Efficient schemes for the weak approximation of reflected diffusions

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Abstract

In this paper, we present two new discretization schemes for reflected stochastic differential equations: their constructions are aimed to achieve the order 1 for the weak convergence, under some conditions, improving the classical order 1/2 obtained with the projected Euler scheme (see Constantini *et al.* [4]). We discuss the approximation of functionals of the reflected SDE, when the time interval is finite or infinite (i.e. stationary problem).

KEY WORDS: reflected SDE, Euler scheme, weak convergence, PDE's with Neumann conditions.

Introduction

We consider $(X_t)_{t\geq 0}$, a reflected stochastic differential equation (RSDE in short) in D, with oblique reflection in the direction γ, i.e. the $\mathbb{R}^d$-valued process which solves

$$X_t = x + \int_0^t B(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \gamma(X_s)\, dk_s, \tag{1}$$

where

- W is a Brownian motion in $\mathbb{R}^d$
- D is a smooth bounded domain of $\mathbb{R}^d$ $(x \in D)$
- k_t is a process increasing only on ∂D: $k_t = \int_0^t \mathbf{1}_{X_t\in\partial D}\, dk_s$
- γ is an unit inward vector.

To ensure the existence of such a process, we now state some hypotheses on B, σ, γ and D, which are assumed to be fulfilled in all the sequel.

Assumption (R):

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(R1) the domain D is bounded and infinitely differentiable: we denote $n(x)$ the unit inward normal vector at $x \in \partial D$.

(R2) the functions B, σ, γ are of class C^∞ on $\overline{D}$.

(R3) the direction γ is uniformly non tangent to ∂D, i.e.

$$\forall x \in \partial D \qquad \gamma(x).n(x) \geq \rho_0 > 0.$$

Under these assumptions[1], there is an unique strong solution to (1) (see Lions and Sznitman [11], Saisho [13] and references therein). Let us denote by $\mathcal{L}$ the second order operator defined on C^2 functions by

$$\mathcal{L}g(x) = \sum_{i=1}^{d} B_i(x)\partial_{x_i}g(x) + \frac{1}{2}\sum_{i,j=1}^{d}[\sigma(x)\sigma^*(x)]_{i,j}\partial^2_{x_i,x_j}g(x).$$

In the following, the regularity of the law of X_t is going to be involved; to ensure it, we assume an uniform strict ellipticity condition (see Cattiaux [3] for hypoelliptic hypotheses):

Assumption (E): for all $x \in D$, one has $\sigma(x)\sigma(x)^* \geq \epsilon\, \mathrm{I}_d$ where $\epsilon > 0$.

Our main objective is to discuss how to accurately evaluate $\mathbb{E}_x[f(X_T)]$, or $\mathbb{E}_x\left[\int_0^\infty f(X_s)\,ds\right]$ when it is well defined, using Monte Carlo simulations.
Let us now stress our attention on the evaluation of $\mathbb{E}_x[f(X_T)]$ for bounded measurable functions f. Generally speaking, the law of X_t is not explicit, hence only numerical procedures are available: we focus on an approach using discretization schemes for the equation (1), which enable to evaluate the expectation of interest using Monte Carlo simulations.
For SDEs without reflection solving simply $X_t = x + \int_0^t B(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, we may use an Euler scheme X^h defined, if we consider N discretization times $t_i = i\,h$ with $h := T/N$, by

$$\text{(2)} \qquad \left\{ \begin{array}{lcl} X_0^h & = & x \\ X_{t_{i+1}}^h & = & X_{t_i}^h + B(X_{t_i}^h)\,(t_{i+1}-t_i) + \sigma(X_{t_i}^h)(W_{t_{i+1}} - W_{t_i}). \end{array} \right.$$

This numerical procedure is easy to implement since at each step, it requires only the additional simulations of d independent Gaussian variables for the Brownian increments. This provides an order 1 scheme for the weak approximation, in the sense that the weak error converges to 0 at the rate h:

$$\mathbb{E}_x\left(f(X_T^h)\right) - \mathbb{E}_x\left(f(X_T)\right) = O(h).$$

Actually, one even has an asymptotic expansion in power of h at any order, under some conditions (for smooth functions f, see Talay *et al.* [15]; for measurable functions f and non degenerate diffusion coefficient σ, see Bally *et al.* [1]).
But, when we consider SDEs with boundary conditions, the derivation of 1-order tractable schemes is not as easy as before.

1. *SDE with killing boundary.* If we are interested in the computation of $\mathbb{E}_x[\mathbf{1}_{T<\tau}f(X_T)]$ where $\tau = \inf\{t : X_t \notin D\}$ is the first exit time from D for X, an Euler scheme X^h defined by (2) with the rough exit time $\tau^h = \inf\{t_i : X_{t_i}^h \notin D\}$ yields only an 1/2-order scheme (for the weak error) (see Gobet [8]) and additional simulations involving Brownian bridge laws are necessary to obtain an 1-order scheme (see Gobet [8] [7]).

[1] the C^∞ condition is too strong for this result, but additional regularity will be needed later in the paper.

2. *SDE with reflection in a half-space.* For RSDE in a half-space with a constant reflecting direction $\gamma(x) = \gamma$, Lépingle [10] suggests to use a reflected Euler scheme, defined by

$$
(3) \qquad \begin{cases} X_0^h &= x \\ X_{t_{i+1}}^h &= X_{t_i}^h + B(X_{t_i}^h)\,(t_{i+1} - t_i) + \sigma(X_{t_i}^h)(W_{t_{i+1}} - W_{t_i}) + \gamma\,(k_{t_{i+1}}^h - k_{t_i}^h). \end{cases}
$$

   The key fact is that once obtained $X_{t_i}^h$, the simulation of $X_{t_{i+1}}^h$ is easy, using d Gaussian variables and an exponential one, all being independently drawn: the precise formulation is stated in Proposition 2.1 below. This scheme is of order 1/2 for the strong error.

3. *SDE with normal reflection in general domain.* In this setting, Constantini *et al.* [4] study an Euler scheme with projection on the boundary. If we denote by $\pi_D(x)$ the orthogonal projection of x on $\overline{D}$, the approximation process is defined by

$$
(4) \qquad \begin{cases} X_0^h &= x \\ X_{t_{i+1}}^h &= \pi_D\left[X_{t_i}^h + B(X_{t_i}^h)\,(t_{i+1} - t_i) + \sigma(X_{t_i}^h)(W_{t_{i+1}} - W_{t_i})\right]. \end{cases}
$$

   They prove that this scheme achieves the 1/2 order convergence for the computation of $\mathbb{E}_x[f(X_T)]$ for smooth functions f with vanishing conditions on ∂D (the approximation of $\mathbb{E}_x[f(X_T)) \exp(-\int_0^T c(X_s)\,ds - \int_0^T \lambda(X_s)dk_s) - \int_0^T h(X_t)\exp(-\int_0^t c(X_s)\,ds - \int_0^t \lambda(X_s)dk_s)dk_t]$ is also studied under some conditions).

   Actually, $\pi_D(x)$ is uniquely defined only in a neighborhood of D and it may happen that $X_{t_i}^h + B(X_{t_i}^h)\,(t_{i+1} - t_i) + \sigma(X_{t_i}^h)(W_{t_{i+1}} - W_{t_i})$ does not belong to this neighborhood. In that case, the choice of the projected point can be arbitrary made, since this event has a probability of occurrence which decreases to 0 as $\exp(-c/h)$ for some $c > 0$, and hence has no incidence on the order of convergence.

Hence, for general RSDEs, an 1/2 order scheme is available, which provides a quite slow convergence for numerical algorithms.

OBJECTIVE. Our aim is to **construct new approximation processes** X^h based on $N = T/h$ regularly spaced discretization times $(t_i)_i$, **for which the weak convergence is of order 1**:

$$
\mathbb{E}_x\left(f(X_T^h)\right) - \mathbb{E}_x\left(f(X_T)\right) = O(h).
$$

It is worth saying that these schemes should be easy to simulate. In this paper, we give two answers to this problem:

1. first, we suggest an Euler scheme with symmetry w.r.t. ∂D (see section 1).
2. second, we propose an Euler scheme with oblique reflection in a half-space, which locally approximates $\overline{D}$ (see section 2).

The first procedure is a bit easier to implement, but it enables to obtain good evaluations only on quantities of the form $\mathbb{E}_x[f(X_T)\exp(-\int_0^T c(X_s)\,ds)]$ under some conditions. The second scheme turns to be more sophisticated and yields good approximations on more general quantities involving in particular the local time $(k_t)_{0\le t\le T}$.

For sake of simplicity, we restrict in section 1 and 2 to the problem of computation of $\mathbb{E}_x[f(X_T)]$, when f is smooth: the case of measurable functions is handled in section 3, whereas the stationary problem (i.e. the evaluation of $\mathbb{E}_x\left[\int_0^\infty f(X_s)\,ds\right]$) is discussed in section 4. Detailed proofs of the theorems presented in this paper and additional results are given in Bossy *et al.* [2] and Gobet [7].

1 Euler scheme with symmetry w.r.t. ∂D

We now recall few basic facts from differential geometry involving the functions *projection on $\overline{D}$ parallel to γ* or *symmetry w.r.t. ∂D parallel to γ*. We follow the notation from Gobet [8].

Property 1.1. *Under* **(R)**, *there is a constant $R = R(D,\gamma) > 0$ (which depends only on the geometry of the domain D and on the constant ρ_0 in (R3) related to γ) such that the following properties hold.*

1. *For any $x \in \{x : d(x,\partial D) \leq R\}$, there is an unique $s \in \partial D$ and $z_1 \in \mathbb{R}$, s.t.*

$$x = s + z_1\,\gamma(s).$$

   The point $s := \pi^\gamma_{\partial D}(x)$ is the projection of x on ∂D parallel to γ.
   The real $z_1 := F^\gamma(x)$ is the algebraic distance (related to the vector fields γ) between x and ∂D.

2. *The functions $\pi^\gamma_{\partial D}(x)$ and $F^\gamma(x)$ are of class C^∞ on the compact set $\{x : d(x,\partial D) \leq R\}$. We arbitrary extend them in C^∞_b on the whole space, with the conditions $F^\gamma(.) > 0$ on D and $F^\gamma(.) < 0$ on $\overline{D}^c$: hence, one has $\partial D = \{x \in \mathbb{R}^d \ : \ F^\gamma(x) = 0\}$.*

3. *The projection of x on $\overline{D}$ parallel to γ is defined on $\{x : d(x,D) \leq R\}$ by:*

$$\pi^\gamma_D(x) = x - (F^\gamma(x))^-\ \gamma(\pi^\gamma_{\partial D}(x)).$$

4. *The symmetric of x w.r.t. ∂D parallel to γ is defined on $\{x : d(x,\partial D) \leq R\}$ by:*

$$\mathrm{Sym}^\gamma_{\partial D}(x) = \pi^\gamma_{\partial D}(x) - F^\gamma(x)\,\gamma(\pi^\gamma_{\partial D}(x)) = x - 2F^\gamma(x)\,\gamma(\pi^\gamma_{\partial D}(x)).$$

5. *We denote by $\mathrm{Sym}^\gamma_D(x)$ the function which is equal to x if $x \in D$, and $\mathrm{Sym}^\gamma_{\partial D}(x)$ if $x \notin D$ and $d(x,\partial D) \leq R$, i.e.*

$$\mathrm{Sym}^\gamma_D(x) = x - 2(F^\gamma(x))^-\ \gamma(\pi^\gamma_{\partial D}(x)).$$

Definition 1.1. Euler scheme $\overline{X}^h$ with symmetry w.r.t. ∂D.
It is defined by:

$$(5)\qquad \begin{cases} \overline{X}^h_0 &= x, \\ \overline{X}^h_{t_{i+1}} &= \mathrm{Sym}^\gamma_D\left[\overline{X}^h_{t_i} + B(\overline{X}^h_{t_i})\,(t_{i+1}-t_i) + \sigma(\overline{X}^h_{t_i})\,(W_{t_{i+1}} - W_{t_i})\right]. \end{cases}$$

Its simulation is straightforward since it only requires realizations of the Brownian increments.

Actually, $\overline{X}^h_{t_i} + B(\overline{X}^h_{t_i})\,(t_{i+1}-t_i) + \sigma(\overline{X}^h_{t_i})\,(W_{t_{i+1}} - W_{t_i})$ may not belong to $\{x : d(x,D) \leq R\}$, the set of definition of Sym^γ_D: in that case, we shall take $\overline{X}^h_{t_{i+1}} = \overline{X}^h_{t_i}$ e.g., this choice having anyhow no incidence on theoretical and numerical convergences. Note that this scheme enables to approximately simulate quantities such as $f(X_T)$ (or $f(X_T)\exp(-\int_0^T c(X_t)\,dt)$) but not $\int_0^T h(X_t)dk_t$.

We now state the main result concerning the analysis of the weak error for $\overline{X}^h$.

Theorem 1.1. *Assume* **(R)**, **(E)** *and that f is a $C^{4+\alpha}_b(\overline{D},\mathbb{R})$ function (for some $\alpha \in (0,1)$) satisfying $\gamma.\nabla f\,|_{\partial D} = \gamma.\nabla(\mathcal{L}f)\,|_{\partial D}$.*

If any case, the convergence is at least of order 1/2 for $\overline{X}^h$:

$$\mathbb{E}_x\left(f(\overline{X}_T^h)\right) - \mathbb{E}_x\left(f(X_T)\right) = O\left(h^{1/2}\right). \tag{6}$$

But if γ is the co-normal vector (i.e. $\gamma(s) \parallel [\sigma\sigma^*](s)n(s)$ for all $s \in \partial D$), $\overline{X}^h$ achieves the 1-order convergence:

$$\mathbb{E}_x\left(f(\overline{X}_T^h)\right) - \mathbb{E}_x\left(f(X_T)\right) = O(h). \tag{7}$$

We shall briefly comments the above result.

1) First, the fact that the co-normal vector plays a key role in this setting is not surprising, since in the PDE's theory, this is also a privileged vector for the reflecting direction. Indeed, if we consider the function $u(t,x) = \mathbb{E}_x(f(X_{T-t}))$ as the solution of the second-order parabolic PDE with a Neumann condition

$$\begin{cases} \partial_t u + \mathcal{L}\, u = 0 \quad (t,x) \in [0,T) \times D \\ u(T,x) = f(x) \quad x \in D \\ \gamma(x).\nabla u(t,x)\,|_{\partial D} = 0, \end{cases}$$

it is well known that the analysis of the above PDE is much easier when the co-normal vector coincides with the reflecting direction (see Freidlin [6] e.g.).

2) Second, we can easily understand why this procedure with a symmetry may work better than those with the projection from Constantini [4]. Indeed, consider the case of a normally reflected ($\gamma = n$) Brownian Motion in $D = \mathbb{R}^+$ starting from, say, 0: this process is equal to

$$\left(B_t - \inf_{s\in[0,t]} B_s\right)_{t\geq 0} \overset{law}{=} (|B_t|)_{t\geq 0} \overset{def}{=} \left(\mathrm{Sym}_D^n(B_t)\right)_{t\geq 0},$$

the equality in law being derived from Lévy's Theorem (see Revuz and Yor [12]).

Elements of proofs of Theorem 1.1 (for a complete proof, see Bossy *et al.* [2]). The assumptions on f ensure that derivatives of u up to some order are uniformly bounded: indeed, one has $u \in C^{2+\alpha/2,4+\alpha}([0,T],\overline{D})$ (see Ladyzenskaja *et al.* [9]). The weak error can be decomposed by writing

$$\mathbb{E}_x\left(f(\overline{X}_T^h)\right) - \mathbb{E}_x\left(f(X_T)\right) = \sum_{i=0}^{N-1} \mathbb{E}_x\left(u(t_{i+1}, \overline{X}_{t_{i+1}}^h) - u(t_i, \overline{X}_{t_i}^h)\right).$$

We analyze each difference using Itô's formula between t_i and t_{i+1}, the key fact being the identification of the semimartingale decomposition of $\mathrm{Sym}_D^\gamma(Y_t)$ where $Y_t = \overline{X}_{t_i}^h + B(\overline{X}_{t_i}^h)\,(t-t_i) + \sigma(\overline{X}_{t_i}^h)\,(W_t - W_{t_i})$. For this, we adapt arguments from Gobet [8] Proposition 3.1, to derive

$$d(\mathrm{Sym}_D^\gamma(Y_t)) = \mathbf{1}_{Y_t\in D}\, dY_t + \mathbf{1}_{Y_t\notin D}\, dY_t^{\partial D} + \gamma(Y_t)\, dL_t^0(F^\gamma(Y)),$$

where $Y_t^{\partial D}$ is an other Itô process and $L_t^0(F^\gamma(Y))$ is the 1-dimensional local time of the continuous semimartingale $F^\gamma(Y)$ at time t and level 0. The estimate (6) now follows quite easily. But the interesting estimate (7) is much trickier to obtain, we refer to Bossy *et al.* [2] for the details.

2 Euler scheme locally reflected in half-space approximation of D

2.1 Exact simulation in a half-space

We first recall the useful result from Lépingle [10], which enables to exactly simulate RSDEs in a half-space when the coefficients B, σ, γ are constant. To state a precise formulation, we define $D = \Big\{z \in \mathbb{R}^d : (z-y).n > 0\Big\}$ and consider the solution of $Y_t = x + B\ t + \sigma\ W_t + \gamma\ k_t$. Here, k_t is explicitly given by:

$$k_t = \frac{1}{n.\gamma} \max\left(0, \sup_{0\le s\le t} -(x + B\ s + \sigma\ W_s - y).n\right).$$

We can simulate Y_t owing the

Proposition 2.1. *Set $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$. If $U \overset{law}{=} \mathcal{N}(0, t\ \mathrm{I}_d)$ and $V \overset{law}{=} \mathcal{E}(1/2t)$ independent of U, one has*

$$\left(W_t, \sup_{0\le s\le t}(a.W_s + c\ s)\right) \overset{law}{=} \left(U, \frac{1}{2}\left[a.U + ct + \sqrt{|a|^2 V + (a.U+ct)^2}\right]\right).$$

2.2 Construction of $(\tilde{X}^h, \tilde{k}^h)$, using an Euler scheme with oblique reflection in a half-space approximation of D

To describe the general procedure, we need to introduce a new uniformly non tangent vector field γ', to which we associate the constant R defined in Property 1.1: the appropriate choice of γ' is discussed in Theorem 2.1 below.
Set $\tilde{X}^h_0 = x$ and $\tilde{k}^h_0 = 0$. We assume that $z := \tilde{X}^h_{t_i} \in \overline{D}$ and $\tilde{k}^h_{t_i}$ are defined and we now construct $\tilde{X}^h_{t_{i+1}}$ and $\tilde{k}^h_{t_{i+1}}$.

a) <u>z is far from ∂D</u>. If $d(z, \partial D) \ge R$, we set $\tilde{X}^h_{t_{i+1}} = z + B(z)\ (t_{i+1} - t_i) + \sigma(z)\ (W_{t_{i+1}} - W_{t_i})$ and $\tilde{k}^h_{t_{i+1}} = \tilde{k}^h_{t_i}$: in other words, we consider that there is no reflection between t_i and t_{i+1}, which is false only with an exponentially small probability w.r.t. $1/h$. If $\tilde{X}^h_{t_{i+1}} \notin D$ (which also occurs with a negligible probability), replace $\tilde{X}^h_{t_{i+1}}$ by its projection on $\overline{D}$.

b) <u>z is not far from ∂D</u>, i.e. $d(z, \partial D) < R$.

   b1) We set $s = \pi^{\gamma'}_{\partial D}(z)$ for the projection of z on ∂D parallel to γ'.

   b2) Let $D_s = \Big\{z \in \mathbb{R}^d : (z - s).n(s) > 0\Big\}$ the half-space delimited by the tangent plane to ∂D, at s.

   b3) Let $(Y_t)_{t_i \le t \le t_{i+1}}$ be the RSDE in D_s defined by

$$Y_t = z + B(z)\ (t - t_i) + \sigma(z)\ (W_t - W_{t_i}) + \gamma(s)\ (\tilde{k}^h_t - \tilde{k}^h_{t_i}).$$

   Note that $(Y^h_{t_{i+1}}, \tilde{k}^h_{t_{i+1}} - \tilde{k}^h_{t_i})$ can be simulated using Proposition 2.1.

   b4) To obtain $\tilde{X}^h_{t_{i+1}}$, project $Y^h_{t_{i+1}}$ on D parallel to γ: $\tilde{X}^h_{t_{i+1}} = \pi^{\gamma}_{D}(Y^h_{t_{i+1}})$. Actually, most of the time, one has $Y^h_{t_{i+1}} \in D$ and the projection is obvious.

If we are interested (as in Constantini *et al.* [4]) in the approximation of $f(X_T)\exp(-Z_T) - \int_0^T h(X_t)$ $\exp(-Z_t)dk_t$ with $Z_t = \int_0^t c(X_s)\,ds + \int_0^t \lambda(X_s)dk_s$, we may use standard discretizations of the integral, which we can simulate since one has obtained realizations of $(\tilde{X}^h_{t_{i+1}}, \tilde{k}^h_{t_{i+1}} - \tilde{k}^h_{t_i})_{0\leq i\leq N-1}$.

We now give the weak error for the computation of $\mathbb{E}_x(f(X_T))$.

Theorem 2.1. *Assume* **(R), (E)** *and that f satisfies the same assumptions as in Theorem 1.1. For any uniformly non tangent vector field γ', the weak convergence for $\tilde{X}^h$ holds with order at least equal to 1/2:*

$$\mathbb{E}\left(f(\tilde{X}^h_T)\right) - \mathbb{E}(f(X_T)) = O\left(h^{1/2}\right). \tag{8}$$

But if γ is the co-normal vector, the choice $\gamma'(.) = \gamma(.)$ leads to the 1-order convergence:

$$\mathbb{E}_x\left(f(\tilde{X}^h_T)\right) - \mathbb{E}_x\left(f(X_T)\right) = O\left(h\right). \tag{9}$$

When the reflecting direction is not the co-normal one, we may use for simplicity $\gamma'(.) = n(.)$.

Elements of proof (for a complete proof, see Gobet [7]). As before, we used the PDE solved by $u(t,x)$ and this leads, after some tricky computations, to the general estimate (8). Actually, the term of order $h^{1/2}$ can be interpreted as an integral on the boundary, involving the explicit transition density function $p_t(x,y)$ of Y_t defined in b3). To remove this term and achieve the order 1, we prove that in the case of co-normal vector with $\gamma' = \gamma$, the function $p_t(x,y)\,|_{y\in\partial D}$ has some nice symmetry properties w.r.t. y. For example, if $D = \{y \in \mathbb{R}^d : y_1 > 0\}$ and $Y_t = x + \sigma\, W_t + \gamma\, k_t$ is reflected in D, we prove that the function $(y_2, \cdots, y_d) \longrightarrow p_t\left(x, \pi^{\gamma}_{\partial D}(x) + (0, y_2, \cdots, y_d)^*\right)$ is an even function in each y_i.

3 Extension of the results when f is measurable

Theorem 3.1. *The theorems 1.1 and 2.1 are still valid if f is a bounded measurable function, satisfying $d(\mathrm{Supp}(f), \partial D) > 0$.*

Elements of proof. Formally, the approach using the PDE solved by u remains the same; actually, the main point consists in deriving time uniform control on quantities such as $\mathbb{E}_x(\partial^k u(t_i, X^h_{t_i}))$, even if derivatives of u for t_i closed to T may explode for irregular functions f.
Following Bally *et al.* [1], we used Malliavin calculus techniques to transform the above expectation using an integration by parts formula. It seems to be especially difficult in our setting because we deal with piecewise RSDEs: the assumption on the support of f however enables to develop a quite easy approach. Indeed, this support condition ensures that derivatives of u are bounded near ∂D, so that we only needs to apply Malliavin Calculus to the law of X^h restricted to the interior of D. But strictly inside D, it behaves like standard Euler scheme without reflection, so that the classical integration by parts formula can apply. This kind of arguments are used in Gobet [8] to deal with killed diffusions.

4 The stationary problems

In this section, we present a numerical procedure for the computation of

$$u(x) = \mathbb{E}_x\left[\int_0^\infty f(X_t)\,dt\right],$$

for $x \in \overline{D}$, using Monte Carlo simulations, where X is the RSDE defined (1). For the details, we refer to Bossy *et al.* [2].

Under **(R)** and **(E)**, $(X_t)_{t\geq 0}$ is ergodic: we denote by μ its invariant probability measure and we set $\mu(f) = \int_D f(x)\mu(dx)$ whenever this quantity is finite. Since the Doeblin's condition is satisfied, one has

$$\sup_{x\in D} |\mathbb{E}_x(f(X_t)) - \mu(f)| \leq C\exp(-\lambda\, t)$$

for some $\lambda > 0$; hence, $u(x)$ is well defined (see Freidlin [6]) if we impose

$$\mu(f) = 0.$$

We assume this assumption in the sequel. Moreover, one knows that u is the solution (up to an additive constant) of the elliptic PDE

$$\begin{cases} \mathcal{L}\, u + f = 0 & x \in \overline{D} \\ \gamma.\nabla u\,|_{\partial D} = 0. \end{cases}$$

The computation of u is motivated by a study by Faugeras *et al.* [5], which deals with the 3-dimensional reconstruction of the electrical activity of the brain from electroencephalography and magnetoencephalography: their approach leads to solve the above PDE (where the domain $D \subset \mathbb{R}^3$ corresponds more or less to the skull of the patient) only at few points $x \in \partial D$, this fact justifying the Monte Carlo approach for a performance purpose.

For the numerical procedure, we adapt ideas from Talay [14], Talay *et al.* [15], who consider the case of SDEs without reflection. If h is a time discretization step, we denote by X^h one of the two schemes $\overline{X}^h$ and $\tilde{X}^h$ studied in sections 1 and 2: as for X, X^h satisfies an ergodic property and we denote by μ^h its invariant probability measure. We write $(X^{h,j})_{j\geq 1}$ for some independent copies of X^h.
Thus, we may evaluate $\int_0^\infty \mathbb{E}_x\left[f\left(X_t\right)\right]dt$ by

$$h\sum_{i=0}^{p}\{\frac{1}{M}\sum_{j=1}^{M}\left[f\left(X_{ih}^{h,j}\right) - f\left(X_{ph}^{h,j}\right)\right]\},$$

since by the ergodic theorem and the weak approximation estimates for X^h, it is approximately equal, for h small, M and ph both large, to

$$h\sum_{i=0}^{p}\left\{\mathbb{E}_x\left[f\left(X_{ih}^h\right) - f\left(X_{ph}^h\right)\right]\right\} \approx h\sum_{i=0}^{p}\left\{\mathbb{E}_x\left[f\left(X_{ih}\right) - f\left(X_{ph}\right)\right]\right\} \approx \int_0^\infty \left(\mathbb{E}_x\left[f\left(X_t\right)\right] - \mu(f)\right)dt = u(x).$$

Actually, the main difficulty consists in justifying that the weak estimates for X^h are somehow valid uniformly in time $ph = T \to \infty$ (see Talay [14]). For this, it is enough to prove that the derivatives of $\mathbb{E}_x\left[f\left(X_t\right)\right]$ converges exponentially fast to 0 when $t \to \infty$. This is the following key result, which seems to be new as far as we know.

Theorem 4.1. *Assume* **(R)**, **(E)**, *and suppose that f is a bounded measurable function satisfying $\mu(f) = 0$. Then, for any multi-index α and any integer k, one has*

$$\forall\, t \geq 1 \qquad \sup_{x\in D}\left|\partial_t^k \partial_x^\alpha \mathbb{E}_x\left(f(X_t)\right)\right| \leq C\exp(-\lambda\, t),$$

for some $\lambda = \lambda(\alpha, k) > 0$.

5 Conclusion

We have proposed two new implementable schemes for the weak approximation of $(X_t)_{t\geq 0}$, a RSDE with oblique reflection. The first one is built using an Euler scheme on which we apply a symmetry procedure at the boundary: it is convenient if we are interested in the simulation of $f(X_T)$ e.g.. The second one consists in locally approximating the domain in a half space, in which a reflected Euler scheme is easy to simulate: this leads to the evaluation of X_T but also of its local time k_t.

Both schemes give an 1-order convergence for the computation of $\mathbb{E}_x(f(X_T))$ when the reflecting direction is the co-normal one. Anyhow, preliminary numerical experiments illustrate that they work better than the usual projected Euler scheme (see Constantini *et al.* [4]). On the figure 5.1, we compare the projected Euler scheme and the Euler scheme with symmetry, in the case of a 2-dimensional Brownian motion ($x = 0$) reflected in the unit sphere : it is clear that the weak convergence is much faster for the Euler scheme with symmetry. Analogous results are available for the scheme using a half-space approximation.

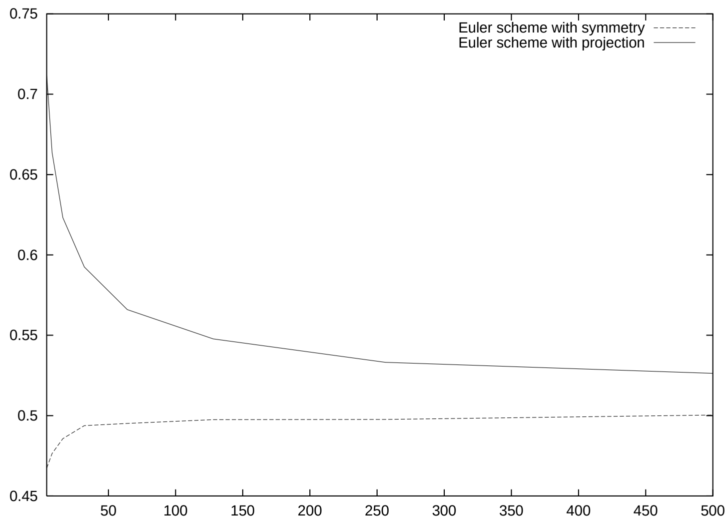

FIG. 5.1: evaluation of $\mathbb{E}_x\|X_1^h\|^2$ w.r.t. the number of discretization times $N = T/h$.

References

[1] V. Bally and D. Talay. The law of the Euler scheme for stochastic differential equations: I. convergence rate of the distribution function. *Probability Theory and Related Fields*, 104-1:43–60, 1996.

[2] M. Bossy, E. Gobet, and D. Talay. Computation of the invariant law of a reflected diffusion process. *in preparation*, 2000.

[3] P. Cattiaux. Stochastic calculus and degenerate boundary value problems. *Annales de l'Institut Fourier*, 42(3), 1992.

[4] C. Constantini, B. Pacchiarotti, and F. Sartoretto. Numerical approximation for functionnals of reflecting diffusion processes. *SIAM J. Appl. Math.*, 58(1):73–102, 1998.

[5] O. Faugeras, F. Clément, R. Deriche, R. Keriven, T. Papadopoulo, J. Roberts, T. Viéville, F. Devernay, J. Gomes, G. Hermosillo, P. Kornprobst, and D. Lingrand. The inverse EEG and MEG problems: the adjoint state approach. I: the continuous case. *Rapport de recherche INRIA N 3673*, 1999.

[6] M. Freidlin. *Functional integration and partial differential equations.* Annals of Mathematics Studies - Princeton University Press, 1985.

[7] E. Gobet. Euler schemes combined with half-space approximation for diffusion with boundary condition. *in preparation*, 2000.

[8] E. Gobet. Euler schemes for the weak approximation of killed diffusion. *Stochastic Processes and its Applications*, 87:167–197, 2000.

[9] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and quasi-linear equations of parabolic type.* Vol.23 of Translations of Mathematical Monographs, American Mathematical Society, Providence, 1968.

[10] D. Lépingle. Euler scheme for reflected stochastic differential equations. *Math. Comput. Simulation*, 38:119–126, 1995.

[11] P.L. Lions and A.S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37:511–537, 1984.

[12] D. Revuz and M. Yor. *Continuous martingales and Brownian motion. 2nd ed.* Grundlehren der Mathematischen Wissenschaften. 293. Berlin: Springer, 1994.

[13] Y. Saisho. Stochastic differential equations for multi-dimensional domain with reflecting boundary. *Probab. Theory Rel. Fields*, 74:455–477, 1957.

[14] D. Talay. Second-order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stochastics and Stochastics Reports*, 29:13–36, 1990.

[15] D. Talay and L.Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Analysis and Applications*, 8-4:94–120, 1990.