upper bound (9) for the law of Y_t and standard arguments involving convolution of Gaussian kernels. □

We now intend to prove the equivalent lemma for the case $(\mathcal{C}_a)$. The techniques are quite similar.

Proof of lemma 3.2. For (15), if we remark that for $\alpha \geq 1$, $\partial_y^\alpha \int_{\mathbb{R}} p_t(y,y')dy' = 0$, we are able to write that

$$\frac{\partial^\alpha v}{\partial y^\alpha}(t,y) = e^{-r(T-t)} \int_{\mathbb{R}} (f(e^{y'}) - f(e^y)) \frac{\partial^\alpha p_{T-t}}{\partial y^\alpha}(y,y')dy'.$$

Now, applying the Hölder properties for the function f and inequality (9), we get

$$\left| \frac{\partial^\alpha v}{\partial y^\alpha}(t,y) \right| \leq \frac{K(T)}{(T-t)^{\frac{\alpha-a}{2}}} \int_{\mathbb{R}} \frac{|e^{y'} - e^y|^a}{(T-t)^{\frac{a}{2}}} e^{-c\frac{(y'-y)^2}{T-t}} \frac{dy'}{\sqrt{T-t}}.$$

The change of variable

$$z = \frac{y'-y}{\sqrt{T-t}} \tag{19}$$

yields (15). For the second inequality, if t is small ($t \leq T/2$), (16) is an immediate consequence of estimate (15). For t large ($t > T/2$), we start from (17) (which is also available in this case) and after an integration by parts, we obtain:

$$\begin{aligned} \frac{\partial^2 v}{\partial y^2}(t,y) &= -\frac{2b(y)}{s^2(y)} \frac{\partial v}{\partial y}(t,y) \\ &\quad - e^{-r(T-t)} \int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) \left(\frac{s^2(y')}{s^2(y)} \frac{\partial p_{T-t}}{\partial y'}(y,y') - \frac{2b(y') - 2s(y')s'(y')}{s^2(y)} p_{T-t}(y,y') \right) dy'. \end{aligned} \tag{20}$$

To go on with the proof of (16), we admit the following lemma, which proof is postponed in Appendix A.

Lemma 3.3. *With the above notation and assumptions, for any bounded function g, for $\beta = 0$ or 1, one has*

$$\left| \int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) g(y,y') \frac{\partial^\beta p_{T-t}}{\partial y'^\beta}(y,y')dy' \right| \leq \frac{K(T)e^{|y|}}{(T-t)^{\frac{\beta+1-a}{2}}} \frac{1}{1 \vee \left| \frac{\log(K)-y}{\sqrt{T-t}} \right|^{1-a}}. \tag{21}$$

We apply (15) to upper bound the first term of (20) and the lemma above for the two last ones. It follows that

$$\mathbb{E}\left[\frac{\partial^2 v}{\partial y^2}(t,Y_t) \right]^2 \leq \frac{K(T)e^{2|y_0|}}{(T-t)^{1-a}} + \frac{K(T)}{(T-t)^{2-a}} \int_{\log(K)}^{+\infty} p_t(y_0,y') \frac{e^{2|y'|}dy'}{1 \vee \left| \frac{\log(K)-y'}{\sqrt{T-t}} \right|^{2-2a}}.$$

The change of variable $z = \frac{y'-\log(K)}{\sqrt{T-t}}$ leads to

$$\mathbb{E}\left[\frac{\partial^2 v}{\partial y^2}(t,Y_t)\right]^2$$
$$\leq \frac{K(T)e^{2|y_0|}}{(T-t)^{1-a}} + \frac{K(T)}{(T-t)^{\frac{3}{2}-a}}\int_0^{+\infty} p_t(y_0, z\sqrt{T-t}+\log(K))\frac{e^{2|z\sqrt{T-t}+\log(K)|}dz}{1\vee|z|^{2-2a}}$$
$$\leq \frac{K(T)e^{2|y_0|}}{\sqrt{T}(T-t)^{\frac{3}{2}-a}}$$

where we use that $p_t(y_0, z\sqrt{T-t}+\log(K))e^{2|z\sqrt{T-t}+\log(K)|}$ is bounded by $K(T)e^{2|y_0|}/\sqrt{T}$, uniformly in z (see inequality (9) with $t \geq T/2$) and that the function $\frac{1}{1\vee|z|^{2-2a}}$ is integrable over $\mathbb{R}$ (a belongs to $(0,1/2)$). □

3.3 Upper bound of the terms A_2 and A_3

Recall that we intend to prove that these terms are negligible w.r.t. the expected order of the term A_1.

3.3.1 Case $(\mathcal{C}_0)$

First, from $v(t,x) = u(t,\exp x)$, we easily deduce

$$\mathbb{E}\left(X_t^2 D_u^2(t,X_t)\right) \leq 2\mathbb{E}\left(\frac{\partial v}{\partial y}(t,Y_t)\right)^2 + 2\mathbb{E}\left(\frac{X_t}{X_{\varphi(t)}}\frac{\partial v}{\partial y}(\varphi(t),Y_{\varphi(t)})\right)^2 \leq \frac{K(T)}{T-t},$$

using (13) and some classical exponential estimates to control $\mathbb{E}(X_t^p + X_t^{-p})$.
Hence, one has

$$|A_2| \leq \int_0^T dt \int_{\varphi(t)}^t d\theta \, \frac{K(T)}{T-\theta} = O\left(\frac{\log(n)}{n}\right).$$

To control A_3, we combine the Cauchy Schwarz inequality with estimates (13) and (14) to have

$$|A_3| \leq K(T)\int_0^T dt \int_{\varphi(t)}^t d\theta \frac{1}{T^{\frac{1}{4}}(T-\theta)^{\frac{3}{4}}} \times \frac{1}{(T-\theta)^{\frac{1}{2}}} = O\left(\frac{1}{n^{3/4}}\right),$$

which proves that A_3 is of order less than the required one, i.e. $n^{-1/2}$. To obtain $A_3 = O\left(\frac{\log(n)}{n}\right)$, we need to apply Itô's formula once again (this replaces the rough estimate given by the Cauchy Schwarz inequality) and develop same arguments as above. We omit the details.

3.3.2 Case $(\mathcal{C}_a)$

Analogous arguments apply to obtain

$$|A_2| \leq \int_0^T dt \int_{\varphi(t)}^t d\theta \, \frac{K(T)}{(T-\theta)^{1-a}} = O\left(\frac{1}{n}\right)$$

since $a > 0$, and

$$|A_3| \leq K(T) \int_0^T dt \int_{\varphi(t)}^t d\theta \frac{1}{T^{\frac{1}{4}}(T-\theta)^{\frac{3}{4}-\frac{a}{2}}} \times \frac{1}{(T-\theta)^{\frac{1-a}{2}}} = o\left(\frac{1}{n^{1/2+a}}\right).$$

3.4 Term A_1

We first rewrite the term A_1 as follows, using the process Y_t:

$$A_1 = \mathbb{E} \int_0^T dt \int_{\varphi(t)}^t d\theta \left(\frac{\partial^2 v}{\partial y^2}(\theta, Y_\theta) - \frac{\partial v}{\partial y}(\theta, Y_\theta)\right)^2 e^{-2r\theta} s^4(Y_\theta).$$

To obtain the expansion result of theorems 2.1 and 2.2, we first state an analysis lemma, which proof is given in Appendix B.

Lemma 3.4. *Let $g : [0,T] \mapsto \mathbb{R}$ be a measurable bounded function which is continuous in T. Then, for all $a \in [0, 1/2)$,*

$$\int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}-a}} = C_a \, g(T) \left(\frac{T}{n}\right)^{1/2+a} + o\left(\frac{1}{n^{1/2+a}}\right)$$

where $C_a := \sum_{k=1}^{+\infty} \int_0^1 ds \int_0^s \frac{dt}{(k-t)^{\frac{3}{2}-a}} \in (0, +\infty)$.

Moreover, if $|g(t) - g(T)| \leq M\sqrt{T-t}$, then

$$\int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}} = C_0 \, g(T) \sqrt{\frac{T}{n}} + O\left(\frac{\log(n)}{n}\right).$$

To complete the proof of theorems, the above lemma (3.4) will be applied with the function g defined by

$$\text{For the case } (\mathcal{C}_0) \quad g(t) = (T-t)^{\frac{3}{2}} e^{-2rt} \mathbb{E}\left(\left(\frac{\partial^2 v}{\partial y^2}(t, Y_t) - \frac{\partial v}{\partial y}(t, Y_t)\right)^2 s^4(Y_t)\right) \tag{22}$$

$$\text{For the case } (\mathcal{C}_a) \quad g(t) = (T-t)^{\frac{3}{2}-a} e^{-2rt} \mathbb{E}\left(\left(\frac{\partial^2 v}{\partial y^2}(t, Y_t) - \frac{\partial v}{\partial y}(t, Y_t)\right)^2 s^4(Y_t)\right) \tag{23}$$

We now intend to prove that g is bounded and has a limit in T (which enables to extend g as a continuous function in T), limit which will give the main term in the expansion of $\mathbb{E}(\Delta_n^2(f))$.

3.4.1 The function g is bounded

For $(\mathcal{C}_0)$, this directly comes from the inequalities (13) and (14), since

$$|g(t)| \leq K(T)(T-t)^{\frac{3}{2}} \left(\frac{1}{(T-t)^{\frac{3}{2}}} + \frac{1}{T-t} \right) \leq K(T).$$

The same statement holds for $(\mathcal{C}_a)$ using (15) and (16).

3.4.2 Calculus of $\lim_{t \to T} g(t)$.

3.4.2.1 Case $(\mathcal{C}_0)$ Actually, our purpose is to prove a little more, i.e. $|g(t) - g(T)| \leq K(T)\sqrt{T-t}$, to ensure that the remainder term is a $O\left(\frac{\log(n)}{n}\right)$. The definition of g (22) and equality (18) yield:

$$\begin{aligned} g(t) = e^{-2rT}(T-t)^{\frac{3}{2}}\mathbb{E}\Bigg[& -2r\frac{\partial v}{\partial y}(t, Y_t)e^{r(T-t)} \\ & + 2\{b(\log(K) - s(\log(K))s'(\log(K))\}p_{T-t}(Y_t, \log(K)) \\ & - s^2(\log(K)) \left(\frac{\partial}{\partial y'}(p_{T-t}(Y_t, y')) \right) \Big|_{y'=\log(K)} \Bigg]^2. \end{aligned}$$

Thus, combining the estimates (9) and (13) with classical calculations involving convolution of Gaussian kernels, verify that

$$\left| g(t) - e^{-2rT}s^4(\log(K))(T-t)^{\frac{3}{2}}\mathbb{E}\left(\frac{\partial}{\partial y'} p_{T-t}(Y_t, y')) \Big|_{y'=\log(K)} \right)^2 \right| \leq \frac{K(T)}{\sqrt{T}}\sqrt{T-t}. \quad (24)$$

To compute some accurate expansions of the derivatives of the transition density of Y_t in small time, we use the standard representation of the transition density of a 1-dimensional diffusion involving some functional of a Brownian bridge. We refer to Rogers [3] for this representation. If we put

$$\begin{aligned} & S(y) = \int_0^y s^{-1}(x)dx; \quad \beta(y) = \left(\frac{b}{s} - \frac{1}{2}s' \right) \circ S^{-1}(y); \quad h(y) = \int_0^y \beta(x)dx; \quad \eta = \beta' + \beta^2; \\ & \gamma(t, y, y') = (2\pi t)^{-\frac{1}{2}} \exp\left(-\frac{(y-y')^2}{2t} \right); \\ & \psi(t, y, y') = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t \eta\left(y + \frac{\theta}{t}(y'-y) + \left(W_\theta - \frac{\theta}{t}W_t \right) \right) d\theta \right) \right], \end{aligned}$$

then, we have

$$p_{T-t}(y, y') = \gamma(T-t, S(y), S(y')) \exp\left(-\int_{S(y)}^{S(y')} \beta(z)dz \right) \psi(T-t, S(y), S(y')).$$

If we differentiate the above expression w.r.t. y', the main term when t is near T comes from $\gamma(t, S(y), S(y'))$ and is equal to

$$\Gamma(T-t, y, y') := -S'(y')\frac{S(y')-S(y)}{T-t}\gamma(T-t, S(y), S(y')) \exp\left(-\int_{S(y)}^{S(y')}\beta(z)dz\right)\psi(T-t, S(y), S(y')). \quad (25)$$

More precisely, since the functions β, h, η and their derivatives are bounded, it is not hard to show that

$$\left|\frac{\partial}{\partial y'}(p_{T-t}(y, y'))\Big|_{y'=\log(K)} - \Gamma(T-t, y, \log(K))\right| \leq K(T)p_{T-t}(y, y'). \quad (26)$$

Combining (9), (24) and (26), we deduce that

$$|g(t) - \hat{g}(t)| \leq K(T)\sqrt{T-t},$$

where the function $\hat{g}$ is defined by

$$\hat{g}(t) := e^{-2rT}s^4(\log(K))(T-t)^{\frac{3}{2}}\mathbb{E}\Gamma^2(T-t, Y_t, \log(K)).$$

Consequently, it remains to prove that $\hat{g}(t)$ has a limit in T (which will be equal to $g(T)$) and that $|\hat{g}(t) - \hat{g}(T)| \leq K(T)\sqrt{T-t}$. We have

$$\hat{g}(t) = \int_{\mathbb{R}} dy'\ p_t(y_0, y')s^2(\log(K))e^{-2rT}\frac{(S(y') - S(\log(K)))^2}{2\pi(T-t)^{\frac{3}{2}}} \exp\left(-\frac{(S(y') - S(\log(K)))^2}{T-t}\right)\exp\left(-2\int_{S(y')}^{S(\log(K))}\beta(\xi)d\xi\right) \psi^2(T-t, S(y'), S(\log(K))).$$

By the change of variables $z = \frac{S(y')-S(\log(K))}{\sqrt{T-t}}$, one has:

$$\hat{g}(t) = \int_{\mathbb{R}} dz\ s\left(S^{-1}\left(z\sqrt{T-t} + S(\log(K))\right)\right)\ p_t\left(y_0, S^{-1}\left(z\sqrt{T-t} + S(\log(K))\right)\right) s^2(\log(K))e^{-2rT}\frac{z^2}{2\pi}e^{-z^2}\ \exp\left(-2\int_{S(\log(K))}^{z\sqrt{T-t}+S(\log(K))}\beta(\xi)d\xi\right) \psi^2\left(T-t, S(\log(K)), z\sqrt{T-t} + S(\log(K))\right).$$

By the Lebesgue dominated convergence theorem, it immediately follows that

$$\begin{aligned}\lim_{t\to T}\hat{g}(t) &= p_T(y_0, \log(K))\ s^3(\log(K))\ e^{-2rT}\ \int_{\mathbb{R}}\frac{z^2}{2\pi}e^{-z^2}\ dz \\ &= p_T(y_0, \log(K))\ s^3(\log(K))\ \frac{e^{-2rT}}{4\sqrt{\pi}} := \hat{g}(T) = g(T).\end{aligned}$$

Furthermore, basic calculations ensure that

$$|\hat{g}(t) - \hat{g}(T)| \leq K(T)\sqrt{T-t}.$$

Hence, by lemma (3.4), we deduce that

$$A_3 = \frac{C_0}{4\sqrt{\pi}} \frac{\sqrt{T}}{\sqrt{n}} \sigma^3(K) e^{-2rT} p_T(y_0, \log(K)) + O\left(\frac{\log(n)}{n}\right),$$

which completes the proof of theorem 2.1, taking into account that $p_T(y_0, \log(K)) = K \times q_T(x_0, K)$.

3.4.2.2 Case $(\mathcal{C}_a)$ From the expression of g, we substitute the derivative of second order with the equation (20), and after some simplifications, g can be written as

$$\begin{aligned} g(t) = e^{-2rT}(T-t)^{\frac{3}{2}-a}\mathbb{E}\Bigg[& -2r\frac{\partial v}{\partial y}(t, Y_t)e^{r(T-t)} \\ & + \int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'})(2b(y') - 2s(y')s'(y'))p_{T-t}(Y_t, y')dy' \\ & - \int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) s^2(y') \frac{\partial p_{T-t}}{\partial y'}(Y_t, y')dy' \Bigg]^2. \end{aligned}$$

Inequality (15) and lemma 3.3 leads to the fact that the main term in the above expression when t is near T is:

$$e^{-2rT}(T-t)^{\frac{3}{2}-a}\mathbb{E}\left[\int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) s^2(y') \frac{\partial p_{T-t}}{\partial y'}(Y_t, y')dy' \right]^2.$$

Keeping the same notation of the last paragraph to describe the representation of the transition density of a 1-dimensional diffusion, we denote

$$\mu_{T-t}(y) = \int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) s^2(y') \Gamma(T-t, y, y')dy'.$$

Therefore, estimates (9), (26) and (21) yield

$$\lim_{t \to T} e^{-2rT}(T-t)^{\frac{3}{2}-a}\mathbb{E}\left[\left(\int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) s^2(y') \frac{\partial p_{T-t}}{\partial y'}(Y_t, y')dy' \right)^2 - \mu_{T-t}^2(Y_t) \right] = 0.$$

Consequently the limit of g when t tend to T is given by the limit of $\hat{g}$ defined by

$$\hat{g}(t) = e^{-2rT}(T-t)^{\frac{3}{2}-a}\mathbb{E}\mu_{T-t}^2(Y_t). \tag{27}$$

Writing the above expression in terms of integral, we have

$$\hat{g}(t) = e^{-2rT}(T-t)^{\frac{3}{2}-a}\int_{\mathbb{R}} dy\; p_t(y_0,y)\left\{\int_{\log(K)}^{+\infty}\frac{ae^{y'}}{(e^{y'}-K)^{1-a}}s^2(y')\Gamma(T-t,y,y')dy'\right\}^2.$$

With the following changes of variable and notation

$$\xi_t(w) = S^{-1}(S(\log(K)) + w\sqrt{T-t}); \quad \omega = \frac{S(y')-S(y)}{\sqrt{T-t}}; \quad \delta = \frac{S(y)-S(\log(K))}{\sqrt{T-t}},$$

$\hat{g}$ can be written as

$$\hat{g}(t) = e^{-2rT}(T-t)^{2-a}\int_{\mathbb{R}} d\delta\; p_t\left(y_0,\xi_t(\delta)\right) s\left(\xi_t(\delta)\right)\left\{\int_{-\delta}^{+\infty}\frac{a}{\left((\delta+\omega)\sqrt{T-t}\right)^{1-a}}\right.$$
$$\left.\frac{e^{\xi_t(\omega+\delta)}s^3\left(\xi_t(\omega+\delta)\right)}{\left(\int_0^1 s\left(\xi_t(\gamma(\omega+\delta))\right)\exp\left(\xi_t\left(\gamma\left(\omega+\delta\right)\right)\right)d\gamma\right)^{1-a}}\sqrt{T-t}\,\Gamma\left(T-t,\xi_t(\delta),\xi_t(\delta+\omega)\right)d\omega\right\}^2.$$

Then, using the definition of Γ (25) and $\lim_{t\to T}\xi_t(\cdot) = \log(K)$, one has

$$\lim_{t\to T}(T-t)\Gamma(T-t,\xi_t(\delta),\xi_t(\delta+\omega)) = \frac{-\omega}{s(\log(K))\sqrt{2\pi}}e^{-\frac{\omega^2}{2}}.$$

Since the function $\int_{-\delta}^{+\infty}\frac{a\omega\,d\omega}{\sqrt{2\pi}(\delta+\omega)^{1-a}}e^{-\frac{\omega^2}{2}}$ is square integrable w.r.t. δ (see estimate (28), Appendix A), the Lebesgue dominated convergence theorem implies

$$\lim_{t\to T}\hat{g}(t) = e^{-2rT}p_T(y_0,\log(K))K^{2a}s^{3+2a}(\log(K))\int_{\mathbb{R}}d\delta\left\{\int_{-\delta}^{+\infty}\frac{a\omega\,d\omega}{\sqrt{2\pi}(\delta+\omega)^{1-a}}e^{-\frac{\omega^2}{2}}\right\}^2$$
$$:= \hat{g}(T) = g(T).$$

Finally, as in the preceding paragraph, we complete the proof of theorem 2.2. □

Appendix A Proof of lemma 3.3

Lemma 3.3. Under (H1) and (H2), for $f(x) = (x-K)_+^a$ (case $(\mathcal{C}_a)$), for any bounded function g, for $\beta = 0$ or 1, one has

$$\left|\int_{\log(K)}^{+\infty} e^{y'}f'(e^{y'})g(y,y')\frac{\partial^\beta p_{T-t}}{\partial {y'}^\beta}(y,y')dy'\right| \le \frac{K(T)e^{|y|}}{(T-t)^{\frac{\beta+1-a}{2}}}\frac{1}{1\vee\left|\frac{\log(K)-y}{\sqrt{T-t}}\right|^{1-a}}.$$

Proof. Estimate (9) combined with the change of variable (19) give:

$$\left|\int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) g(y,y') \frac{\partial^\beta p_{T-t}}{\partial y'^\beta}(y,y') dy'\right| \leq \frac{K(T)}{(T-t)^{\frac{\beta}{2}}} \int_{\frac{\log(K)-y}{\sqrt{T-t}}}^{+\infty} \frac{e^{z\sqrt{T-t}+y}}{\left(e^{z\sqrt{T-t}+y}-K\right)^{1-a}} e^{-cz^2} dz.$$

Note that

$$e^{z\sqrt{T-t}+y} - K = \int_{\log(K)}^{z\sqrt{T-t}+y} e^{\gamma} d\gamma \geq K(z\sqrt{T-t}+y-\log(K)),$$

and $e^{z\sqrt{T-t}} e^{-cz^2} \leq K(T) e^{-c'z^2}$, where c' is an other positive constant. This readily implies that

$$\left|\int_{\log(K)}^{+\infty} e^{y'} f'(e^{y'}) g(y,y') \frac{\partial^\beta p_{T-t}}{\partial y'^\beta}(y,y') dy'\right| \leq \frac{K(T)e^y}{(T-t)^{\frac{\beta+1-a}{2}}} \underbrace{\int_{\frac{\log(K)-y}{\sqrt{T-t}}}^{+\infty} \frac{e^{-c'z^2}}{\left(z-\frac{\log(K)-y}{\sqrt{T-t}}\right)^{1-a}} dz}_{I\left(\frac{\log(K)-y}{\sqrt{T-t}}\right)}.$$

To complete the proof of lemma 3.3, it remains to prove that

$$\forall \lambda \in \mathbb{R} \quad I(\lambda) = \int_{\lambda}^{+\infty} \frac{e^{-c'z^2}}{(z-\lambda)^{1-a}} dz \leq \frac{C_{(a,c')}}{1 \vee |\lambda|^{1-a}}, \tag{28}$$

for some positive constant $C_{(a,c')}$. In fact if $\lambda \geq 0$, we have

$$I(\lambda) = \int_0^{+\infty} \frac{e^{-c'(z+\lambda)^2}}{z^{1-a}} dz \leq e^{-c'\lambda^2} \int_0^{+\infty} \frac{e^{-c'z^2}}{z^{1-a}} dz \leq C_{(a,c')} e^{-c'\lambda^2},$$

which implies (28). If λ is negative, one has

$$I(\lambda) = \int_{\lambda}^{\frac{\lambda}{2}} \frac{e^{-c'z^2}}{(z-\lambda)^{1-a}} dz + \int_{\frac{\lambda}{2}}^{+\infty} \frac{e^{-c'z^2}}{(z-\lambda)^{1-a}} dz \leq C e^{-\frac{c'}{4}\lambda^2} |\lambda|^a + \frac{C}{|\lambda|^{1-a}}.$$

This concludes the proof of (28) and therefore the proof of lemma (3.3). □

Appendix B Proof of lemma 3.4

Lemma 3.4. Let $g : [0,T] \mapsto \mathbb{R}$ be a measurable bounded function which is continuous in T. Then, for all $a \in [0, 1/2)$,

$$\int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}-a}} = C_a\, g(T) \left(\frac{T}{n}\right)^{1/2+a} + o\left(\frac{1}{n^{1/2+a}}\right) \tag{29}$$

where $C_a := \sum_{k=1}^{+\infty} \int_0^1 ds \int_0^s \frac{dt}{(k-t)^{\frac{3}{2}-a}} \in (0, +\infty)$.

Moreover, if $|g(t) - g(T)| \leq M\sqrt{T-t}$, then

$$\int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}} = C_0\, g(T)\sqrt{\frac{T}{n}} + O\left(\frac{\log(n)}{n}\right). \tag{30}$$

Proof. 1. Suppose first that g is constant.
We can assume $g \equiv 1$ e.g. . A simple change of variables leads to

$$\int_0^T ds \int_{\varphi(s)}^s \frac{dt}{(T-t)^{\frac{3}{2}-a}} = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} ds \int_{t_k}^s \frac{dt}{(T-t)^{\frac{3}{2}-a}}$$
$$= \left(\frac{T}{n}\right)^{\frac{1}{2}+a} \left(\sum_{k=1}^{n} \int_0^1 ds \int_0^s \frac{dt}{(k-t)^{\frac{3}{2}-a}}\right).$$

The series above is convergent because its terms decrease like $n^{-3/2+a}$: we denote by C_a its limit. This completes the proof of (29) in that case.

2. Suppose now that g is a bounded measurable function, continuous in T.
There's no restriction to assume that $g(T) = 0$, up to replacing g by $g - g(T)$ and applying the first case. The proof of (29) now consists in showing that

$$\lim_n \left(\left(\frac{n}{T}\right)^{1/2+a} \int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}-a}} \right) = 0. \tag{31}$$

Fix $\delta > 0$. Since g is continuous in T, there exists $\eta > 0$ such that $\forall t \in [T-\eta, T]$, $|g(t)| \leq \frac{\delta}{C_a}$. Thus, we deduce that

$$\left| \int_{T-\eta}^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}-a}} \right| \leq \delta \left(\frac{T}{n}\right)^{1/2+a},$$

and for $0 \leq s \leq T - \eta$, since $T - s \geq \eta$, we obtain that

$$\left| \int_0^{T-\eta} ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}-a}} \right| \leq \frac{T^2}{n} \frac{\|g\|_\infty}{\eta^{\frac{3}{2}-a}}.$$

Therefore, for n large enough,

$$\left| \left(\frac{n}{T}\right)^{1/2+a} \int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}-a}} \right| \leq 2\delta,$$

which completes the proof of (31) and consequently (29), when $g(T) = 0$. To prove (30), we simply remark that

$$\left| \int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}} \right| \leq M \int_0^T ds \int_{\varphi(s)}^s \frac{dt}{T-t} = O\left(\frac{\log(n)}{n}\right).$$

□

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