BENCHMARK : THE 2D DISCRETE DUALITY FINITE VOLUME METHOD

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PROBLEM

The aim of this work is to study, from a computational point of view, the behaviour of the steady ($\theta = 0$) or unsteady ($\theta = 1$) incompressible Stokes ($\chi = 0$) or Navier-Stokes ($\chi = 1$) equations in the framework of the Discrete Duality Finite Volume method (DDFV for short).
Let $\mathcal{D} \subset \mathbb{R}^2$ be a connected and bounded polygonal domain (here $\mathcal{D}$ is the unit square). For a given final time $T > 0$ the problem we are interested in is the following: Find the velocity $\mathbf{u} : (0,T) \times \mathcal{D} \to \mathbb{R}^2$ and the pressure $p : (0,T) \times \mathcal{D} \to \mathbb{R}$ satisfying

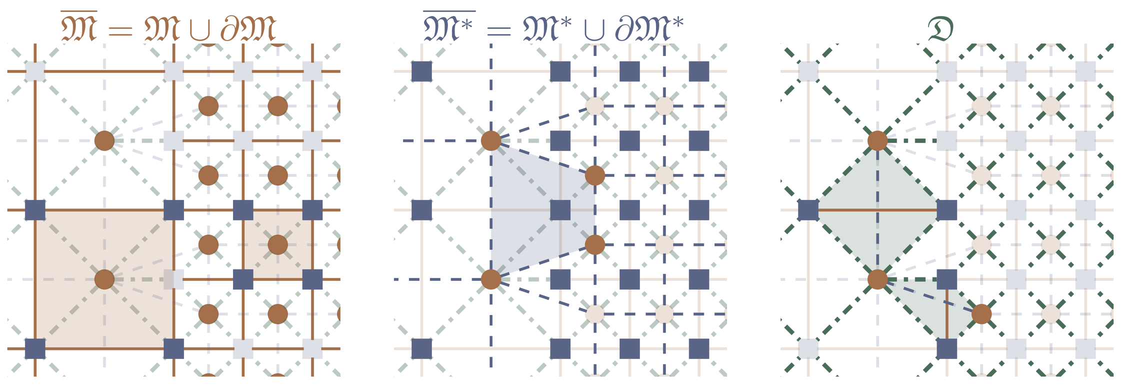
$$\begin{cases} \theta \mathbf{u}_t - \nu \Delta \mathbf{u} + \chi (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, & (t,\mathbf{x}) \in (0,T) \times \mathcal{D}, \\ \mathrm{div}(\mathbf{u}) = 0, & (t,\mathbf{x}) \in (0,T) \times \mathcal{D}, \\ \int_{\mathcal{D}} p(t,\mathbf{x}) \mathrm{d}\mathbf{x} = 0, & t \in (0,T), \\ (\text{if } \theta = 1) \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \mathcal{D}, \\ \mathbf{u} = \mathbf{g}, & (t,\mathbf{x}) \in (0,T) \times \partial\mathcal{D}, \end{cases}$$

where ν stands for the viscosity, $\mathbf{f} \in (L^2((0,T) \times \mathcal{D}))^2$ and $\mathbf{g} \in (L^2((0,T) \times \partial\mathcal{D}))^2$ and $\mathbf{u}_0 \in L^2(\mathcal{D})$.

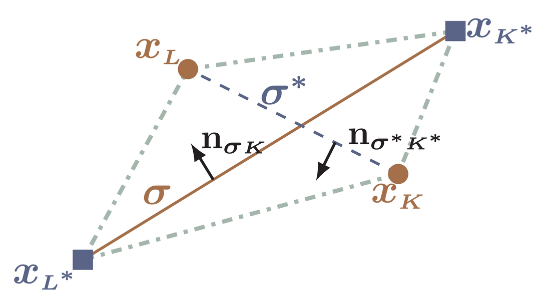
DDFV FRAMEWORK

- **The DDFV meshes:** primal, dual and "diamond"

Primal mesh $\overline{\mathfrak{M}} = \mathfrak{M} \cup \partial\mathfrak{M}$ — Dual mesh $\overline{\mathfrak{M}^*} = \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ — Diamond mesh $\mathfrak{D}$



- **Zoom on a diamond cell** D:



- **Discrete unknowns:**

$$\mathbf{u}_{\mathcal{T}} = ((\mathbf{u}_K)_K, (\mathbf{u}_{K^*})_{K^*}) \in (\mathbb{R}^2)^{\mathcal{T}} = (\mathbb{R}^2)^{\overline{\mathfrak{M}}} \times (\mathbb{R}^2)^{\overline{\mathfrak{M}^*}},$$

$$p_{\mathfrak{D}} = (p_D)_{D \in \mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$$

- **Discrete gradient:** $\nabla^{\mathfrak{D}}$ constant on each diamond cell,

$$\begin{cases} \nabla^D \mathbf{u}_{\mathcal{T}} (x_L - x_K) = \mathbf{u}_L - \mathbf{u}_K, \\ \nabla^D \mathbf{u}_{\mathcal{T}} (x_{L^*} - x_{K^*}) = \mathbf{u}_{L^*} - \mathbf{u}_{K^*}, \end{cases}$$

$$\nabla^D \mathbf{u}_{\mathcal{T}} = \frac{1}{2m_D} \left[m_\sigma (\mathbf{u}_L - \mathbf{u}_K) \otimes \mathbf{n}_{\sigma K} + m_{\sigma^*} (\mathbf{u}_{L^*} - \mathbf{u}_{K^*}) \otimes \mathbf{n}_{\sigma^* K^*} \right].$$

- **Discrete velocity divergence on the diamond mesh:**

$$\mathrm{div}^D \mathbf{u}_{\mathcal{T}} = \mathrm{Tr}(\nabla^D \mathbf{u}_{\mathcal{T}}), \quad \forall D \in \mathfrak{D}.$$

- **Discrete divergence on the primal and dual meshes :**

$$\mathbf{div}^{\mathcal{T}}(\xi_{\mathfrak{D}}) = \begin{cases} \mathbf{div}^K(\xi_{\mathfrak{D}}) = \frac{1}{m_K} \sum_{\sigma \in \partial K} m_\sigma \xi_D . \mathbf{n}_{\sigma K}, & \forall K \in \mathfrak{M}, \\ \mathbf{div}^{K^*}(\xi_{\mathfrak{D}}) = \frac{1}{m_{K^*}} \sum_{\sigma^* \in \partial K^*} m_{\sigma^*} \xi_D . \mathbf{n}_{\sigma^* K^*}, & \forall K^* \in \mathfrak{M}^*. \end{cases}$$

Pressure gradient $\rightsquigarrow$ $\nabla^{\mathcal{T}} p_{\mathfrak{D}} = \mathbf{div}^{\mathcal{T}}(p_{\mathfrak{D}} \mathrm{Id})$.

APPROXIMATION OF THE TERM $(\mathbf{u} \cdot \nabla)\mathbf{u}$

For any $\mathbf{v}, \mathbf{w}$ s.t. $\mathrm{div}\mathbf{v} = 0$,

$$\int_K (\mathbf{v} \cdot \nabla)\mathbf{w} = \sum_{\sigma \in \partial K} \int_\sigma (\mathbf{v} \cdot \mathbf{n}_{\sigma K}) \mathbf{w}, \ \forall K \in \mathfrak{M},$$

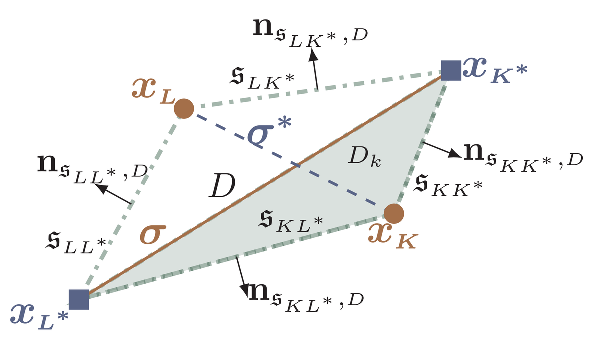
$$\int_{K^*} (\mathbf{v} \cdot \nabla)\mathbf{w} = \sum_{\sigma^* \in \partial K^*} \int_{\sigma^*} (\mathbf{v} \cdot \mathbf{n}_{\sigma^* K^*}) \mathbf{w}, \ \forall K^* \in \mathfrak{M}^*.$$

Definition: New operator $b_{\mathcal{T}}(\mathbf{v}_{\mathcal{T}}, \mathbf{w}_{\mathcal{T}})$

$$b_K(\mathbf{v}_{\mathcal{T}}, \mathbf{w}_{\mathcal{T}}) = \frac{1}{m_K} \sum_{\sigma \in \partial K} F_{\sigma,K}(\mathbf{v}_{\mathcal{T}}) \mathbf{w}_\sigma, \quad \text{with } \mathbf{w}_\sigma = \frac{\mathbf{w}_K + \mathbf{w}_L}{2};$$

$$b_{K^*}(\mathbf{v}_{\mathcal{T}}, \mathbf{w}_{\mathcal{T}}) = \frac{1}{m_{K^*}} \sum_{\sigma^* \subset \partial K^*} F_{\sigma^*,K^*}(\mathbf{v}_{\mathcal{T}}) \mathbf{w}_{\sigma^*}, \quad \text{with } \mathbf{w}_{\sigma^*} = \frac{\mathbf{w}_{K^*} + \mathbf{w}_{L^*}}{2};$$

$$\text{where} \quad F_{\sigma,K}(\mathbf{v}_{\mathcal{T}}) \sim \int_\sigma \mathbf{v} \cdot \mathbf{n}_{\sigma K} \quad \text{and} \quad F_{\sigma^*,K^*}(\mathbf{v}_{\mathcal{T}}) \sim \int_{\sigma^*} \mathbf{v} \cdot \mathbf{n}_{\sigma^* K^*}.$$



$$0 = \int_{D_k} \mathrm{div}\mathbf{v} = \int_\sigma \mathbf{v} \cdot \mathbf{n}_{\sigma K} + \underbrace{\int_{\mathfrak{s}_{KK^*}} \mathbf{v} \cdot \mathbf{n}_{\mathfrak{s}_{KK^*},D}}_{\sim F_{\mathfrak{s}_{KK^*},D}(\mathbf{v}_{\mathcal{T}})} + \underbrace{\int_{\mathfrak{s}_{KL^*}} \mathbf{v} \cdot \mathbf{n}_{\mathfrak{s}_{KL^*},D}}_{\sim F_{\mathfrak{s}_{KL^*},D}(\mathbf{v}_{\mathcal{T}})}$$

$$F_{\sigma,K}(\mathbf{v}_{\mathcal{T}}) = -\left(F_{\mathfrak{s}_{KK^*},D}(\mathbf{v}_{\mathcal{T}}) + F_{\mathfrak{s}_{KL^*},D}(\mathbf{v}_{\mathcal{T}})\right)$$

with

$$F_{\mathfrak{s}_{KK^*},D}(\mathbf{v}_{\mathcal{T}}) = m_{\mathfrak{s}_{KK^*}} \frac{\mathbf{v}_K + \mathbf{v}_{K^*}}{2} \cdot \mathbf{n}_{\mathfrak{s}_{KK^*},D}.$$

Properties:

- Conservativity: $0 = \mathrm{div}^D \mathbf{v}_{\mathcal{T}} = \frac{1}{m_D} \sum_{\mathfrak{s} \in \partial D} F_{\mathfrak{s},D}(\mathbf{v}_{\mathcal{T}})$

$$\Rightarrow \quad F_{\sigma,K}(\mathbf{v}_{\mathcal{T}}) = -F_{\sigma,L}(\mathbf{v}_{\mathcal{T}}) \ \text{and} \ F_{\sigma^*,K^*}(\mathbf{v}_{\mathcal{T}}) = -F_{\sigma^*,L^*}(\mathbf{v}_{\mathcal{T}}).$$

- Cancellation property:

$$[\![b_{\mathcal{T}}(\mathbf{v}_{\mathcal{T}}, \mathbf{w}_{\mathcal{T}}), \mathbf{w}_{\mathcal{T}}]\!]_{\mathcal{T}} = 0,$$

$$\text{where} \quad [\![\mathbf{u}_{\mathcal{T}}, \mathbf{v}_{\mathcal{T}}]\!]_{\mathcal{T}} = \frac{1}{2} \left(\sum_{K \in \mathfrak{M}} m_K \mathbf{u}_K \cdot \mathbf{v}_K + \sum_{K^* \in \mathfrak{M}^*} m_{K^*} \mathbf{u}_{K^*} \cdot \mathbf{v}_{K^*} \right).$$

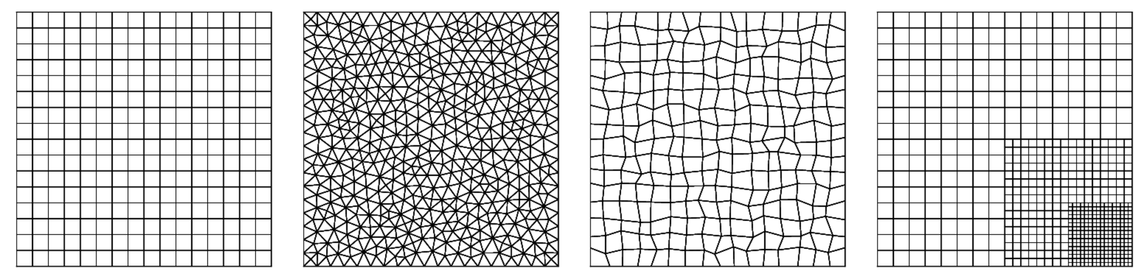
THE DDFV SCHEME

For $\mathbf{u}_{\mathcal{T}}^{\mathbf{n-1}} \in (\mathbb{R}^2)^{\mathcal{T}}$ and $\mathbf{u}_{\mathcal{T}}^{\mathbf{n}} \in (\mathbb{R}^2)^{\mathcal{T}}$ given, we define $(\mathbf{u}_{\mathcal{T}}^{\mathbf{n+1}}, p_{\mathfrak{D}}^{n+1}) \in (\mathbb{R}^2)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$ to be a solution to the following problem

$$\begin{cases} \theta \frac{\frac{3}{2}\mathbf{u}_{\mathcal{T}}^{\mathbf{n+1}} - 2\mathbf{u}_{\mathcal{T}}^{\mathbf{n}} + \frac{1}{2}\mathbf{u}_{\mathcal{T}}^{\mathbf{n-1}}}{\Delta t} - \nu \mathbf{div}^{\mathcal{T}}(\nabla^{\mathfrak{D}} \mathbf{u}_{\mathcal{T}}^{\mathbf{n+1}}) \\ \qquad + \chi \mathbf{b}^{\mathcal{T}}(2\mathbf{u}_{\mathcal{T}}^{\mathbf{n}} - \mathbf{u}_{\mathcal{T}}^{\mathbf{n-1}}, \widetilde{\mathbf{u}}_{\mathcal{T}}^{\mathbf{n+1}}) + \nabla^{\mathcal{T}} p_{\mathfrak{D}}^{n+1} = \mathbf{f}_{\mathcal{T}}^{\mathbf{n+1}}, \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}_{\mathcal{T}}^{\mathbf{n+1}}) = 0, \qquad \sum_{D \in \mathfrak{D}} m_D p_D^{n+1} = 0, \\ \mathbf{u}_{\partial\mathfrak{M}}^{\mathbf{n+1}} = \mathbf{g}_{\partial\mathfrak{M}}^{\mathbf{n+1}}, \qquad \mathbf{u}_{\partial\mathfrak{M}^*}^{\mathbf{n+1}} = \mathbf{g}_{\partial\mathfrak{M}^*}^{\mathbf{n+1}}, \end{cases}$$

$$\text{where} \quad \widetilde{\mathbf{u}}_{\mathcal{T}}^{\mathbf{n+1}} = \begin{cases} (2\mathbf{u}_{\mathcal{T}}^{\mathbf{n}} - \mathbf{u}_{\mathcal{T}}^{\mathbf{n-1}}), & \text{if } \Delta t \|\mathbf{u}_{\mathcal{T}}^{\mathbf{n}}\|_{L^\infty}^2 \le 2\nu, \\ \mathbf{u}_{\mathcal{T}}^{\mathbf{n+1}}, & \text{otherwise.} \end{cases}$$

THE MESHES

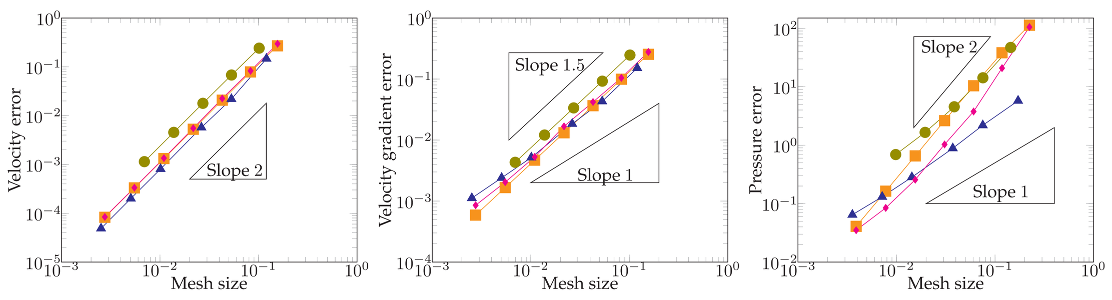


Uniform cartesian ■ Triangular ▲ Quadrangles ♦ Locally refined ●

ERROR ANALYSIS

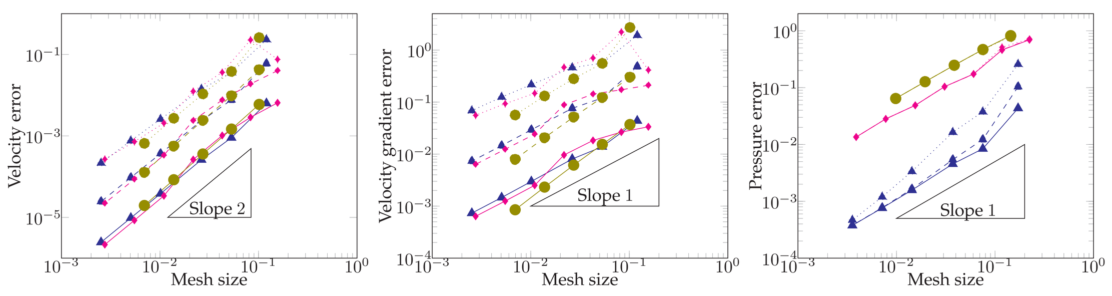
- **2D Steady Stokes tests: Bercovier-Engelman test case**

$\theta = 0$, $\chi = 0$, $\nu = 1$.



- **Steady 2D Navier-Stokes tests**

$\theta = 0$, $\chi = 1$, $\nu = 10^{-1}$, 10^{-2} (dash line), 10^{-3} (dot line).

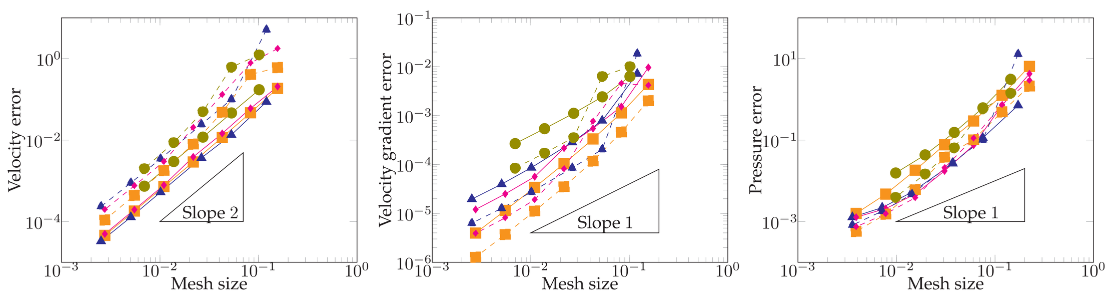


mesh	errgu	erru	errp	ordp
1	8.11406e-16	1.48627e-16	0.678997	nan
2	3.39314e-15	3.47145e-16	0.379343	0.908988
3	1.20171e-14	6.46699e-16	0.199709	0.965408
4	5.95251e-14	1.74916e-15	0.102346	0.985671
5	2.88888e-14	5.78869e-16	0.051792	0.993596
6	5.07121e-14	1.10146e-15	0.0260502	0.99699
7	1.38585e-13	4.21429e-15	0.0130635	0.998543

Rectangular mesh for $\nu = 10^{-3}$

- **Unsteady 2D Navier-Stokes tests**

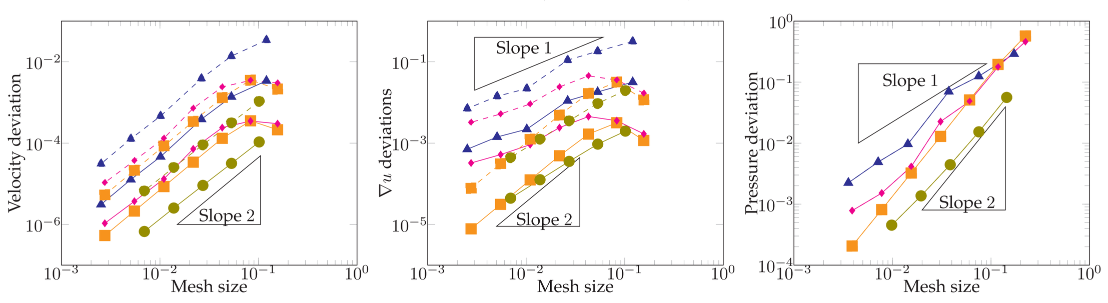
$\theta = 1$, $\chi = 1$, $\nu = 10^{-1}$, 10^{-2} (dash line), $T = \frac{1}{10\nu}$, $\Delta t = 10^{-3}$.



- **Robustness with respect to the invariance property**

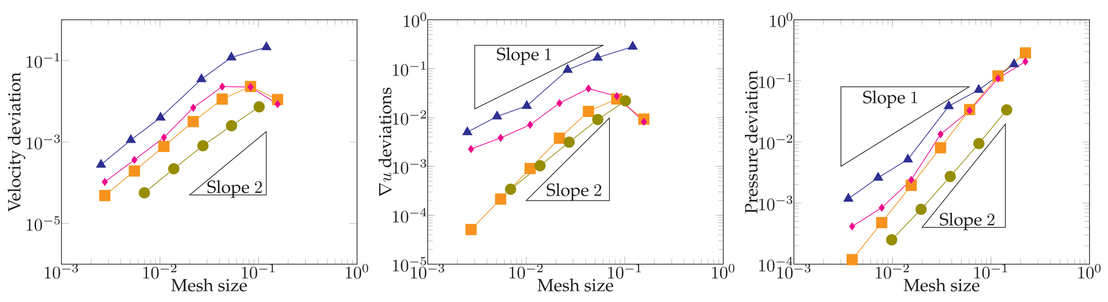
$\rightsquigarrow$ Test on the 2D steady Stokes system:

$\theta = 0$, $\chi = 0$, $\nu = 10^{-1}$, 10^{-2} (dash line).



$\rightsquigarrow$ Test on the 2D steady Navier-Stokes system:

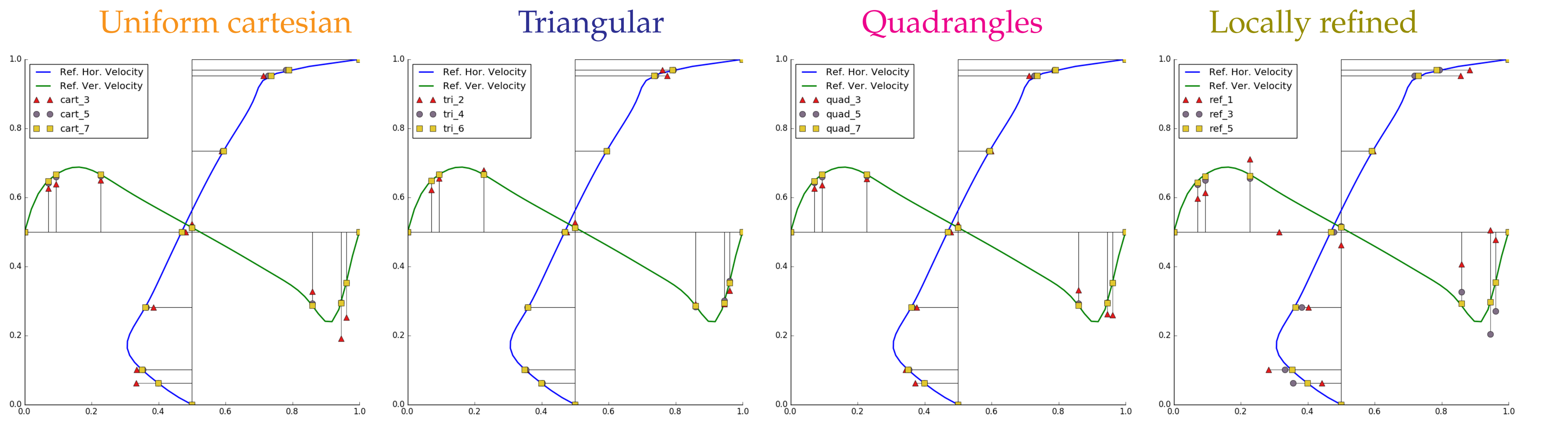
$\theta = 0$, $\chi = 1$, $\nu = \frac{1}{400}$.



2D LID DRIVEN CAVITY TESTS

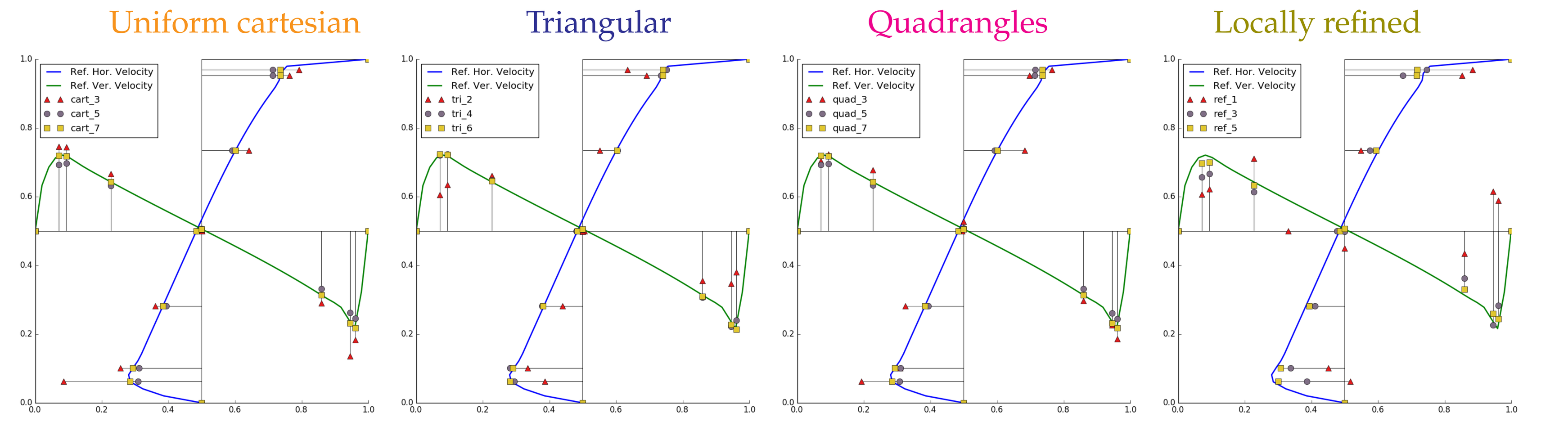
$Re = 1000$

Uniform cartesian — Triangular — Quadrangles — Locally refined

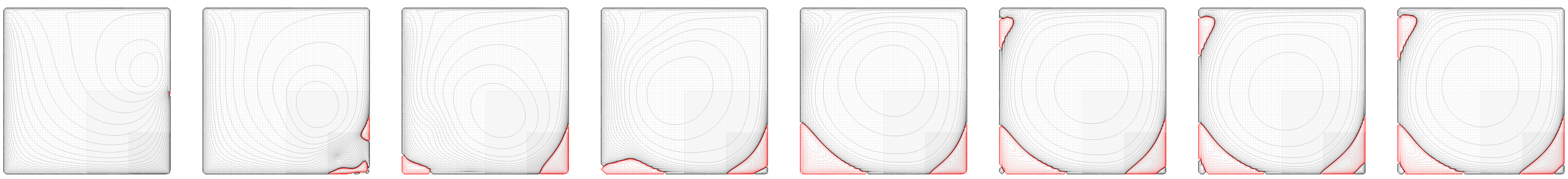


$Re = 5000$

Uniform cartesian — Triangular — Quadrangles — Locally refined



$Re = 10000$



$t = 5$ — $t = 10$ — $t = 15$ — $t = 20$ — $t = 25$ — $t = 30$ — $t = 35$ — $t = 40$

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