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ERRATUM: end of Section 2, stopping criterium “noeffectcoord”: “Stop if adding 0.2-standard deviation in each coordinate does change. . . ” should be “Stop if adding 0.2-standard deviation in any coordinate does not change. . . ”

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Abstract- One natural question when testing performance of global optimization algorithm is: how performances compare to a restart local search algorithm. One purpose of this paper is to provide results for such comparisons. To this end, the performances of a restart (advanced) local-search strategy, the CMA-ES with small initial step-size, are investigated on the 25 functions of the CEC 2005 real-parameter optimization test suit. The second aim is to clarify the theoretical background of the performance criterion proposed to quantitatively compare the search algorithms. The theoretical analysis allows us to generalize the criterion proposed and to define a new criterion that can be applied more appropriate in a different context.

1 Introduction

This paper introduces the restart version of the so-called (μ_W, λ)-CMA-ES (referred to as LR-CMA-ES) and applies it to the CEC 2005 real-parameter optimization benchmark function suit [7]. The LR-CMA-ES is a quasi parameter free, comparatively simple global optimization algorithm exploiting the advanced local search properties of the (μ_W, λ)-CMA-ES. Beyond the interest of the LR-CMA-ES as a global optimization algorithm, the purpose of investigating a restart strategy of a competitive local search evolutionary algorithm is to provide baseline results for the CEC 2005 real-parameter optimization benchmark function suit. The second contribution of this paper is the theoretical analysis of the (success) performance criterion proposed for reporting the results [7]. This analysis allows us to clarify and generalize the criterion.

The CMA-ES was originally introduced to improve the *local* search performances of evolution strategies [5]. It reduces the number of function evaluations to solve badly scaled quadratic problems by several orders of magnitude [6]. Compared to other evolutionary algorithms, an important property of the CMA-ES is its invariance against linear transformations of the search space. Surprisingly, also global search properties can be improved by the CMA [6], and, depending on the population size, the CMA-ES even reveals competitive global search performances [4].

In order to stress the local search characteristics of the CMA-ES, we use a hundred times smaller initial step-size than is recommended as default. Moreover we stick to the default population size (between 10 and 15 for the search space dimensions in this paper), which leads to competitive local search performances. The resulting algorithm can be then regarded as an *advanced* local search, because (a) the complete covariance matrix of the search distribution is efficiently adapted to the local topography of the objective function and (2) step-size adaptation can result in comparatively large steps even when the initial step-size is chosen to be small.

The remainder of this paper is organized as follows: Section 2 analyzes and generalizes the success performance criterion given in [7], Section 3 presents the bottom line of the algorithm, Section 4 explains the experimental procedure and Section 5 presents the experimental results.

2 Success performances

Comparing the performances of different algorithms on multi-modal problems implies to take into account that some algorithms may have a small probability of success but converge fast whereas others may have a larger probability of success but be slower. Thereby, one way to measure (success) performance of an Algorithm **A** is to investigate the expected number of function evaluations to reach a certain function value (the success criterion) by conducting independent restarts of **A**. We consider that **A** has a probability of success $p_{\mathrm{s}} \in (0, 1]$ (or success rate) and define T_A^{us} as the random variable measuring the running time (here, number of function evaluations) for unsuccessful runs of **A** stopped after a reasonable stopping criterion and T_A^s the number of evaluations for a successful run of **A**. Let T be the random variable measuring the overall running time (here: overall number of function evaluations) until the success criterion is met by independent restarts of **A**,

$$T = \sum_{k=1}^{N-1} (T_A^{us})_k + T_A^s \ , \qquad (1)$$

where $(T_A^{us})_k$ are independent random variables with the same distribution as T_A^{us} and N the random variable measuring the number of runs of **A** ($N-1$ unsuccessful, and 1 successful run). The random variable N follows a geometric distribution with parameter p_{s}. To compute the expectation $\mathbb{E}(T)$ of T, we first take the conditional expectation of Eq. 1 with respect to the random variable N and use the fact that $(T_A^{us})_k$ are i.i.d as T_A^{us}

$$\mathbb{E}(T|N) = (N-1)\mathbb{E}(T_A^{us}) + \mathbb{E}(T_A^s)$$

Taking now the expectation again, we obtain the general expression for the expectation of T

$$\begin{aligned} \mathbb{E}(T) &= (\mathbb{E}(N) - 1)\mathbb{E}(T_A^{us}) + \mathbb{E}(T_A^s) \\ &= \left(\frac{1-p_{\mathrm{s}}}{p_{\mathrm{s}}}\right)\mathbb{E}(T_A^{us}) + \mathbb{E}(T_A^s) \qquad (2) \end{aligned}$$

where we have used the property that the expectation of a geometric distribution of parameter $p_{\rm s}$ is equal to $1/p_{\rm s}$.

We can in the same way derive the variance of T starting from Eq. 1. We omit the intermediate steps and give the final expression:

$$\mathrm{var}(T) = \Big(\frac{1-p_{\rm s}}{p_{\rm s}}\Big)\mathrm{var}(T_A^{us}) + \Big(\frac{1-p_{\rm s}}{p_{\rm s}^2}\Big)\mathbb{E}\Big((T_A^{us})^2\Big) + \mathrm{var}(T_A^s) \quad (3)$$

where we use the notation var to denote the variance of a random variable.

From the general expression Eq. 2 for the expectation of T we now derive two specific (success) performance criteria for stochastic search algorithms. Moreover from Eq. 3 we will derive the variance of the second success performance criterion.

First, making the assumption that the expected number of evaluations for successful and unsuccessful runs is the same, *i.e* $\mathbb{E}(T_A^{us}) = \mathbb{E}(T_A^s)$, the RHS of Eq. 2 simplifies to the following expression that corresponds to the performance criterion defined in [7] (already proposed in [4])

$$\mathrm{SP1} = \frac{\mathbb{E}(T_A^s)}{p_{\rm s}}. \quad (4)$$

Second, we use the facts that the algorithm **A** investigated in this paper is a restart strategy, and that the maximum number of function evaluations is given as $FE_{\max} = n \times 10^4$ [7]. Then, because **A** proceeds with a restart whenever a stopping criterion is met, any unsuccessful run of **A** reaches the maximum number of function evaluations allowed. Therefore, $\mathbb{E}(T_A^{us}) = FE_{\max}$ and $\mathrm{var}(T_A^{us}) = 0$ where var denotes the variance of a random variable. These expressions lead to a different simplification of Eq. 2, and to the definition of the second success performance:

$$\mathrm{SP2} = \Big(\frac{1-p_{\rm s}}{p_{\rm s}}\Big)FE_{\max} + \mathbb{E}(T_A^s) \quad (5)$$

with its variance

$$\mathrm{var}(\mathrm{SP2}) = \Big(\frac{1-p_{\rm s}}{p_{\rm s}^2}\Big)(FE_{\max})^2 + \mathrm{var}(T_A^s)\ , \quad (6)$$

derived from Eq. 3.

Estimating the success performances The estimator of SP1 proposed in [7] can be derived by first estimating the probability of success $p_{\rm s}$ as

$$\widehat{p_{\rm s}} = \frac{\text{Nbr. successful runs}}{\text{Nbr. runs}}.$$

This estimator is a maximum likelihood estimator for p_s and is unbiased. Second the estimation of the expected number of function evaluations for successful runs is

$$\widehat{\mathbb{E}(T_A^s)} = \frac{\text{Nbr. of evaluations for successful runs}}{\text{Nbr. successful runs}}$$

When $\widehat{p_{\rm s}} \neq 0$, an estimator $\widehat{SP1}$ for SP1 is then

$$\widehat{SP1} = \frac{\widehat{\mathbb{E}(T_A^s)}}{\widehat{p_{\rm s}}}\ . \quad (7)$$

This estimator is asymptotically convergent.

Asymptotically convergent estimators for SP2 and $\mathrm{var}(\mathrm{SP2})$ can be derived using as well $\widehat{p_{\rm s}}$:

$$\widehat{\mathrm{SP2}} = \Big(\frac{1-\widehat{p_{\rm s}}}{\widehat{p_{\rm s}}}\Big)FE_{\max} + \widehat{\mathbb{E}(T_A^s)} \quad (8)$$

$$\widehat{\mathrm{var}(\mathrm{SP2})} = \Big(\frac{1-\widehat{p_{\rm s}}}{\widehat{p_{\rm s}}^2}\Big)(FE_{\max})^2 + \widehat{\mathrm{var}(T_A^s)} \quad (9)$$

with $\widehat{\mathrm{var}(T_A^s)}$ the classical unbiased estimator for the variance of the number of successful runs.

A more natural estimator for $\mathbb{E}(N) = \frac{1}{p_s}$ consists in sampling a fixed number of geometric random variables of parameter p_s and computing the empirical mean. In particular this estimator is unbiased. As sampling a geometric random variable of parameter p_s means restarting the algorithm **A** until success, sampling $N_{\rm success}$ geometric random variables implies doing runs of **A** until a fixed number of successes $N_{\rm success}$ is reached. Therefore the number of runs of **A** performed is not a fixed number (like for $\widehat{SP1}$ and $\widehat{SP2}$) but a random variable. This random variable depends on the probability of success and will increase in expectation for decreasing probability of success. The drawback of such an estimator in practice is that the number of runs performed is not fixed. However, since such an estimator is the empirical mean of independent identically distributed random variables, asymptotic confidence intervals can be derived from the Central Limit Theorem. Let $\widetilde{SP2}$ denote the estimator, with a probability of 0.95 we have asymptotically that $\mathrm{SP2} \in \widetilde{SP2} \pm \frac{1.96\sqrt{\widetilde{\mathrm{var}(\mathrm{SP2})}}}{\sqrt{N_{\rm success}}}$, where $N_{\rm success}$ is the number of geometric random variables sampled (corresponding to doing runs of **A** until $N_{\rm success}$ success are reached). This confidence interval suggests that confidence intervals for $\widehat{SP1}$ and $\widehat{SP2}$ scale like $\frac{1}{\sqrt{\widehat{p_{\rm s}} \times \text{Nbr. runs}}}$.

3 The restart CMA-ES

The (μ_W, λ)-CMA-ES In this paper we use the (μ_W, λ)-CMA-ES thoroughly described in [4]. We outline the general principle of the algorithm in short and refer to [4] for the details.

For generation $g+1$, λ offspring are sampled independently according to

$$\vec{x}_k^{(g+1)} \sim \mathcal{N}\left(\langle\vec{x}\rangle_{\rm W}^{(g)}, (\sigma^{(g)})^2\boldsymbol{C}^{(g)}\right) \text{ for } k = 1, \ldots, \lambda$$

where $\mathcal{N}(\vec{m}, \boldsymbol{C})$ denotes a normally distributed random vector with mean $\vec{m}$ and covariance matrix $\boldsymbol{C}$. The μ best offspring are recombined into $\langle\vec{x}\rangle_{\rm W}^{(g+1)} = \sum_{i=1}^{\mu} w_i \vec{x}_{i:\lambda}^{(g+1)}$, where the positive weights $w_i \in \mathbb{R}$ sum to one. The equations for updating the remaining parameters of the normal distribution are given in [4]: Eqs. 2 and 3 for the covariance matrix $\boldsymbol{C}$, Eqs. 4 and 5 for the step-size σ (cumulative path length control).[1] On convex quadratic functions, the adaptation mechanisms for σ and $\boldsymbol{C}$ allow to achieve log-linear convergence[2] after an adaptation time which can scale between 0 and n^2.

The default parameters for the strategy are given in [4], Eqs. 6-8. The default population size grows with $\log n$ and equals to $\lambda = 10, 14, 15$ for $n = 10, 30, 50$. Only $\langle\vec{x}\rangle_{\mathrm{W}}^{(0)}$ and $\sigma^{(0)}$ have to be set depending on the problem.

The local restart (μ_W, λ)-CMA-ES (LR-CMA-ES) For the restart strategy the (μ_W, λ)-CMA-ES is stopped whenever a stopping criterion is met, and a restart is launched. The new run uses the same strategy parameters and the same initialization procedure, and it is independent of all other runs.

To decide when to restart, the following stopping criteria are used.[3]

- Stop if the range of the best objective function values of the last $10 + \lceil 30n/\lambda \rceil$ generations is zero (`equalfunvalhist`), or the range of these function values and all function values of the last generation is below `Tolfun`$= 10^{-12}$.
- Stop if the standard deviation of the normal distribution is smaller than `TolX` in all coordinates and if $\sigma\vec{p}_c$ (the evolution path from Eq. 2 in [4]) is smaller than `TolX` in all components. We set `TolX`$= 10^{-12}\sigma^{(0)}$.
- Stop if adding a 0.1-standard deviation vector in a principal axis direction of $\boldsymbol{C}^{(g)}$ does not change $\langle\vec{x}\rangle_{\mathrm{W}}^{(g)}$ (`noeffectaxis`). More formally, stop if $\langle\vec{x}\rangle_{\mathrm{W}}^{(g)}$ equals to $\langle\vec{x}\rangle_{\mathrm{W}}^{(g)} + 0.1\,\sigma^{(g)}\sqrt{\lambda_i}\vec{u}_i$, where $i = (g \mod n) + 1$, and λ_i and $\vec{u}_i$ are respectively the ith eigenvalue and eigenvector of $\boldsymbol{C}^{(g)}$, with $\|\vec{u}_i\| = 1$.
- Stop if adding 0.2-standard deviation in each coordinate does change $\langle\vec{x}\rangle_{\mathrm{W}}^{(g)}$ (`noeffectcoord`).
- Stop if the condition number of the covariance matrix exceeds 10^{14} (`conditioncov`).

The distribution of the starting points $\langle\vec{x}\rangle_{\mathrm{W}}^{(0)}$ and the initial step-size $\sigma^{(0)}$ are problem dependent and their setting is described in the next section, as well as the overall stopping criteria for the LR-CMA-ES.

4 Experimental procedure

The LR-CMA-ES has been investigated on the 25 test functions described in [7] for dimension 10, 30 and 50. For each function a bounded subset $[A, B]^n$ of $\mathbb{R}^n$ is prescribed. The initial starting points $\langle\vec{x}\rangle_{\mathrm{W}}^{(0)}$ for each restart are sampled uniformly within this subset and the initial step-size $\sigma^{(0)}$ for each restart is equal to $10^{-2}(B - A)/2$. The overall stopping criteria for the algorithm prescribed in [7] are: stop before $n \cdot 10^4$ function evaluations or stop if the error in the function values is below 10^{-8}. The boundary handling is done according to the standard implementation of CMA-ES and consists in penalizing the individuals in the infeasible region.[4] For each test function, 25 runs are performed. All performance criteria were evaluated based on the same runs. In particular, the times when to measure the objective function error value (namely at 10^3, 10^4, 10^5 function evaluations) were not used as input parameter to the algorithm (*e.g.,* to set the maximum number of function evaluations to adjust an annealing rate).

Table 1: Measured CPU-seconds, according to [7], using MATLAB 7.0.1, Red Hat Linux 2.4, 1GByte RAM, Pentium 4 3GHz processor. Time T2 is the CPU-time for running the restart CMA-ES until 2×10^5 function evaluations on function 3. The smaller number for T2 for $n = 30$ compared to $n = 10$ is caused by the 1.4 times larger population size for $n = 30$. Because each population is evaluated (serially) within a single function call the number of function calls to reach 2×10^5 function evaluations is smaller

	T0	T1	T2
$n = 10$	0.4s	32s	51s
$n = 30$	0.4s	41s	45s
$n = 50$	0.4s	49s	68s

Test functions The complete definition of the test suit is available in [7]. The definition of functions 1 to 12 is based on classical benchmark functions, that we will refer in the sequel also by their name. Functions 1 to 5 are unimodal and functions 6 to 12 are multi-modal. Functions 13 to 25 result from the composition of several functions. To prevent exploitation of symmetry of the search space and of the typical zero value associated with the global optimum, the local optimum is shifted to a value different from zero and the function values of the global optima are non zero.

5 Results

Figure 1 presents the convergence graphs of objective function error values. The steps in the graphs are caused by the restarts that improve the performance on the noisy function 4 and on the multi-modal functions 11–13, and 15–17.

According to the requirements, Table 1 reports CPU-time measurements, Table 2 gives the number of function evaluations to reach the success criterion (if successful), the success rate, and the success performances as defined in Section 2. The objective function error values after 10^3, 10^4, 10^5 and $n \times 10^4$ function evaluations are presented in Table 3, 4, and 5.

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[1] A more elaborated algorithm description can be accessed via http://www.bionik.tu-berlin.de/user/niko/cmatutorial.pdf.

[2] On a log scale the performance is linear with respect to the number of function evaluations.

[3] These stopping criteria were developed before the benchmark function suit used in this paper was assembled.

[4] For details refer to the used MATLAB code, cmaes.m, Version 2.35, see http://www.bionik.tu-berlin.de/user/niko/formersoftwareversions.html

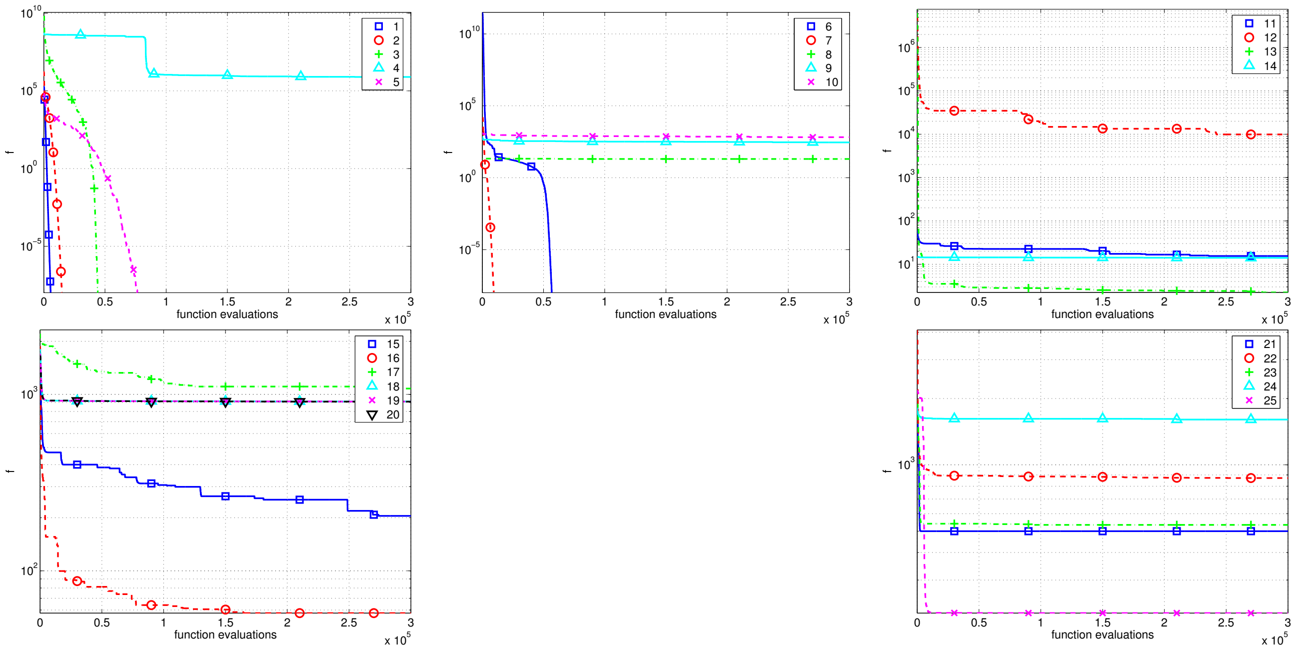

Figure 1: Best objective function error value (log scale) versus number of function evaluations for the 25 benchmark functions in dimension $n = 30$. For each run the best individual found until the current generation is considered and shown is the median value of 25 runs at each generation. The respective problem number is given in the legend.

In some cases (function 13, $n = 10$; function 21, $n = 10, 50$, and function 23, $n = 50$), the LR-CMA-ES significantly outperforms IPOP-CMA-ES [1], a strategy with successively increased population size.[5] Furthermore, the diversity between the runs is often larger for the LR-CMA-ES, and the best of 25 runs is (sligthly) better in most cases on functions 15, 18–23, and 25.

The performance on functions 1–3, and 5–7 is highly competitive (besides $n = 50$ for function 5). Even on the multi-modal Griewank function 7 the success rate is 100% (see Table 2). Here we observe an initial increase of the step-size by about two orders of magnitude almost up to a step-size that is typically used as initial step-size. If the upper bound for the step-size is set to the initial step-size, results on the Griewank function become significantly worse (not shown). This clearly indicates that, due to its step-size adaptation, the LR-CMA-ES allows yet to search more globally than a pure local search method.

On the remaining multi-modal functions LR-CMA-ES fails to locate the global optimum. While on Rastrigin function 9 and 10 the CMA-ES with large population size is able to locate the global optimum [4, 1], for the composite functions 13–25 the feasibility of locating the global optimum has yet to be shown. The failure on the noisy function 4 can be explained by the small initial step-size. In a highly noisy environment, the cumulative step-size adaptation of the CMA fails to enlarge the step-size [2].

[5]The statistical test tables for the non-parametric Wilcoxon rank sum test for different median values are given in [1]. To judge the significance level, we apply the (most conservative) Bonferroni correction for multiple testing.

6 Summary and conclusions

In this paper empirical results on the CEC 2005 real parameter optimization benchmark function suit are presented for the LR-CMA-ES, a local restart strategy of a competitive local search evolutionary algorithm. The purpose of the results is to be a baseline for comparison. On the non-noisy unimodal and on few multi-modal functions the LR-CMA-ES reveals competitive performance. The results on the remaining multi-modal functions serve as a benchmark that any ambitious global search algorithm has to beat.

Second, we have introduced a general performance criterion based on the success rate and the running times of successful and unsuccessful runs (*e.g.* in terms of number of function evaluations). This criterion reveals a meaningful *single* number that can be used to *quantitatively* compare search algorithms on functions, where at least one run reaches a given success criterion (*e.g.* a given function value).

Finally we emphasize two important aspects that have to be taken into account when judging the performance of search algorithms. First, the LR-CMA-ES is quasi parameter free:[6] in the presented experiments only the initial search

[6]Remark that the number of parameters in the description of an algorithm is somewhat arbitrary: the more general the description, the more parameters appear. Therefore, the existence or absence of parameters in the algorithm description cannot have influence on the assessment of the number of parameters that need to be (empirically or heuristically) determined each time the algorithm is applied. For the CMA-ES, strategy parameters have been chosen in advance, based on principle algorithmic considerations and in-depth empirical investigations on a few simple test functions. To our experience the strategy parameters (*e.g.* a learning rate, a time horizon, or a damping factor) mostly depend on algorithmic internal considerations and on the search space dimension, and to a much lesser extend on

Table 2: Performance measures for successfully optimized problems. Prob.: Problem number; Tol: success criterion on function value error; 3rd–9th column: number of function evaluations (minimal, 7^{th}, median, 19^{th}, maximal, mean and standard deviation) to reach the success criterion Tol; 10th–14th column: Empirical estimators for the success probability p_s, for the success performance criteria SP1 and SP2, and for $\mathrm{std}(\mathrm{SP2})$

$n = 10$

Prob.	Tol	min	7^{th}	median	19^{th}	max	mean	std	$\widehat{p_s}$	$\widehat{\mathrm{SP1}}$	$\widehat{\mathrm{SP2}}$	$\sqrt{\mathrm{var}(\widehat{\mathrm{SP2}})}$
1	1e−6	1.60e+3	1.65e+3	1.71e+3	1.82e+3	1.91e+3	1.74e+3	1.02e+2	1.00	1.74e+3	1.74e+3	1.02e+2
2	1e−6	2.35e+3	2.50e+3	2.61e+3	2.71e+3	2.83e+3	2.61e+3	1.36e+2	1.00	2.61e+3	2.61e+3	1.36e+2
3	1e−6	6.46e+3	6.62e+3	6.82e+3	7.08e+3	7.46e+3	6.84e+3	2.64e+2	1.00	6.84e+3	6.84e+3	2.64e+2
4	1e−6	6.10e+3	9.99e+4	-	-	-	5.39e+4	4.23e+4	0.28	1.93e+5	3.11e+5	3.06e+5
5	1e−6	5.30e+3	5.67e+3	5.80e+3	5.95e+3	6.79e+3	5.86e+3	3.68e+2	1.00	5.86e+3	5.86e+3	3.68e+2
6	1e−2	5.43e+3	6.55e+3	7.80e+3	9.45e+3	1.80e+4	9.13e+3	3.63e+3	1.00	9.13e+3	9.13e+3	3.63e+3
7	1e−2	1.55e+3	2.00e+3	2.12e+3	6.25e+3	2.17e+4	5.50e+3	5.55e+3	1.00	5.50e+3	5.50e+3	5.55e+3
8	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
9	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
10	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
11	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
12	1e−2	2.99e+3	4.26e+4	-	-	-	4.53e+4	2.89e+4	0.48	9.45e+4	1.54e+5	1.53e+5

$n = 30$

Prob.	Tol	min	7^{th}	median	19^{th}	max	mean	std	$\widehat{p_s}$	$\widehat{\mathrm{SP1}}$	$\widehat{\mathrm{SP2}}$	$\sqrt{\mathrm{var}(\widehat{\mathrm{SP2}})}$
1	1e−6	4.50e+3	4.69e+3	4.75e+3	4.85e+3	5.05e+3	4.78e+3	1.40e+2	1.00	4.78e+3	4.78e+3	1.40e+2
2	1e−6	1.29e+4	1.32e+4	1.35e+4	1.39e+4	1.43e+4	1.36e+4	4.37e+2	1.00	1.36e+4	1.36e+4	4.37e+2
3	1e−6	4.23e+4	4.30e+4	4.34e+4	4.38e+4	4.50e+4	4.34e+4	6.52e+2	1.00	4.34e+4	4.34e+4	6.52e+2
4	1e−6	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
5	1e−6	4.72e+4	6.22e+4	7.20e+4	7.74e+4	9.90e+4	7.10e+4	1.10e+4	1.00	7.10e+4	7.10e+4	1.10e+4
6	1e−2	4.25e+4	4.90e+4	5.28e+4	7.11e+4	1.30e+5	6.41e+4	2.52e+4	1.00	6.41e+4	6.41e+4	2.52e+4
7	1e−2	5.08e+3	5.31e+3	5.61e+3	5.80e+3	1.83e+4	7.00e+3	4.07e+3	1.00	7.00e+3	7.00e+3	4.07e+3
8	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
9	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
10	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
11	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
12	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-

$n = 50$

Prob.	Tol	min	7^{th}	median	19^{th}	max	mean	std	$\widehat{p_s}$	$\widehat{\mathrm{SP1}}$	$\widehat{\mathrm{SP2}}$	$\sqrt{\mathrm{var}(\widehat{\mathrm{SP2}})}$
1	1e−6	7.01e+3	7.18e+3	7.30e+3	7.38e+3	7.70e+3	7.30e+3	1.57e+2	1.00	7.30e+3	7.30e+3	1.57e+2
2	1e−6	3.10e+4	3.21e+4	3.23e+4	3.33e+4	3.43e+4	3.26e+4	7.64e+2	1.00	3.26e+4	3.26e+4	7.64e+2
3	1e−6	1.15e+5	1.17e+5	1.18e+5	1.18e+5	1.19e+5	1.17e+5	1.05e+3	1.00	1.17e+5	1.17e+5	1.05e+3
4	1e−6	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
5	1e−6	4.72e+5	-	-	-	-	4.84e+5	1.70e+4	0.08	6.04e+6	6.23e+6	5.99e+6
6	1e−2	1.11e+5	1.28e+5	1.36e+5	1.47e+5	4.27e+5	1.66e+5	7.41e+4	1.00	1.66e+5	1.66e+5	7.41e+4
7	1e−2	7.86e+3	8.36e+3	8.56e+3	8.71e+3	9.18e+3	8.54e+3	3.50e+2	1.00	8.54e+3	8.54e+3	3.50e+2
8	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
9	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
10	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
11	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-
12	1e−2	-	-	-	-	-	-	-	0.00	-	0.00e+0	-

region was defined problem dependent and no (further) parameter tuning was performed. Second, the LR-CMA-ES has several invariance properties [3], like invariance against order preserving transformations of the objective function values and invariance against linear transformations of the search space [6]. Invariance properties are highly desirable, because they imply uniform performance on classes of functions and therefore allow for generalization of the empirical results.

Invariances and the procedure to adjust parameters need to be carefully regarded for a conclusive performance evaluation of search algorithms.

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the specific objective function the algorithm is applied to. Nevertheless, it is possible to improve the performance by tuning strategy parameters and stopping criteria on most (all?) functions.

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Table 3: Best objective function error value reached after 10^3, 10^4 and 10^5 function evaluations (FES) respectively (rows) on the 25 test problems (columns) in dimension $n = 10$. Given are minimum, 7^{th}, median, 19^{th}, and maximum value from 25 runs, as well as mean and standard deviation. A run is stopped whenever the objective function error value drops below 10^{-8} and its final value is used for all larger FES

FES	Prob.	1	2	3	4	5	6	7	8	9	10	11	12
1e3	min	1.35e−2	1.78e+1	7.59e+5	1.07e+7	1.66e+1	3.18e+1	8.65e−1	2.04e+1	3.40e+1	4.78e+1	4.83e+0	1.75e+1
	7^{th}	3.41e−2	7.08e+1	1.44e+6	1.68e+7	6.27e+1	7.97e+2	1.04e+0	2.07e+1	8.76e+1	2.02e+2	9.23e+0	6.15e+3
	med.	6.39e−2	1.15e+2	2.28e+6	1.77e+7	2.18e+2	3.83e+3	1.16e+0	2.08e+1	1.46e+2	2.75e+2	1.18e+1	1.35e+4
	19^{th}	1.49e−1	2.38e+2	1.19e+7	1.94e+7	3.60e+2	5.82e+3	1.36e+0	2.09e+1	1.57e+2	3.30e+2	1.30e+1	3.67e+4
	max	7.71e−1	1.80e+3	6.44e+7	2.34e+7	3.05e+3	1.69e+4	2.07e+0	2.09e+1	2.57e+2	5.59e+2	1.86e+1	1.34e+5
	mean	1.45e−1	2.60e+2	9.33e+6	1.74e+7	4.74e+2	4.37e+3	1.26e+0	2.07e+1	1.31e+2	2.76e+2	1.15e+1	2.96e+4
	std	1.94e−1	3.96e+2	1.41e+7	3.19e+6	7.02e+2	4.08e+3	3.06e−1	1.43e−1	5.81e+1	1.29e+2	3.45e+0	3.71e+4
1e4	min	1.81e−9	2.41e−9	1.35e−9	7.35e−9	3.35e−9	2.31e−9	2.46e−9	2.00e+1	3.40e+1	2.29e+1	2.56e+0	3.79e−9
	7^{th}	3.83e−9	3.80e−9	4.30e−9	4.22e+4	5.19e−9	4.69e−9	3.79e−9	2.00e+1	5.57e+1	1.03e+2	7.53e+0	4.22e+2
	med.	5.39e−9	4.99e−9	5.58e−9	1.46e+7	6.83e−9	6.59e−9	7.65e−9	2.05e+1	8.16e+1	1.48e+2	8.71e+0	1.11e+4
	19^{th}	6.58e−9	6.48e−9	5.97e−9	1.81e+7	7.61e−9	1.44e−4	9.86e−3	2.06e+1	1.04e+2	2.39e+2	1.10e+1	2.72e+4
	max	8.59e−9	8.76e−9	7.01e−9	2.24e+7	9.83e−9	4.38e+0	1.72e−2	2.07e+1	1.70e+2	3.03e+2	1.38e+1	1.32e+5
	mean	5.14e−9	5.31e−9	4.94e−9	1.05e+7	6.57e−9	9.62e−1	4.84e−3	2.04e+1	8.60e+1	1.68e+2	8.61e+0	2.41e+4
	std	1.82e−9	1.77e−9	1.45e−9	8.93e+6	1.88e−9	1.75e+0	6.06e−3	2.56e−1	3.84e+1	8.13e+1	3.08e+0	3.60e+4
1e5	min	1.81e−9	2.41e−9	1.35e−9	3.99e−9	3.35e−9	2.31e−9	2.32e−9	2.00e+1	1.69e+1	6.96e+0	1.30e−1	1.88e−9
	7^{th}	3.83e−9	3.80e−9	4.30e−9	2.12e−7	5.19e−9	4.22e−9	3.71e−9	2.00e+1	3.58e+1	1.39e+1	2.56e+0	5.12e−9
	med.	5.39e−9	4.99e−9	5.58e−9	2.45e+4	6.83e−9	4.77e−9	4.79e−9	2.00e+1	4.78e+1	2.29e+1	3.42e+0	1.00e+1
	19^{th}	6.58e−9	6.48e−9	5.97e−9	2.25e+5	7.61e−9	6.84e−9	6.22e−9	2.00e+1	5.27e+1	7.59e+1	4.80e+0	3.55e+1
	max	8.59e−9	8.76e−9	7.01e−9	1.65e+7	9.83e−9	9.65e−9	7.83e−9	2.00e+1	6.27e+1	1.04e+2	6.77e+0	1.56e+3
	mean	5.14e−9	5.31e−9	4.94e−9	1.79e+6	6.57e−9	5.41e−9	4.91e−9	2.00e+1	4.49e+1	4.08e+1	3.65e+0	2.09e+2
	std	1.82e−9	1.77e−9	1.45e−9	4.66e+6	1.88e−9	1.81e−9	1.68e−9	0.00e+0	1.36e+1	3.35e+1	1.66e+0	4.69e+2

FES	Prob.	13	14	15	16	17	18	19	20	21	22	23	24	25
1e3	min	2.61e+0	4.11e+0	2.48e+2	1.68e+2	2.72e+2	4.07e+2	3.94e+2	3.22e+2	4.11e+2	7.13e+2	5.54e+2	2.00e+2	4.12e+2
	7^{th}	3.38e+0	4.48e+0	5.23e+2	2.04e+2	1.12e+3	9.42e+2	8.31e+2	8.01e+2	5.06e+2	8.18e+2	1.13e+3	1.05e+3	4.51e+2
	med.	3.70e+0	4.84e+0	9.07e+2	2.71e+2	1.43e+3	1.00e+3	9.76e+2	9.69e+2	1.23e+3	8.97e+2	1.26e+3	1.56e+3	5.06e+2
	19^{th}	4.11e+0	4.97e+0	1.25e+3	6.43e+2	1.71e+3	1.04e+3	1.05e+3	1.04e+3	1.45e+3	9.97e+2	1.77e+3	1.79e+3	2.16e+3
	max	5.92e+0	5.00e+0	2.22e+3	2.27e+3	2.47e+3	2.14e+3	2.16e+3	2.24e+3	2.37e+3	2.26e+3	2.17e+3	2.34e+3	2.78e+3
	mean	3.87e+0	4.73e+0	9.43e+2	4.98e+2	1.46e+3	9.92e+2	9.81e+2	9.88e+2	1.10e+3	1.11e+3	1.36e+3	1.39e+3	1.13e+3
	std	7.95e−1	2.75e−1	4.73e+2	4.66e+2	5.69e+2	3.09e+2	2.94e+2	3.66e+2	5.70e+2	4.94e+2	4.91e+2	6.29e+2	8.99e+2
1e4	min	5.08e−1	3.84e+0	1.17e+2	9.39e+1	1.39e+2	3.00e+2	3.00e+2	3.00e+2	3.00e+2	5.00e+2	5.54e+2	2.00e+2	3.71e+2
	7^{th}	6.87e−1	4.39e+0	4.00e+2	1.19e+2	6.59e+2	8.00e+2	8.00e+2	7.86e+2	4.10e+2	7.92e+2	8.34e+2	5.00e+2	3.81e+2
	med.	8.24e−1	4.50e+0	4.00e+2	1.29e+2	1.05e+3	9.21e+2	8.40e+2	8.00e+2	8.00e+2	8.00e+2	1.13e+3	1.51e+3	3.83e+2
	19^{th}	9.16e−1	4.84e+0	8.75e+2	1.40e+2	1.36e+3	9.77e+2	9.80e+2	9.34e+2	1.17e+3	8.88e+2	1.57e+3	1.78e+3	3.93e+2
	max	1.60e+0	4.98e+0	1.35e+3	5.00e+2	2.31e+3	1.10e+3	1.45e+3	1.17e+3	1.85e+3	2.08e+3	2.17e+3	2.30e+3	2.52e+3
	mean	8.79e−1	4.57e+0	6.06e+2	1.49e+2	1.08e+3	8.40e+2	8.16e+2	7.75e+2	8.32e+2	9.12e+2	1.22e+3	1.22e+3	7.16e+2
	std	3.23e−1	3.28e−1	3.81e+2	8.01e+1	6.59e+2	2.17e+2	2.74e+2	2.60e+2	4.53e+2	3.33e+2	5.16e+2	7.14e+2	7.11e+2
1e5	min	1.88e−1	3.36e+0	7.35e+1	6.14e+1	1.23e+2	3.00e+2	3.00e+2	3.00e+2	2.00e+2	5.00e+2	4.25e+2	2.00e+2	2.00e+2
	7^{th}	4.16e−1	3.67e+0	1.42e+2	9.82e+1	2.14e+2	3.00e+2	3.00e+2	3.00e+2	3.00e+2	7.38e+2	5.59e+2	2.00e+2	3.75e+2
	med.	4.79e−1	4.05e+0	2.00e+2	1.06e+2	4.91e+2	3.78e+2	4.19e+2	3.55e+2	4.10e+2	7.43e+2	5.60e+2	5.00e+2	3.77e+2
	19^{th}	5.60e−1	4.25e+0	2.12e+2	1.14e+2	8.11e+2	8.00e+2	8.00e+2	5.00e+2	4.10e+2	7.48e+2	1.09e+3	1.50e+3	3.82e+2
	max	8.17e−1	4.43e+0	4.42e+2	1.21e+2	1.33e+3	8.00e+2	9.76e+2	9.07e+2	8.00e+2	9.00e+2	1.26e+3	2.18e+3	2.15e+3
	mean	4.94e−1	4.01e+0	2.11e+2	1.05e+2	5.49e+2	4.97e+2	5.16e+2	4.42e+2	4.04e+2	7.40e+2	7.91e+2	8.65e+2	4.42e+2
	std	1.38e−1	3.14e−1	1.02e+2	1.26e+1	3.49e+2	2.18e+2	2.34e+2	2.03e+2	1.23e+2	5.94e+1	2.79e+2	6.39e+2	3.58e+2

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Table 4: Best objective function error values reached in dimension $n = 30$, see caption of Table 3 for details

FES	Prob.	1	2	3	4	5	6	7	8	9	10	11	12
1e3	min	1.06e+3	3.99e+4	6.89e+7	3.33e+8	1.23e+4	6.06e+7	5.32e+2	2.11e+1	2.62e+2	6.76e+2	3.19e+1	1.91e+5
	7^{th}	3.49e+3	5.04e+4	1.77e+8	3.93e+8	1.61e+4	1.89e+8	6.89e+2	2.11e+1	4.01e+2	9.14e+2	3.60e+1	3.16e+5
	med.	4.74e+3	6.21e+4	2.19e+8	4.30e+8	1.92e+4	2.74e+8	7.99e+2	2.12e+1	4.44e+2	1.10e+3	4.01e+1	4.67e+5
	19^{th}	5.43e+3	7.67e+4	3.01e+8	4.41e+8	2.20e+4	4.90e+8	9.07e+2	2.12e+1	5.78e+2	1.29e+3	4.39e+1	5.56e+5
	max	9.11e+3	9.69e+4	4.24e+8	4.82e+8	2.43e+4	2.05e+9	1.26e+3	2.13e+1	7.29e+2	1.99e+3	4.94e+1	7.69e+5
	mean	4.54e+3	6.38e+4	2.36e+8	4.20e+8	1.89e+4	4.88e+8	8.24e+2	2.12e+1	4.74e+2	1.14e+3	4.01e+1	4.54e+5
	std	1.83e+3	1.81e+4	9.82e+7	3.95e+7	3.32e+3	5.29e+8	1.89e+2	6.51e−2	1.19e+2	3.20e+2	4.65e+0	1.57e+5
1e4	min	2.98e−9	6.69e−3	3.42e+5	9.21e+4	7.06e+2	2.17e+1	4.87e−9	2.09e+1	2.59e+2	6.56e+2	1.34e+1	3.97e+3
	7^{th}	4.81e−9	3.41e−2	7.71e+5	3.48e+8	1.22e+3	2.59e+1	5.95e−9	2.11e+1	3.44e+2	8.39e+2	2.46e+1	1.51e+4
	med.	5.40e−9	5.10e−2	1.10e+6	4.03e+8	1.81e+3	5.73e+1	7.06e−9	2.11e+1	4.15e+2	9.25e+2	3.00e+1	3.74e+4
	19^{th}	5.82e−9	1.27e−1	1.79e+6	4.35e+8	2.15e+3	2.40e+2	8.79e−9	2.12e+1	4.70e+2	1.10e+3	3.59e+1	9.02e+4
	max	7.06e−9	2.80e−1	3.17e+6	4.71e+8	3.46e+3	1.22e+3	1.48e−2	2.12e+1	6.49e+2	1.45e+3	4.07e+1	3.14e+5
	mean	5.28e−9	9.47e−2	1.28e+6	3.50e+8	1.75e+3	1.96e+2	2.46e−3	2.11e+1	4.19e+2	9.73e+2	2.91e+1	7.55e+4
	std	9.82e−10	8.76e−2	7.13e+5	1.44e+8	6.88e+2	3.05e+2	4.72e−3	6.69e−2	1.02e+2	2.12e+2	7.54e+0	8.78e+4
1e5	min	2.98e−9	4.90e−9	2.98e−9	2.31e+1	5.23e−9	4.28e−9	4.46e−9	2.00e+1	2.53e+2	5.44e+2	9.84e+0	1.15e+3
	7^{th}	4.81e−9	6.49e−9	4.55e−9	1.94e+5	7.39e−9	5.55e−9	5.56e−9	2.00e+1	2.96e+2	7.15e+2	1.73e+1	5.50e+3
	med.	5.40e−9	6.91e−9	5.24e−9	1.09e+6	8.65e−9	6.71e−9	6.44e−9	2.00e+1	3.26e+2	7.73e+2	2.26e+1	1.90e+4
	19^{th}	5.82e−9	7.24e−9	5.74e−9	4.08e+8	9.55e−9	7.68e−9	7.13e−9	2.05e+1	3.64e+2	8.37e+2	2.56e+1	3.55e+4
	max	7.06e−9	8.77e−9	7.57e−9	4.57e+8	5.85e−7	3.99e+0	8.79e−9	2.11e+1	4.30e+2	9.96e+2	2.82e+1	2.50e+5
	mean	5.28e−9	6.93e−9	5.18e−9	1.93e+8	3.13e−8	3.64e−1	6.48e−9	2.03e+1	3.28e+2	7.81e+2	2.15e+1	3.39e+4
	std	9.82e−10	8.27e−10	1.03e−9	2.08e+8	1.15e−7	1.04e+0	1.14e−9	4.14e−1	4.46e+1	1.13e+2	5.12e+0	5.08e+4
3e5	min	2.98e−9	4.90e−9	2.98e−9	2.31e+1	5.23e−9	4.28e−9	4.46e−9	2.00e+1	2.43e+2	4.20e+1	6.85e+0	1.29e+0
	7^{th}	4.81e−9	6.49e−9	4.55e−9	1.85e+5	7.39e−9	5.55e−9	5.56e−9	2.00e+1	2.60e+2	5.80e+2	1.31e+1	3.67e+3
	med.	5.40e−9	6.91e−9	5.24e−9	7.84e+5	8.65e−9	6.35e−9	6.44e−9	2.00e+1	2.86e+2	6.56e+2	1.55e+1	9.91e+3
	19^{th}	5.82e−9	7.24e−9	5.74e−9	3.36e+6	9.34e−9	7.38e−9	7.13e−9	2.00e+1	3.13e+2	7.42e+2	1.75e+1	1.87e+4
	max	7.06e−9	8.77e−9	7.57e−9	4.48e+8	9.99e−9	8.44e−9	8.79e−9	2.00e+1	3.66e+2	7.92e+2	2.26e+1	3.50e+4
	mean	5.28e−9	6.93e−9	5.18e−9	9.26e+7	8.30e−9	6.31e−9	6.48e−9	2.00e+1	2.91e+2	5.63e+2	1.52e+1	1.32e+4
	std	9.82e−10	8.27e−10	1.03e−9	1.68e+8	1.38e−9	1.14e−9	1.14e−9	9.62e−15	3.54e+1	2.48e+2	3.51e+0	1.15e+4

FES	Prob.	13	14	15	16	17	18	19	20	21	22	23	24	25
1e3	min	2.98e+1	1.41e+1	5.24e+2	3.25e+2	1.51e+3	1.02e+3	1.01e+3	1.04e+3	1.02e+3	1.08e+3	1.06e+3	1.36e+3	1.99e+3
	7^{th}	6.37e+1	1.45e+1	7.18e+2	3.97e+2	1.76e+3	1.08e+3	1.08e+3	1.09e+3	1.13e+3	1.23e+3	1.26e+3	1.63e+3	2.05e+3
	med.	1.32e+2	1.48e+1	8.40e+2	4.81e+2	1.96e+3	1.15e+3	1.17e+3	1.15e+3	1.19e+3	1.31e+3	1.34e+3	1.69e+3	2.09e+3
	19^{th}	2.15e+2	1.49e+1	9.65e+2	6.09e+2	2.14e+3	1.30e+3	1.38e+3	1.19e+3	1.22e+3	1.51e+3	1.47e+3	1.80e+3	2.17e+3
	max	1.19e+3	1.50e+1	1.36e+3	1.56e+3	2.36e+3	1.74e+3	1.78e+3	1.49e+3	1.25e+3	2.51e+3	1.93e+3	1.91e+3	2.41e+3
	mean	2.06e+2	1.47e+1	8.60e+2	5.96e+2	1.94e+3	1.22e+3	1.27e+3	1.17e+3	1.17e+3	1.41e+3	1.40e+3	1.70e+3	2.12e+3
	std	2.54e+2	2.60e−1	2.35e+2	2.90e+2	2.37e+2	1.87e+2	2.29e+2	1.29e+2	6.26e+1	3.17e+2	2.26e+2	1.29e+2	9.93e+1
1e4	min	2.61e+0	1.36e+1	2.40e+2	5.77e+1	4.33e+2	9.10e+2	9.11e+2	9.09e+2	5.00e+2	8.79e+2	5.34e+2	2.00e+2	2.10e+2
	7^{th}	3.05e+0	1.40e+1	4.00e+2	8.19e+1	1.54e+3	9.16e+2	9.16e+2	9.20e+2	5.00e+2	9.27e+2	5.34e+2	1.54e+3	2.12e+2
	med.	3.67e+0	1.44e+1	4.69e+2	1.56e+2	1.88e+3	9.22e+2	9.20e+2	9.23e+2	5.00e+2	9.43e+2	5.41e+2	1.63e+3	2.14e+2
	19^{th}	4.19e+0	1.45e+1	8.00e+2	4.00e+2	1.93e+3	9.27e+2	9.36e+2	9.28e+2	5.00e+2	9.83e+2	9.44e+2	1.69e+3	1.87e+3
	max	5.02e+0	1.49e+1	1.34e+3	8.02e+2	2.28e+3	1.49e+3	1.73e+3	9.42e+2	1.16e+3	1.21e+3	1.78e+3	1.87e+3	2.07e+3
	mean	3.64e+0	1.43e+1	6.07e+2	2.52e+2	1.71e+3	9.64e+2	1.03e+3	9.24e+2	6.18e+2	9.63e+2	7.51e+2	1.42e+3	7.25e+2
	std	7.27e−1	3.45e−1	3.23e+2	2.08e+2	4.42e+2	1.46e+2	2.41e+2	7.64e+0	2.28e+2	7.04e+1	3.30e+2	5.49e+2	8.09e+2
1e5	min	2.09e+0	1.35e+1	1.20e+2	5.05e+1	2.68e+2	8.00e+2	9.07e+2	8.00e+2	4.09e+2	8.51e+2	5.34e+2	2.00e+2	2.10e+2
	7^{th}	2.52e+0	1.39e+1	2.10e+2	5.84e+1	6.40e+2	9.14e+2	9.10e+2	9.11e+2	5.00e+2	8.71e+2	5.34e+2	1.52e+3	2.11e+2
	med.	2.82e+0	1.42e+1	3.06e+2	6.40e+1	1.17e+3	9.15e+2	9.15e+2	9.13e+2	5.00e+2	8.85e+2	5.34e+2	1.62e+3	2.12e+2
	19^{th}	3.01e+0	1.45e+1	3.52e+2	7.65e+1	1.57e+3	9.18e+2	9.19e+2	9.17e+2	5.00e+2	8.96e+2	5.41e+2	1.67e+3	1.84e+3
	max	3.82e+0	1.47e+1	5.00e+2	1.68e+2	2.17e+3	9.22e+2	9.30e+2	9.22e+2	5.00e+2	9.34e+2	1.56e+3	1.86e+3	2.05e+3
	mean	2.84e+0	1.42e+1	2.86e+2	7.41e+1	1.13e+3	9.07e+2	9.15e+2	9.05e+2	4.96e+2	8.85e+2	6.44e+2	1.41e+3	7.03e+2
	std	4.69e−1	3.76e−1	9.95e+1	2.84e+1	5.58e+2	3.22e+1	5.58e+0	3.19e+1	1.81e+1	2.08e+1	2.68e+2	5.45e+2	8.05e+2
3e5	min	1.48e+0	1.28e+1	1.08e+2	4.34e+1	2.66e+2	8.00e+2	8.00e+2	8.00e+2	4.09e+2	8.25e+2	5.34e+2	2.00e+2	2.10e+2
	7^{th}	2.10e+0	1.36e+1	1.41e+2	5.42e+1	6.39e+2	9.10e+2	9.09e+2	9.09e+2	5.00e+2	8.58e+2	5.34e+2	1.52e+3	2.11e+2
	med.	2.24e+0	1.40e+1	2.05e+2	5.77e+1	1.08e+3	9.12e+2	9.11e+2	9.10e+2	5.00e+2	8.71e+2	5.34e+2	1.61e+3	2.12e+2
	19^{th}	2.52e+0	1.43e+1	2.90e+2	6.18e+1	1.43e+3	9.13e+2	9.13e+2	9.13e+2	5.00e+2	8.81e+2	5.34e+2	1.67e+3	1.71e+3
	max	2.98e+0	1.45e+1	4.00e+2	7.65e+1	2.17e+3	9.18e+2	9.23e+2	9.16e+2	5.00e+2	9.20e+2	5.41e+2	1.86e+3	2.05e+3
	mean	2.32e+0	1.40e+1	2.16e+2	5.84e+1	1.07e+3	8.90e+2	9.03e+2	8.89e+2	4.85e+2	8.71e+2	5.35e+2	1.41e+3	6.91e+2
	std	3.46e−1	4.04e−1	8.29e+1	7.28e+0	5.13e+2	4.60e+1	3.11e+1	4.55e+1	3.39e+1	2.15e+1	1.53e+0	5.44e+2	7.87e+2

Table 5: Best objective function error values reached in dimension $n = 50$, see caption of Table 3 for details

FES	Prob.	1	2	3	4	5	6	7	8	9	10	11	12
	min	2.02e+4	1.11e+5	2.63e+8	1.67e+9	2.78e+4	2.82e+9	2.70e+3	2.11e+1	6.50e+2	1.46e+3	6.14e+1	1.49e+6
	7^{th}	3.25e+4	1.38e+5	8.13e+8	1.83e+9	3.17e+4	7.49e+9	3.62e+3	2.13e+1	7.35e+2	1.89e+3	7.20e+1	2.62e+6
	med.	3.57e+4	1.60e+5	1.02e+9	1.94e+9	3.59e+4	9.11e+9	4.20e+3	2.13e+1	8.21e+2	2.08e+3	7.51e+1	3.29e+6
1e3	19^{th}	4.09e+4	1.95e+5	1.20e+9	2.03e+9	4.01e+4	1.08e+10	4.69e+3	2.14e+1	1.03e+3	2.24e+3	7.72e+1	4.12e+6
	max	7.02e+4	2.92e+5	1.75e+9	2.20e+9	4.29e+4	3.06e+10	6.09e+3	2.14e+1	1.18e+3	3.00e+3	8.62e+1	5.55e+6
	mean	3.82e+4	1.76e+5	9.97e+8	1.93e+9	3.58e+4	9.75e+9	4.21e+3	2.13e+1	8.60e+2	2.09e+3	7.43e+1	3.32e+6
	std	1.02e+4	4.97e+4	3.25e+8	1.43e+8	4.62e+3	5.37e+9	8.94e+2	6.85e−2	1.52e+2	3.45e+2	6.16e+0	1.06e+6
	min	4.34e−9	2.00e+3	4.88e+6	1.63e+9	2.77e+3	4.46e+1	1.02e−4	2.11e+1	6.20e+2	1.43e+3	2.84e+1	1.97e+4
	7^{th}	5.71e−9	3.06e+3	8.49e+6	1.82e+9	5.24e+3	1.61e+2	4.53e−4	2.12e+1	7.14e+2	1.86e+3	4.80e+1	2.10e+5
	med.	6.23e−9	3.91e+3	1.19e+7	1.89e+9	5.97e+3	1.04e+3	6.93e−4	2.12e+1	7.85e+2	2.05e+3	5.46e+1	2.80e+5
1e4	19^{th}	6.74e−9	6.25e+3	1.58e+7	1.98e+9	6.33e+3	7.14e+3	1.60e−3	2.13e+1	9.85e+2	2.20e+3	5.87e+1	4.30e+5
	max	7.50e−9	1.06e+4	2.24e+7	2.19e+9	7.61e+3	1.42e+4	7.93e−3	2.13e+1	1.15e+3	2.93e+3	6.29e+1	1.31e+6
	mean	6.20e−9	4.89e+3	1.26e+7	1.90e+9	5.78e+3	3.48e+3	1.69e−3	2.13e+1	8.32e+2	2.06e+3	5.24e+1	4.00e+5
	std	8.04e−10	2.51e+3	5.12e+6	1.45e+8	1.09e+3	4.37e+3	2.34e−3	4.65e−2	1.50e+2	3.38e+2	8.53e+0	3.33e+5
	min	4.34e−9	5.38e−9	2.49e−1	1.42e+5	5.51e+2	1.60e+0	6.50e−9	2.00e+1	4.35e+2	1.35e+3	2.83e+1	7.27e+3
	7^{th}	5.71e−9	7.48e−9	3.42e+0	1.13e+6	1.15e+3	8.24e+0	7.06e−9	2.12e+1	5.96e+2	1.51e+3	4.05e+1	1.33e+5
	med.	6.23e−9	8.04e−9	4.71e+0	3.59e+8	1.35e+3	1.18e+1	7.53e−9	2.12e+1	6.27e+2	1.69e+3	4.22e+1	2.51e+5
1e5	19^{th}	6.74e−9	8.66e−9	1.06e+1	1.82e+9	1.85e+3	1.33e+1	7.81e−9	2.12e+1	6.83e+2	1.95e+3	4.70e+1	3.31e+5
	max	7.50e−9	9.31e−9	3.83e+1	2.06e+9	2.99e+3	2.44e+1	8.26e−9	2.13e+1	7.63e+2	2.93e+3	5.20e+1	1.01e+6
	mean	6.20e−9	7.96e−9	7.88e+0	8.51e+8	1.48e+3	1.16e+1	7.49e−9	2.11e+1	6.36e+2	1.79e+3	4.26e+1	3.13e+5
	std	8.04e−10	8.32e−10	8.24e+0	9.00e+8	6.19e+2	5.65e+0	5.25e−10	2.80e−1	7.35e+1	3.71e+2	6.05e+0	2.61e+5
	min	4.34e−9	5.38e−9	4.87e−9	1.42e+5	9.40e−8	6.04e−9	6.50e−9	2.00e+1	4.20e+2	1.27e+3	2.65e+1	3.72e+3
	7^{th}	5.71e−9	7.48e−9	5.48e−9	4.87e+5	4.46e−3	6.64e−9	7.06e−9	2.00e+1	5.34e+2	1.39e+3	2.97e+1	3.37e+4
	med.	6.23e−9	8.04e−9	5.90e−9	9.96e+5	7.08e−2	7.12e−9	7.53e−9	2.00e+1	5.79e+2	1.43e+3	3.47e+1	7.26e+4
5e5	19^{th}	6.74e−9	8.66e−9	6.54e−9	3.39e+8	6.67e−1	7.43e−9	7.81e−9	2.00e+1	6.14e+2	1.56e+3	3.81e+1	1.15e+5
	max	7.50e−9	9.31e−9	8.05e−9	2.06e+9	4.79e+1	8.87e−9	8.26e−9	2.12e+1	6.65e+2	1.89e+3	4.27e+1	3.21e+5
	mean	6.20e−9	7.96e−9	6.04e−9	4.46e+8	3.27e+0	7.12e−9	7.49e−9	2.00e+1	5.67e+2	1.48e+3	3.41e+1	8.93e+4
	std	8.04e−10	8.32e−10	7.67e−10	7.93e+8	9.94e+0	7.08e−10	5.25e−10	2.34e−1	6.05e+1	1.37e+2	4.97e+0	7.86e+4

FES	Prob.	13	14	15	16	17	18	19	20	21	22	23	24	25
	min	2.01e+3	2.41e+1	5.45e+2	5.54e+2	1.43e+3	1.20e+3	1.16e+3	1.15e+3	1.29e+3	1.36e+3	1.62e+3	1.38e+3	2.08e+3
	7^{th}	1.08e+4	2.44e+1	6.87e+2	6.72e+2	1.72e+3	1.25e+3	1.24e+3	1.25e+3	1.36e+3	1.54e+3	1.76e+3	1.78e+3	2.25e+3
	med.	1.88e+4	2.46e+1	8.46e+2	7.45e+2	1.81e+3	1.32e+3	1.32e+3	1.34e+3	1.40e+3	1.68e+3	1.79e+3	1.87e+3	2.29e+3
1e3	19^{th}	2.59e+4	2.48e+1	1.17e+3	9.71e+2	1.98e+3	1.37e+3	1.40e+3	1.42e+3	1.50e+3	1.84e+3	1.85e+3	1.90e+3	2.33e+3
	max	8.60e+4	2.50e+1	1.57e+3	1.91e+3	2.29e+3	1.64e+3	1.68e+3	1.73e+3	1.79e+3	2.71e+3	1.99e+3	2.08e+3	2.44e+3
	mean	2.74e+4	2.46e+1	9.58e+2	8.79e+2	1.85e+3	1.34e+3	1.34e+3	1.35e+3	1.43e+3	1.74e+3	1.80e+3	1.83e+3	2.28e+3
	std	2.55e+4	2.38e−1	3.24e+2	3.51e+2	2.27e+2	1.05e+2	1.28e+2	1.35e+2	1.04e+2	2.93e+2	8.32e+1	1.60e+2	8.54e+1
	min	4.60e+0	2.37e+1	3.05e+2	6.09e+1	1.11e+3	8.79e+2	8.93e+2	8.00e+2	5.00e+2	9.78e+2	5.39e+2	2.00e+2	2.24e+2
	7^{th}	6.53e+0	2.40e+1	4.00e+2	9.14e+1	1.56e+3	9.32e+2	9.33e+2	9.31e+2	5.00e+2	1.00e+3	5.43e+2	2.00e+2	2.75e+2
	med.	7.33e+0	2.43e+1	4.00e+2	2.06e+2	1.72e+3	9.36e+2	9.40e+2	9.39e+2	5.00e+2	1.02e+3	8.93e+2	1.15e+3	5.87e+2
1e4	19^{th}	8.54e+0	2.44e+1	9.00e+2	4.00e+2	1.86e+3	9.46e+2	9.47e+2	9.52e+2	5.00e+2	1.03e+3	1.79e+3	1.71e+3	2.10e+3
	max	2.01e+1	2.50e+1	1.50e+3	1.33e+3	2.25e+3	9.66e+2	9.84e+2	9.65e+2	8.00e+2	1.07e+3	1.94e+3	1.87e+3	2.28e+3
	mean	8.16e+0	2.43e+1	6.68e+2	3.34e+2	1.70e+3	9.36e+2	9.39e+2	9.32e+2	5.36e+2	1.02e+3	1.12e+3	9.44e+2	1.16e+3
	std	3.08e+0	3.45e−1	4.31e+2	2.96e+2	2.83e+2	2.13e+1	1.62e+1	3.38e+1	9.95e+1	2.36e+1	6.20e+2	7.02e+2	8.98e+2
	min	4.25e+0	2.31e+1	1.81e+2	6.09e+1	3.87e+2	8.79e+2	8.89e+2	8.00e+2	5.00e+2	8.88e+2	5.39e+2	2.00e+2	2.14e+2
	7^{th}	4.99e+0	2.39e+1	3.42e+2	7.22e+1	8.55e+2	9.16e+2	9.14e+2	9.28e+2	5.00e+2	9.12e+2	5.43e+2	2.00e+2	2.16e+2
	med.	5.80e+0	2.42e+1	3.63e+2	7.96e+1	1.22e+3	9.32e+2	9.31e+2	9.32e+2	5.00e+2	9.22e+2	8.93e+2	2.00e+2	2.17e+2
1e5	19^{th}	6.47e+0	2.43e+1	4.00e+2	1.03e+2	1.66e+3	9.35e+2	9.35e+2	9.39e+2	5.00e+2	9.30e+2	1.79e+3	1.70e+3	2.18e+2
	max	7.68e+0	2.48e+1	4.05e+2	2.13e+2	2.20e+3	9.46e+2	9.42e+2	9.60e+2	5.00e+2	9.57e+2	1.94e+3	1.85e+3	2.17e+3
	mean	5.73e+0	2.41e+1	3.53e+2	9.77e+1	1.26e+3	9.24e+2	9.25e+2	9.22e+2	5.00e+2	9.21e+2	1.12e+3	8.46e+2	5.60e+2
	std	9.62e−1	4.50e−1	5.43e+1	3.97e+1	4.97e+2	1.94e+1	1.55e+1	3.75e+1	4.39e−13	1.61e+1	6.20e+2	7.18e+2	7.07e+2
	min	3.76e+0	2.31e+1	1.15e+2	5.61e+1	3.81e+2	8.71e+2	8.63e+2	8.00e+2	5.00e+2	8.82e+2	5.39e+2	2.00e+2	2.14e+2
	7^{th}	4.37e+0	2.37e+1	1.88e+2	6.58e+1	7.73e+2	8.90e+2	9.03e+2	8.95e+2	5.00e+2	8.98e+2	5.40e+2	2.00e+2	2.16e+2
	med.	4.56e+0	2.40e+1	2.46e+2	7.22e+1	9.31e+2	9.04e+2	9.14e+2	9.20e+2	5.00e+2	9.10e+2	5.42e+2	2.00e+2	2.16e+2
5e5	19^{th}	5.05e+0	2.42e+1	3.13e+2	7.63e+1	1.19e+3	9.26e+2	9.26e+2	9.29e+2	5.00e+2	9.18e+2	5.46e+2	1.70e+3	2.18e+2
	max	5.60e+0	2.47e+1	3.55e+2	8.16e+1	2.20e+3	9.34e+2	9.30e+2	9.32e+2	5.00e+2	9.57e+2	1.84e+3	1.85e+3	2.17e+3
	mean	4.70e+0	2.39e+1	2.50e+2	7.09e+1	1.05e+3	9.06e+2	9.11e+2	9.01e+2	5.00e+2	9.10e+2	6.37e+2	8.43e+2	4.77e+2
	std	5.09e−1	3.97e−1	6.74e+1	6.99e+0	4.56e+2	2.05e+1	1.84e+1	3.93e+1	2.26e−13	1.72e+1	2.77e+2	7.15e+2	6.15e+2