Non-local interaction equations: Stationary states and stability analysis

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Abstract: In this paper, we are interested in the long-time behavior of solutions to a non-local interaction equationin dimension 1. We show that up to an extraction, the solution converges to a steady-state. Then, we study the structure of stable steady-states.

1 Introduction

We are interested in the asymptotic behaviour of a density $\rho(t,x)$ of particles or individuals at position $x \in \mathbb{R}^d$ $(d \geq 1)$ and at time $t \geq 0$, which evolves according to the nonlocal aggregation equation:

$$\partial_t \rho = \nabla_x \cdot (\rho\, \nabla_x[W * \rho + V]), \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1}$$

This equation can be seen as a many particles limit of discret processes where particles (or individuals) can interact at a large distance, through an interaction potential W (see [27, 20]), and may be subjected to an external potential V. Such equations appear in various biological phenomenons like swarming (see [7, 15]), distribution of actin-filament networks (see [16, 19]), as well as in physical problems, for example in the field of granular media (see [2, 32, 14]). For some interaction potentials, this equation can lead to surprisingly complicated paterns, such as solutions converging to singular steady states, as shown in [4, 29, 17, 18], or more recently in [23, 8, 1].

Many of the above models couple the long-range interaction between particles with a diffusive term. Nevertheless, in this paper we shall not consider a diffusion term, and focus our study on the effect of a long-range interaction.

Let us now describe typical interaction potentials W which appear in the models quoted above:

- In [21, 29], interaction potentials are regular, repulsive at short range and attractive when particles are far apart, typically $W(x) = -x^2 + x^4$. In this case, the solution typically concentrates and tends to a finite number of Dirac masses, when time goes to infinity. This type of potentials have been studied in [12, 10, 14], but we don't know any general study of the case of regular interaction potentials so far.

- In chemotaxis models (see [28, 22, 5]), interaction potentials are singular at $x = 0$ and attractive, typically, in dimension 2, $W(x) := -\frac{1}{2\pi}\log|x|$. In this case, the solution usually (if there is no diffusion) blows-up in finite time. Potentials singular at $x = 0$ and attractive have been widely studied both with a diffusion term (see [6, 9]), or without diffusion (see [11, 24, 13, 4, 3]), for various types of attractive singularities.

- In swarming models (see [15, 26, 31]), interaction potentials are usually singular at $x = 0$ and repulsive, typical examples are the repulsive Morse potential $W(x) = -e^{-|x|}$, or the attractive-repulsive Morse potentials $W(x) = -C_a\, e^{-|x|/l_a} + C_r\, e^{-|x|/l_r}$ and $W(x) = -C_a\, e^{-|x|^2/l_a} + C_r\, e^{-|x|^2/l_r}$. Related interpolation potentials in physics are, for instance, the Lennard-Jones potential [30]. We don't know any qualitative study of such models.

We will show in this article that the asymptotic behaviour of the solution of (1) highly depends on the type of singularity of W at point $x = 0$.

In the present article, we shall focus on the one-dimensional case. We aim at understanding the dynamical behavior presented by a non-local interaction operator with even potential. This assumption is not crutial in this study, but is satisfied by the interaction kernels used in practice for this model.

Assumption 1:

$$\forall x \in \mathbb{R}, W(x) = W(-x). \tag{2}$$

In this study, we shall focus on compactly supported densities, we shall thus only consider situations where a confinement exists, either from the external potential, or from the interaction potential itself. We shall assume that:

Assumption 2: One of the two following conditions is satisfied:
There exists $C > 0$ such that

$$\|W'\|_{L^\infty([-2C,2C])} < \min\left(V'(C),\, -V'(-C)\right), \tag{3}$$

or

$$V = 0, \quad \exists C_1,\, C_2 > 0,\, \forall x \geq C_1 : \quad W'(x) \geq C_2\, x. \tag{4}$$

Assumption 3:

$$\rho^0 \in M^1(\mathbb{R}),\ \text{supp}\ \rho^0 \subset [-C, C]. \tag{5}$$

where $C < \infty$. If $V \neq 0$, C must satisfy (3).

It has been proven in [11] that Assumptions 2 and 3 ensure that the support of $\rho(t, \cdot)$ is (uniformly w.r.t. time) bounded:

$$\exists C > 0,\ \forall t \geq 0, \quad \text{supp}\, \rho(t, \cdot) \subset [-C, C]. \tag{6}$$

Note that (1) formally conserves the total mass $\int \rho(t, x)\, dx$, which w.l.o.g. we shall assume to be normalized $\int_{\mathbb{R}} \rho(x)\, dx = 1$. The quantity $\rho(t, \cdot)$ is then interpreted as a probability density. In particular in the one-dimensional case, this enables a change of variables in which one introduces the pseudo-inverse of the distribution function $\int_{-\infty}^{x} d\rho$, i.e.

$$u(t, z) = \inf \left\{ x \in \mathbb{R} : \int_{(-\infty, x]} \rho(t, y)\, dy > z \right\} \qquad z \in [0, 1], \tag{7}$$

which transforms the evolution equations (1) for measure solutions $\rho(t, \cdot)$ into an integral equation for the non-decreasing pseudo-inverse $u(t, z)$ satisfying (see, e.g. [25, 5, 10])

$$\partial_t u(t, z) = \int W' \left(u(t, \xi) - u(t, z) \right)\, d\xi - V'(u(t, z)), \qquad \forall z \in [0, 1]. \tag{8}$$

Since eq. (8) is much more convenient than eq. (1) for stability analysis, we shall often use it in this paper. In particular, atomic parts of measure solutions $\rho(x)$ correspond to constant parts of the pseudo-inverse $u(z)$. Notice also the useful change of variable $\int g(x)\rho(x)\, dx = \int_0^1 g(u(\xi))\, d\xi$, which holds for any $g \in L^1(\text{supp}\ \rho)$.

In the absence of a confining potential V (and if W is symmetric), the center of mass $\int_{\mathbb{R}} x\, \rho(t, x)\, dx$ is conserved by eq. (1), or equivalently, $\int_0^1 u$ is preserved by (8):

$$\frac{d}{dt} \int_{\mathbb{R}} x \rho(t, x)\, dx = 0, \quad \frac{d}{dt} \int_0^1 u(t, z)\, dz = 0. \tag{9}$$

Note that eq. (1) can be seen as a gradient-flow equation for the following energy (see [11]):

$$E(t) := \frac{1}{2} \int \int \rho(t, x)\rho(t, y) W(x - y)\, dx\, dy + \int_{\mathbb{R}} \rho(t, x) V(x)\, dx. \tag{10}$$

In section 2, we shall consider regular interaction potentials W. Prop. 1 shows that $\rho(t,\cdot)$ converges (in a sense to be precised then) to a set of steady-states, as time goes to infinity. This result emphasizes the importance of steady-states, when one wishes to understand the long-time behavior of solutions to (1).

In subsection 2.2, we show that stable steady-states of (1) are generically sums of Dirac masses. More precisely, we show in Prop. 2 that for analytic V, W, the steady-states of (1) are necessarily finite sums of Dirac masses. If V, W are only C^2, continuous steady-states may exist, but they cannot be linearly stable.

In Section 3, we consider interaction potentials having a singularity at $x = 0$.

In Subsection 3.1, we consider the steady-states of (1) for an interaction potential W having an attractive singularity at $x = 0$. Since (1) may develop blow-ups in L^∞ in finite time (see [4, 3]), we consider (following [11]), the extension (23) of (1) to measure-valued solutions. In Prop. 4, we show that a steady-state $\bar{\rho}$ of (23) such that supp $\bar{\rho}$ has an accumulation point (and a bit more, see (25)) is nonlinearly unstable.

In Subsection 3.2, we consider the steady-states of (1) for an interaction potential W having a repulsive singularity at $x = 0$. In Prop. 5, we provide an existence proof for (1) with a regular initial condition (until now, no existence result had been written down for such interaction potentials). In particular, Prop. 5 provides a uniform bound on the solution in $L^\infty(\mathbb{R})$. The situation is therefore completely different from the two other cases: no blow-up can occur.

2 Regular interaction potentials

In this first section, we make the following regularity assumptions on V and W:
Assumption 4:

$$V \in C^2(\mathbb{R}),\, W \in C^2(\mathbb{R}), \tag{11}$$

$$W \in W^{2,\infty}(\mathbb{R}). \tag{12}$$

We shall use in the following the measure space

$$\mathcal{P}_\infty(\mathbb{R}) := \{\rho \in M^1(\mathbb{R});\ \text{supp}\ \rho \text{ is bounded}\},$$

together with the Wasserstein distance

$$W_\infty(\rho_1, \rho_2) := \|u_1 - u_2\|_\infty, \tag{13}$$

where u_1, u_2 are the pseudo-inverses of ρ_1, ρ_2.

Under Assumption 1 to 4, it has been proven in [10] that a unique solution $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ to (1) exists. The support of $\rho(t,\cdot)$ is uniformly bounded w.r.t. time thanks to [11]

2.1 Asymptotic behavior of the solution

In this subsection, we show that we cannot expect the solution to converge to anything else than a set of steady-states, using a energy dissipation argument. In particular, no periodic limit cycles exist. We define a steady-state of (1) as a probabitity measure $\bar{\rho} \in \mathcal{P}_\infty(\mathbb{R})$ such that the velocity field it generates is equal to 0 on the support of $\bar{\rho}$, that is:

$$\nabla_x[W * \bar{\rho} + V] = 0 \quad \text{on supp } \bar{\rho}.$$

Proposition 1. *Let ρ^0, V, W satisfy Assumptions 1 to 4. Let $\rho \in Lip_{loc}([0,\infty), \mathcal{P}_\infty(\mathbb{R}))$ be the unique solution of* (1) *given by [10]. Then,*

1. $$\int \rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 dx \to 0 \text{ as } t \to \infty.$$

2. *For any sequence $t_k \to \infty$, there exists a subsequence, still denoted (t_k), such that:*
$$W_1\left(\rho(t_k,\cdot), \bar{\rho}\right) \to 0 \quad \text{as } k \to \infty, \tag{14}$$
where W_1 denotes the $1-$Wasserstein distance, and $\bar{\rho}$ is a steady-state of (1).

Remark 1. *The limit $\bar{\rho}$ of $\rho(t_k,\cdot)$ in* (14) *is not necessarily unique : it may depend both on the sequence (t_k) and the extracted sequence.*

Proof of Prop. 1

Step 1: Proof of 1.

We first show that the energy (10) is non-increasing in time, using integrations by parts:

$$\begin{aligned}
\frac{dE}{dt}(t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x \left(\rho(t,x) \left(\int W'(x-z)\rho(t,z)\,dz + V'(x) \right)(t,x) \right) \\
&\quad \rho(t,y)W(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}} \partial_x \left(\rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right) \right) V(x)\,dx \\
&= -\int \rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 dx \\
&\le 0.
\end{aligned} \tag{15}$$

Next, we have the following estimate on the regularity of the energy dissipation:

$$
\begin{aligned}
\frac{d^2E}{dt^2} =& - \int \partial_x \left(\rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right) \right) \\
& \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 \,dx \\
& - 2 \int \rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right) \int W'(x-y) \\
& \partial_y \left(\rho(t,y) \left(\int W'(y-z)\rho(t,z)\,dz + V'(y) \right) \right) \,dy\,dx \\
=& 2 \int \rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right)^2 \\
& \partial_x \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right) \,dx \\
& + 2 \int \rho(t,x) \left(\int W'(x-y)\rho(t,y)\,dy + V'(x) \right) \int \partial_y (W'(x-y)) \\
& \left(\rho(t,y) \left(\int W'(y-z)\rho(t,z)\,dz + V'(y) \right) \right) \,dy\,dx.
\end{aligned}
$$

Since V, $W \in C^2(\mathbb{R})$, we can estimate $\frac{d^2E}{dt^2}$ as follows:

$$
\begin{aligned}
\left| \frac{d^2E}{dt^2} \right| &\leq 2 \left(\|V\|_{W^{2,\infty}(-C,C)} + \|W\|_{W^{2,\infty}(-2C,2C)} \right) \left(\|W\|_{W^{2,\infty}(-2C,2C)} + \|V\|_{W^{2,\infty}(-C,C)} \right)^2 \\
&\leq C, \qquad (16)
\end{aligned}
$$

where $C < +\infty$ is a constant.

Finally, notice that the energy is bounded from below:

$$
E \geq - \left(\frac{1}{2} \|W\|_{L^\infty(-2C,2C)} + \|V\|_{L^\infty(-C,C)} \right). \qquad (17)
$$

To prove that $\frac{dE}{dt}(t) \to 0$, we use an interpolation between $E(t) \to \bar{E}$ and $\frac{d^2}{dt^2}E(t)$ bounded:

Let $\varepsilon > 0$. Since the energy E is non increasing (15) and bounded from below (17), E has a limit $\bar{E}$ when $t \to \infty$. Let $t > 0$ and $\tau \in (0, \frac{t}{2}]$. Then,

$$\begin{aligned}
\left|\frac{dE}{dt}(t)\right| &= \left|\frac{1}{\tau}\int_{t-\tau}^{t}\left[\frac{dE}{dt}(s)+\int_s^t \frac{d^2E}{dt^2}(\sigma)\,d\sigma\right]\,ds\right| \\
&= \left|\frac{1}{\tau}\left[E(t)-E(t-\tau)\right]+\frac{1}{\tau}\int_{t-\tau}^{t}\int_s^t \frac{d^2E}{dt^2}(\sigma)\,d\sigma\,ds\right| \\
&\leq \frac{2}{\tau}\|E-\bar{E}\|_{L^\infty([\frac{t}{2},\infty))}+\frac{\tau}{2}\left\|\frac{d^2E}{dt^2}\right\|_{L^\infty([0,\infty))}.
\end{aligned}$$

For $t>0$ large enough, $\tau := \frac{\|E-\bar{E}\|^{\frac{1}{2}}_{L^\infty([\frac{t}{2},\infty))}}{\left\|\frac{d^2E}{dt^2}\right\|^{\frac{1}{2}}_{L^\infty([0,\infty))}} < \frac{t}{2}$, and then,

$$\left|\frac{dE}{dt}(t)\right| \leq \frac{5}{2}\|E-\bar{E}\|^{\frac{1}{2}}_{L^\infty([\frac{t}{2},\infty))}\left\|\frac{d^2E}{dt^2}\right\|^{\frac{1}{2}}_{L^\infty([0,\infty))},$$

which implies $\frac{dE}{dt}(t)\to 0$ as $t\to\infty$.

Step 2: Proof of 2.

The pseudo-inverse $u(t,\cdot)$ of $\rho(t,\cdot)$ is an increasing function, and is uniformly bounded thanks to (6). The sequence $u(t_k,\cdot)$ is then a uniformly bounded sequence of $BV([0,1])$. There exists then a subsequence, still denoted $u(t_k,\cdot)$, that converges in L^1 to a limit denoted by $\bar{u}$:

$$\|u(t_k,\cdot)-\bar{u}\|_{L^1}\to 0.$$

Our aim is to prove that $\bar{u}$ is a steady-state of (8). In order to prove that, we shall use the estimate obtained above, $\frac{dE}{dt}(t_k)\to 0$. Let us write this estimate in the pseudo-inverse setting:

$$\begin{aligned}
\frac{dE}{dt}(t_k) &= -\int \rho(t_k,x)\left(\int W'(x-y)\rho(t_k,y)\,dy+V'(x)\right)^2\,dx \\
&= -\int_0^1\left(\int W'(u(t_k,z)-u(t_k,\xi))\,d\xi+V'(u(t_k,z))\right)^2\,dz.
\end{aligned}$$

We define $\bar{F} := -\int_0^1 \left(\int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\right)^2\,dz$. Then,

$$
\begin{aligned}
\bar{F} - \frac{dE}{dt} &= \int_0^1 \left(\int W'(u(z)-u(\xi))\,d\xi + V'(u(z))\right)^2 \\
&\quad - \left(\int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\right)^2\,dz \\
&= \int_0^1 \Bigg(\int W'(u(z)-u(\xi))\,d\xi + V'(u(z)) \\
&\qquad + \int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi + V'(\bar{u}(z))\Bigg) \\
&\qquad \cdot \Bigg(\int W'(u(z)-u(\xi))\,d\xi - \int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi \\
&\quad + V'(u(z)) - V'(\bar{u}(z))\Bigg)\,dz \\
&\le C\left\|\int W'(u(z)-u(\xi))\,d\xi - \int W'(\bar{u}(z)-\bar{u}(\xi))\,d\xi\right\|_{L^1} \\
&\quad + C\|u-\bar{u}\|_{L^1} \\
&\le C\|W'\|_{L^\infty(-2C,2C)}\,\|u-\bar{u}\|_{L^1} + C\|u-\bar{u}\|_{L^1}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\bar{F} &\le \frac{dE}{dt}(t_k) + C\|u(t_k,\cdot)-\bar{u}\|_{L^1} \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}
$$

Then, $\bar{F} = 0$, that is:

$$
\operatorname{supp}\bar{\rho} \subset \left\{x \in \mathbb{R};\ \int W'(x-y)\bar{\rho}(y)\,dy + V'(x) = 0\right\},
$$

and $\bar{\rho}$ is a steady-state of (1).

□

2.2 Study of the steady states

In the previous subsection, we showed that for any regular potential W satisfying Assumption 4, the sequence $\rho(t_k,\cdot)$ converges, up to an extraction, to a steady solution of (1). In this subsection, we shall try to characterize the steady-states of (1).

In the case of an analytical interaction potentials W and analytical external field V, we show in the following proposition that steady-states are necessarily finite sums of Dirac masses:

Proposition 2. *Assume W and V are analytical. Then, every steady state $\bar{\rho} \in M^1(\mathbb{R})$ of (1) with bounded support is a finite sum of Dirac masses:*

$$\bar{\rho} = \sum_{i=1}^{N} \bar{\rho}_i \delta_{\bar{u}_i},$$

with $\bar{\rho}_1, \ldots, \bar{\rho}_N > 0$, $\bar{u}_1, \ldots, \bar{u}_N \in \mathbb{R}$.

Proof of Prop. 2

Let us consider a steady solution $\bar{\rho}$ of (1). For $x \in \operatorname{supp} \bar{\rho}$,

$$\begin{aligned} 0 &= \nabla \left[\int W(x-y)\, d\bar{\rho}(y) - V(x) \right] \\ &= -(W' * \bar{\rho} + V')(x). \end{aligned}$$

Since W and V are analytic, so is $W' * \bar{\rho} + V'$, and if supp $\bar{\rho}$ has an accumulation point, then

$$\forall x \in \mathbb{R}, \quad (W' * \bar{\rho})(x) + V'(x) = 0,$$

which is not possible since V, W satisfy (3) or (4). Then, supp $\bar{\rho}$ cannot have any accumulation point, and is thus a finite set of points.

□

For less regular potentials, for instance when W is only C^2, the same result cannot be expected to hold anymore, as the following example shows.

Example 1. *Consider the interaction potential $W(x) := (\operatorname{dist}(x, [-1, 1]))^3$, where $\operatorname{dist}(x, y) := |x - y|$, and $V = 0$. W is C^2 (one could even consider a smoothed (C^∞) version of the potential), but (1) admits the $L^1(\mathbb{R})$ steady state:*

$$\bar{\rho} = \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}.$$

Nevertheless, the following proposition shows that steady states which are linearly stable (in a sense made clear in the following proposition) have to be sums of Dirac masses:

Proposition 3. *Let V, W satisfy Assumptions 1 and 4. Let $\bar\rho \in M^1(\mathbb{R})$ be a compactly supported steady state of* (1)*, and $\bar u$ be its pseudo-inverse. If $\bar\rho$ is such that* $\mathrm{supp}(\bar\rho)$ *has an accumulation point x_0, then the pseudo-inverse equation* (8) *linearized around $\bar u$ in L^1 has no spectral gap.*

Remark 2. *Since the perturbations u^ε of $\bar u$ used in the proof of Prop. 3 satisfy $\int_0^1 u^\varepsilon = \int_0^1 \bar u$, Prop. 3 remains true if we only consider perturbations preserving the center of mass $\int x\bar\rho(x)\,dx$ of $\bar\rho$ (this is important since* (1) *is invariant w.r.t. translations along x).*

Remark 3. *For a stability analysis of steady-states $\bar\rho$ that are sums of Dirac masses, see [17, 18]. In [17], a necessary and sufficient condition for local stability of such steady-states with respect to perturbations ρ of $\bar\rho$ such that $W_\infty(\bar\rho,\rho)$ is small (where W_∞ denotes the $\infty-$Wasserstein distance) is discussed.*

Recently, several papers have shown that in several dimensions, interaction potentials that are singular repulsive locally and attractive at long range can lead to very complicated paterns, see [8, 23, 1].

The idea of the proof is to construct a measure ρ^ε arbitrarily close to $\bar\rho$ by collapsing the mass of $\bar\rho$ around x_0 into a single Dirac mass (A similar construction will be employed in the proof of Prop. 4). If we denote by L the linearization of (8) around $\bar u$, then we show that $\|L(v^\varepsilon)\|_{L^1} = o_\varepsilon(1)\|v^\varepsilon\|_{L^1}$, which implies that L cannot have any spectral gap.

Proof of Prop. 3

We begin by linearizing (8) around $\bar u$, with $u = \bar u + \delta v$, $\delta > 0$:

$$\begin{aligned}
\partial_t u(t,z) &= \int_0^1 W'(u(t,\xi)-u(t,z))\,d\xi - V'(u(t,z)) \\
&= \int_0^1 W'(\bar u(t,\xi)-\bar u(t,z))\,d\xi - V'(\bar u(t,z)) \\
&\quad +\delta\left(\int_0^1 W''(\bar u(\xi)-\bar u(z))(v(t,\xi)-v(t,z))\,d\xi - V''(\bar u(z))v(t,z)\right) + o(\delta) \\
&= \delta\Bigg(\int_0^1 W''(\bar u(\xi)-\bar u(z))v(t,\xi)\,d\xi - \int_0^1 W''(\bar u(\xi)-\bar u(z))\,d\xi v(t,z) \\
&\qquad -V''(\bar u(z))v(t,z)\Bigg) + o(\delta),
\end{aligned}$$

so that the linearization of (8) around $\bar u$ yields the linear operator $L : L^1([0,1]) \to L^1([0,1])$:

$$L(v)(z) = \int_0^1 W''(\bar u(\xi)-\bar u(z))v(\xi)\,d\xi - \left[\int_0^1 W''(\bar u(\xi)-\bar u(z))\,d\xi + V''(\bar u(z))\right] v(z). \tag{18}$$

We now shall show that if supp $\bar{\rho}$ has an accumulation point x_0, then we can build a sequence (v^ε) of perturbations of u such that:

$$\frac{\|L(v^\varepsilon)\|_{L^1}}{\|v^\varepsilon\|_{L^1}} \to 0,$$

which shows that the linear operator L does not have any spectral gap. Since we are dealing with pseudo-inverses, we must however restrict to perturbations v such that for some $\alpha > 0$, $u = \bar{u} + \alpha v$ is non decreasing.

We assume without any loss of generality that x_0 is an accumulation point of $\mathrm{supp}(\bar{\rho}) \cap [x_0, \infty)$. Then, for any $\varepsilon > 0$,

$$\int_{(x_0, x_0+\varepsilon)} d\bar{\rho} > 0. \tag{19}$$

For a given $\varepsilon > 0$, we define

$$\begin{aligned} z_0 &:= \inf\left\{z \in (0,1); \bar{u}(z) > x_0\right\}, \\ z_1^\varepsilon &:= \sup\left\{z \in (0,1); \bar{u}(z) < x_0 + \varepsilon\right\}, \\ Z^\varepsilon &:= [z_0, z_1^\varepsilon]. \end{aligned}$$

We define the following perturbation u^ε of $\bar{u}$:

$$u^\varepsilon(z) := \left| \begin{array}{l} \bar{u}(z) \text{ on } (Z^\varepsilon)^c, \\ \frac{1}{|Z^\varepsilon|}\int_{Z^\varepsilon} \bar{u}(y)\,dy \text{ on } Z^\varepsilon, \end{array}\right.$$

and we write $v^\varepsilon := u^\varepsilon - \bar{u}$. The function u^ε is then the pseudo-inverse of the measure:

$$\rho^\varepsilon = \bar{\rho}|_{[x_0,x_0+\varepsilon]^c} + \left(\int_{[x_0,x_0+\varepsilon]} \bar{\rho}(x)\,dx\right)\delta_{\tilde{x}},$$

where $\tilde{x} = \frac{1}{|Z^\varepsilon|}\int_{Z^\varepsilon} \bar{u}(y)\,dy = \int_{[x_0,x_0+\varepsilon]} x\bar{\rho}(x)\,\frac{dx}{\int_{[x_0,x_0+\varepsilon]}\bar{\rho}(x)\,dx}$.

- We estimate $\int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))v^\varepsilon(\xi)\,d\xi$:

$$\begin{aligned} \int_0^1 W''(\bar{u}(\xi) - \bar{u}(z))v^\varepsilon(\xi)\,d\xi &= \int_0^1 W''(\bar{u}(\xi) - x_0)v^\varepsilon(\xi)\,d\xi + \int_0^1 o_\varepsilon(1)v^\varepsilon(\xi)\,d\xi \\ &= o_\varepsilon(1)\|v^\varepsilon\|_{L^1}. \end{aligned} \tag{20}$$

- We estimate $\left[\int_0^1 W''(\bar{u}(\xi)-\bar{u}(z))\,d\xi + V''(\bar{u}(z))\right] v^\varepsilon(z)$:

  Since $\bar{u}$ is a steady state of (8),

$$\forall x \in \text{supp } \bar{\rho}, \quad (W' * \bar{\rho})(x) + V'(x) = 0.$$

  Thanks to Assumption 4, $W' * \bar{\rho} + V' \in C^1(\mathbb{R})$ is differentiable at $x = x_0$. Since x_0 is an accumulation point of supp $\bar{\rho}$, there exists a sequence $(x^k)_k \in (\text{supp } \bar{\rho})^{\mathbb{N}}$ such that $x^k \to x_0$. Then,

$$\begin{aligned}(W'' * \bar{\rho})(x_0) + V''(x_0) &= \lim_{k\to\infty} \frac{((W' * \bar{\rho})(x_0) + V'(x_0)) - \big((W' * \bar{\rho})(x^k) + V'(x^k)\big)}{x_0 - x_k} \\ &= \lim_{k\to\infty} 0 \\ &= 0.\end{aligned}$$

  Since $W'' * \rho + V''$ is continuous, and thanks to the definition of z_0, z_1^ε, for any $z \in \text{supp}(v) \subset [z_0, z_1^\varepsilon]$,

$$[(W'' *_x \bar{\rho})(\bar{u}(z)) + V''(\bar{u}(z))]\, v^\varepsilon(z) = \big(0 + o_{\bar{u}(z)-x_0}(1)\big)\, v^\varepsilon(z) = o_\varepsilon(1) v^\varepsilon(z). \tag{21}$$

Finally, using (21) and (20) in (18), we get:

$$\|L(v^\varepsilon)\|_{L^1} = o_\varepsilon(1)\|v^\varepsilon\|_{L^1},$$

which proves the proposition.

□

3 Singular interaction potentials

In this section, we shall consider interaction potentials having a singularity at $x = 0$:

- Interaction potentials having an attractive singularity at $x = 0$, satisfying Assumption 5 (see Subsection 3.1),
- Interaction potentials having a repulsive singularity at $x = 0$, satisfying Assumption 6 (see Subsection 3.2).

(6) shows that the support of $\rho(t,\cdot)$ is uniformly bounded w.r.t. time, we shall therefore only consider compactly supported solutions. We shall show that those two cases have a very different dynamics : If Assumption 5 is satisfied, every steady-state apart from sums of Dirac masses are nonlinearly unstable, whereas if Assumption 6 is satisfied, the solution (of the time-dependant equation) is uniformly bounded in $L^\infty(\mathbb{R})$.

3.1 Interaction potentials having an attractive singularity at $x = 0$

We shall consider in this section potentials having an attractive singularity at $x = 0$, that is interaction potentials W such that $W'(0) > 0$:

Assumption 5

$$V \in C^2(\mathbb{R}), \quad W \in C^0(\mathbb{R}),$$

and there exist $W'(0^+) > 0$ such that

$$x \mapsto \tilde{W}(x) := W(x) - W'(0^+)|x| \in C^2(\mathbb{R}). \tag{22}$$

It is well known that in this case, classical solutions of (1) may blow up in finite time (see [4, 3]). Following [11], we extend (1) to measure-valued solutions with the following equation:

$$\partial_t \rho(t,x) = \partial_x \left[\rho(t,x) \left(\int_{y \neq x} W'(x-y)\rho(t,y)\,dy + V'(x) \right) \right], \tag{23}$$

where we write (with a slight abuse of notation) $\rho(t,y)\,dy$ instead of $d\rho(t,\cdot)(y)$. If Assumptions 1 to 3 and 5 are satisfied, then it has been proven in [11] that a unique solution $\rho \in \mathrm{AC}_{loc}([0,\infty), \mathcal{P}_2(\mathbb{R}))$ to (23) exist. Note that the energy (10) is also a Lyapounov functional for (23).

One can check that the pseudo-inverse $u(t,z)$ of the solution $\rho(t,x)$ to (23) satisfies:

$$\partial_t u(t,z) = \int_{\{\xi \in [0,1];\, u(t,\xi) \neq u(t,z)\}} W'(u(t,\xi) - u(t,z))\,d\xi - V'(u(t,z)). \tag{24}$$

For regular potentials, we showed that if a (compactly supported) steady-state $\bar{\rho} \in M^1(\mathbb{R})$ of (1) is such that supp $\bar{\rho}$ has an accumulation point, then $\bar{\rho}$ cannot be linearly stable (in a sense defined in Prop. 3). In the case of interaction potentials having an attractive singularity at $x = 0$, we shall show that if a (compactly supported) steady-state $\bar{\rho} \in M^1(\mathbb{R})$ of (23) is such that supp $\bar{\rho}$ has an accumulation point (and a bit more, see (25)), then $\bar{\rho}$ is actually nonlinearly unstable in the sense that there exists arbitrarily close measures of strictly smaller energy, as we show in the proposition below:

Proposition 4. *Let V, W satisfy Assumptions 1 and 5. Let $\bar{\rho}$ be a compactly supported steady-state of* (23)*. If supp $\bar{\rho}$ has an accumulation point x_0 such that:*

$$\exists C > 0,\ \exists \eta > 0,\ \forall \gamma \in (0,\eta), \quad \frac{1}{\gamma}\int_{x_0}^{x_0+\gamma} \bar{\rho}(y)\,dy \geq C \tag{25}$$

(or the same estimate with $-\eta < \varepsilon < 0$), then it is locally unstable: For any $\varepsilon > 0$, there exists $\rho^\varepsilon \in M^1(\mathbb{R})$, such that $W_1(\rho^\varepsilon, \bar\rho) \le \varepsilon$ and

$$E(\rho^\varepsilon) < E(\bar\rho), \tag{26}$$

where E is the energy defined by (10)*.*

Remark 4. *As in the case of regular potentials, there may exist L^1 steady-states of* (23)*: For example, if $V(x) := \frac{-x^2}{2}$, $W(x) := |x|$,*

$$\bar\rho := \frac{1}{2}\mathbb{I}_{[-1,1]} \tag{27}$$

is a steady-state of (23)*. Prop. 4 shows that such steady-states are unstable.*

Eq. (24) *is not linearisable around steady-states (in L^1) in general. As a consequence, in order to define the nonlinear instability of steady-states like* (27)*, we use the energy E (which is a Lyapounov functional of* (1)*), see* (26)*.*

Sketch of the proof of Prop. 4 :

The main idea is to construct a measure ρ^ε arbitrarily close to $\bar\rho$ by collapsing the mass of $\bar\rho$ around x_0 into a single Dirac mass (see (30)):

$$\rho^\varepsilon = \bar\rho|_{[x_0,x_0+\varepsilon]^c} + \left(\int_{[x_0,x_0+\varepsilon]} \bar\rho(x)\,dx\right)\delta_{\bar x^\varepsilon}.$$

Then, in Step 2, we estimate the difference of energy of $\bar\rho$ and ρ^ε to get (see(33)):

$$\begin{aligned} E(\rho^\varepsilon) - E(\bar\rho) &= -\frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\ &\quad + \frac{1}{2}\left(-\omega^\varepsilon + o_\varepsilon(1)\right)\|v^\varepsilon\|_{L^2}^2. \end{aligned}$$

In steps 3 and 4, we estimate resp. $\|v^\varepsilon\|_{L^2}^2$ and ω^ε, to show that the first termon the right hand side of (28), which is negative, is dominant, and thus, $E(\rho^\varepsilon) - E(\bar\rho) < 0$.

Proof of Prop. 4

Step 1 : We define a sequence of measures (ρ^ε) approaching $\bar\rho$.

We assume w.l.o.g. that x_0 is an accumulation point of $\operatorname{supp}\bar\rho \cap [x_0, \infty)$ such that (25) is satisfied. We define for $\varepsilon > 0$ such that $x_0 + \varepsilon \in \operatorname{supp}\bar\rho$:

$$\begin{gathered} z_0 := \inf\{z \in (0,1); \bar u(z) \ge x_0\}, \\ z_1^\varepsilon := \sup\{z \in (0,1); \bar u(z) \le x_0 + \varepsilon\}, \\ Z^\varepsilon := [z_0, z_1^\varepsilon]. \end{gathered}$$

Since x_0, $x_0+\varepsilon \in$ supp $\bar\rho$ and $\bar\rho$ is a steady-state of (23),

$$\int_{\{y\notin[x_0,x_0+\varepsilon]\}} W'(x_0-y)\bar\rho(y)\,dy + V'(x_0) = -\int_{y\in(x_0,x_0+\varepsilon]} W'(x_0-y)\bar\rho(y)\,dy,$$

$$\int_{\{y\notin[x_0,x_0+\varepsilon]\}} W'(x_0+\varepsilon-y)\bar\rho(y)\,dy + V'(x_0+\varepsilon) = -\int_{y\in[x_0,x_0+\varepsilon)} W'(x_0+\varepsilon-y)\bar\rho(y)\,dy.$$

If $\varepsilon>0$ is small enough, then, $\mathrm{sign}(W'(x)) = \mathrm{sign}(x)$ for $x\in[-\varepsilon,\varepsilon]$. Then,

$$\int_{\{y\notin[x_0,x_0+\varepsilon]\}} W'(x_0-y)\bar\rho(y)\,dy+V'(x_0) > 0 > \int_{\{y\notin[x_0,x_0+\varepsilon]\}} W'(x_0+\varepsilon-y)\bar\rho(y)\,dy+V'(x_0+\varepsilon).$$

On $[x_0,x_0+\varepsilon]$,

$$\begin{aligned} F(x) &= \int_{\{y\notin[x_0,x_0+\varepsilon]\}} W'(x-y)\bar\rho(y)\,dy + V'(x) \\ &= W'(0^+)\int_{(-\infty,x_0)} \bar\rho(y)\,dy - W'(0^+)\int_{(x_0,+\infty)} \bar\rho(y)\,dy \\ &\quad + \int_{\{y\notin[x_0,x_0+\varepsilon]\}} \tilde W'(x-y)\bar\rho(y)\,dy + V'(x), \end{aligned}$$

where $\tilde W$ is defined in (22), and F is then continuous on $[x_0,x_0+\varepsilon]$. There exists then $\bar x^\varepsilon\in[x_0,x_0+\varepsilon]$ such that

$$\int_{\{y\notin[x_0,x_0+\varepsilon]\}} W'(\bar x^\varepsilon-y)\bar\rho(y)\,dy + V'(\bar x^\varepsilon) = 0. \tag{28}$$

We define the following perturbation u^ε of $\bar u$:

$$u^\varepsilon(z) := \left|\begin{array}{l} \bar u(z) \text{ on } (Z^\varepsilon)^c, \\ \bar x^\varepsilon \text{ on } Z^\varepsilon, \end{array}\right. , \quad v^\varepsilon := u^\varepsilon - \bar u. \tag{29}$$

u^ε is then the pseudo-inverse of the measure:

$$\rho^\varepsilon = \bar\rho|_{[x_0,x_0+\varepsilon]^c} + \left(\int_{[x_0,x_0+\varepsilon]} \bar\rho(x)\,dx\right)\delta_{\bar x^\varepsilon}. \tag{30}$$

Notice that $W_1(\rho^\varepsilon,\bar\rho)\le\varepsilon$.

Step 2: We estimate $E(\rho^\varepsilon)-E(\bar\rho)$.

We use the symmetry of W and the fact that $u^\varepsilon=\bar u$ on $(Z^\varepsilon)^c$ to compute:

$$
\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) &= \frac{1}{2}\int\int_{(Z^\varepsilon)^2} W(u^\varepsilon(\xi) - u^\varepsilon(z))\, d\xi\, dz - \frac{1}{2}\int\int_{(Z^\varepsilon)^2} W(\bar u(\xi) - \bar u(z))\, d\xi\, dz \\
&+ \int_{Z^\varepsilon}\int_{(Z^\varepsilon)^c} W(u^\varepsilon(z) - u^\varepsilon(\xi))\, d\xi\, dz - \int_{Z^\varepsilon}\int_{(Z^\varepsilon)^c} W(\bar u(z) - \bar u(\xi))\, d\xi\, dz \\
&+ \int_{Z^\varepsilon} V(u^\varepsilon(z))\, dz - \int_{Z^\varepsilon} V(\bar u(z))\, dz.
\end{aligned}
$$

Since u^ε is constant on Z^ε (see (29)), the first term can be computed. We estimate the second term using the expansion $W(x) = W(0)+W'(0)|x|+\tilde W'(0)x+O(x^2)$ (thanks to Assumption 5), where the notation $O(x^2)$ stands for a term such that $\frac{1}{x^2}O(x^2)$ is bounded when x is in a neighbourhood of 0. Notice that $\tilde W'(0) = 0$ thanks to Assumption 1. We use Taylor expansions on the fourth and sixth terms to get:

$$
\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) &= \frac{W(0)}{2}\left(|Z^\varepsilon|^2 - |Z^\varepsilon|^2\right) \\
&-\frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\, d\xi\, dz \\
&+ \int_{Z^\varepsilon}\left\{\left[\int_{(Z^\varepsilon)^c} W(u^\varepsilon(z) - u^\varepsilon(\xi))\, d\xi\right] + V(u^\varepsilon(z))\right\} dz \\
&- \int_{Z^\varepsilon}\left\{\left[\int_{(Z^\varepsilon)^c} W(\bar x^\varepsilon - \bar u(\xi))\, d\xi\right] + V(\bar x^\varepsilon)\right\} dz \\
&+ \int_{Z^\varepsilon}\left\{\left[\int_{(Z^\varepsilon)^c} W'(\bar x^\varepsilon - \bar u(\xi))\, d\xi\right] + V'(\bar x^\varepsilon)\right\} (\bar x^\varepsilon - \bar u(z))\, dz \\
&-\frac{1}{2}\int_{Z^\varepsilon}\left\{\left[\int_{(Z^\varepsilon)^c} W''(\theta_1(\xi,z) - \bar u(\xi)))\, d\xi\right] + V''(\theta_2(z))\right\} (\bar x^\varepsilon - \bar u(z))^2\, dz,
\end{aligned}
$$

where $\theta_1(\xi, z)$, $\theta_2(z) \in [(\bar u(z), \bar x^\varepsilon)]$. Since $u^\varepsilon(z) = \bar x^\varepsilon$ on Z^ε, the third and fourth line cancel. The fifth line is equal to 0 thanks to the definition of $\bar x^\varepsilon$ (see (28)). Then,

$$
\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) &= -\frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2} |\bar u(\xi) - \bar u(z)|\, d\xi\, dz \\
&-\frac{1}{2}\int_{Z^\varepsilon}\left\{\left[\int_{(Z^\varepsilon)^c} W''(\theta_1(\xi,z) - \bar u(\xi))\, d\xi\right] + V''(\theta_2(z))\right\} (\bar x^\varepsilon - \bar u(z))^2\, dz.
\end{aligned}
$$

Since $\bar{\rho}$ is compactly supported, W'', V'' are continuous, and $\theta_1(\xi, z)$, $\theta_2(z) \in [(\bar{u}(z), \bar{x}^\varepsilon)]$, we have uniform estimates:

$$\begin{aligned} \sup_{\{\xi \in (Z^\varepsilon)^c,\, z \in Z^\varepsilon\}} |W''(\theta_1(\xi, z) - \bar{u}(\xi)) - W''(\bar{x}^\varepsilon - \bar{u}(\xi))| = o_\varepsilon(1), \\ \sup_{\{z \in Z^\varepsilon\}} |V''(\theta_2(z)) - V''(\bar{x}^\varepsilon)| = o_\varepsilon(1), \end{aligned} \tag{31}$$

where the notation $o_\varepsilon(1)$ stands for a a term such that $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Then, if we define

$$\omega^\varepsilon := \int_{(Z^\varepsilon)^c} W''(\bar{u}(\xi) - \bar{x}^\varepsilon)\, d\xi + V''(\bar{x}^\varepsilon), \tag{32}$$

we get:

$$\begin{aligned} E(\rho^\varepsilon) - E(\bar{\rho}) \;=\; & -\frac{W'(0^+) + O(\varepsilon)}{2} \int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)|\, d\xi\, dz \\ & + \frac{1}{2}\left(-\omega^\varepsilon + o_\varepsilon(1)\right) \|v^\varepsilon\|_{L^2}^2. \end{aligned} \tag{33}$$

In order to prove the proposition, we shall show that the first term of (33) is strictly negative and dominates the second term (which is strictly positive). Then, $E(\rho^\varepsilon) - E(\bar{\rho}) < 0$ if $\varepsilon > 0$ is small enough. However, the two terms of (33) are of the same order in ε, we shall thus need to estimate precisely the second term.

Step 3: We estimate $\|v^\varepsilon\|_{L^2}^2$.

Since $\bar{u}$ is a steady-state, for any $z \in Z^\varepsilon$,

$$\begin{aligned} 0 \;&=\; \int_{\{\xi;\, \bar{u}(\xi) \neq \bar{u}(z)\}} W'(\bar{u}(\xi) - \bar{u}(z))\, d\xi - V'(\bar{u}(z)) \\ &=\; \left[\int_{(Z^\varepsilon)^c} W'(\bar{u}(\xi) - \bar{u}(z))\, d\xi - V'(\bar{u}(z))\right] \\ &\quad + \int_{\{\xi \in Z^\varepsilon;\, \bar{u}(\xi) \neq \bar{u}(z)\}} W'(\bar{u}(\xi) - \bar{u}(z))\, d\xi. \end{aligned}$$

We estimate the first term through Taylor expansions of $x \mapsto W'(\bar{u}(\xi) - x)$, $x \mapsto V'(x)$ around $\bar{x}^\varepsilon$ (the rest term is estimated as in (31)), and the second term using $W'(x) = W'(0^+)\mathrm{sign}(x) + \tilde{W}'(x) = W'(0^+)\mathrm{sign}(x) + \tilde{W}''(\theta)x$ and $\mathrm{sign}(0) = 0$ to get:

$$
\begin{aligned}
0 &= \left[\int_{(Z^\varepsilon)^c} W'(\bar u(\xi)-\bar x^\varepsilon)\,d\xi - V'(\bar x^\varepsilon)\right]\\
&\quad+\left[\int_{(Z^\varepsilon)^c} W''(\bar u(\xi)-\bar x^\varepsilon)\,d\xi + V''(\bar x^\varepsilon)\right](\bar x^\varepsilon-\bar u(z)) + o_\varepsilon(1)(\bar x^\varepsilon-\bar u(z))\\
&\quad+W'(0^+)\int_{Z^\varepsilon}\mathrm{sign}(\bar u(\xi)-\bar u(z))\,d\xi + \int_{Z^\varepsilon} W''(\theta)(\bar u(\xi)-\bar u(z))\,d\xi\\
&= 0+\omega^\varepsilon\, v^\varepsilon(z) + W'(0^+)\int_{Z^\varepsilon}\mathrm{sign}(\bar u(\xi)-\bar u(z))\,d\xi\\
&\quad+O(1)\int_{Z^\varepsilon}|\bar u(\xi)-\bar u(z)|\;d\xi + o_\varepsilon(1)v^\varepsilon(z),
\end{aligned}
$$

thanks to the definition of $\bar x^\varepsilon$. We can then estimate v^ε (see (29), we also recall the definition of ω^ε (32)) as follows:

$$
\begin{aligned}
\|v^\varepsilon\|_{L^2}^2 &= \int_{Z^\varepsilon} v^\varepsilon(z)^2\,dz\\
&= \int_{Z^\varepsilon}\left[\frac{W'(0^+)}{-\omega^\varepsilon}\int_{z_0}^{z_1^\varepsilon}\mathrm{sign}(\bar u(\xi)-\bar u(z))\,d\xi\right] v^\varepsilon(z)\,dz\\
&\quad+\frac{1}{-\omega^\varepsilon}O(1)\|v^\varepsilon\|_\infty \int\!\!\int_{(Z^\varepsilon)^2}|\bar u(\xi)-\bar u(z)|\;d\xi\,dz + \frac{o_\varepsilon(1)}{\omega^\varepsilon}\|v^\varepsilon\|_{L^2}^2. \quad (34)
\end{aligned}
$$

Let $z\in[0,1]$, and $\zeta := \inf\{\xi\in[z_0,z_1^\varepsilon];\ \bar u(\xi)=\bar u(z)\}$, $\zeta' := \sup\{\xi\in[z_0,z_1^\varepsilon];\ \bar u(\xi)=\bar u(z)\}$. Then,

$$
\begin{aligned}
\int_{z_0}^{z_1^\varepsilon}\mathrm{sign}(\bar u(\xi)-\bar u(z))\,d\xi &= \int_{[z_0,z_1^\varepsilon]\setminus(\zeta,\zeta')}\mathrm{sign}(\xi-z)\,d\xi + \int_\zeta^{\zeta'} 0\,d\xi\\
&= \int_{z_0}^{z_1^\varepsilon}\mathrm{sign}(\xi-z)\,d\xi - \int_\zeta^{\zeta'}\mathrm{sign}(\xi-z)\,d\xi\\
&= [(z_1^\varepsilon-z)-(z-z_0)] - [(\zeta'-z)-(z-\zeta)]\\
&= -2\left[z-\frac{z_0+z_1^\varepsilon}{2}\right] + 2\left[z-\frac{\zeta+\zeta'}{2}\right].
\end{aligned}
$$

Then, since $\bar u$ is constant on (ζ,ζ'), so is $z\mapsto v^\varepsilon(z)=\bar x^\varepsilon-\bar u(z)=v^\varepsilon\left(\frac{\zeta+\zeta'}{2}\right)$, and

$$\int_{\zeta}^{\zeta'} \left[\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\bar{u}(\xi) - \bar{u}(z))\, d\xi \right] v^\varepsilon(z)\, dz$$
$$= -2 \int_{\zeta}^{\zeta'} \left[z - \frac{z_0 + z_1^\varepsilon}{2} \right] v^\varepsilon(z)\, dz + 2 v^\varepsilon \left(\frac{\zeta + \zeta'}{2} \right) \int_{\zeta}^{\zeta'} \left[z - \frac{\zeta + \zeta'}{2} \right] dz$$
$$= -2 \int_{\zeta}^{\zeta'} \left[z - \frac{z_0 + z_1^\varepsilon}{2} \right] v^\varepsilon(z)\, dz. \tag{35}$$

We consider

$$\Omega := \Big\{ (\zeta, \zeta') \subset Z^\varepsilon;\ \bar{u} \text{ is constant on } (\zeta, \zeta'),$$
$$(\zeta, \zeta') \text{ being the maximal interval such that this is true} \Big\}.$$

Since each element of Ω contains a rational number, Ω is at most countable, and then, thanks to (35),

$$\int_{Z^\varepsilon} \left[\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\bar{u}(\xi) - \bar{u}(z))\, d\xi \right] v^\varepsilon(z)\, dz$$
$$= \int_{Z^\varepsilon \setminus \left(\cup_{(\zeta,\zeta') \in \Omega} (\zeta, \zeta') \right)} \left[\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\bar{u}(\xi) - \bar{u}(z))\, d\xi \right] v^\varepsilon(z)\, dz$$
$$+ \sum_{(\zeta,\zeta') \in \Omega} \int_{\zeta}^{\zeta'} \left[\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\bar{u}(\xi) - \bar{u}(z))\, d\xi \right] v^\varepsilon(z)\, dz$$
$$= \int_{Z^\varepsilon \setminus \left(\cup_{(\zeta,\zeta') \in \Omega} (\zeta, \zeta') \right)} \left[\int_{z_0}^{z_1^\varepsilon} \operatorname{sign}(\xi - z)\, d\xi \right] v^\varepsilon(z)\, dz$$
$$+ \sum_{(\zeta,\zeta') \in \Omega} -2 \int_{\zeta}^{\zeta'} \left[z - \frac{z_0 + z_1^\varepsilon}{2} \right] v^\varepsilon(z)\, dz$$
$$= -2 \int_{Z^\varepsilon} \left[z - \frac{z_0 + z_1^\varepsilon}{2} \right] v^\varepsilon(z)\, dz. \tag{36}$$

Thanks to (36), (34) becomes:

$$\left(1 - \frac{o_\varepsilon(1)}{\omega^\varepsilon} \right) \|v^\varepsilon\|_{L^2}^2 \;=\; -2 \frac{W'(0^+)}{-\omega^\varepsilon} \int_{Z^\varepsilon} \left(z - \frac{z_0 + z_1^\varepsilon}{2} \right) v^\varepsilon(z)\, dz$$
$$+ \frac{1}{-\omega^\varepsilon} O(\varepsilon) \int\!\!\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)|\, d\xi\, dz. \tag{37}$$

We notice that:

$$\int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)|\, d\xi\, dz = 2\int\int_{(Z^\varepsilon)^2,\, \xi\geq z} [\bar{u}(\xi) - \bar{u}(z)]\, d\xi\, dz$$
$$= 2\int_{Z^\varepsilon} [(z - z_0)\bar{u}(z) - (z_1^\varepsilon - z)\bar{u}(z)]\, dz$$
$$= 4\int_{Z^\varepsilon} \left(z - \frac{z_0 + z_1^\varepsilon}{2}\right) \bar{u}(z)\, dz,$$

and since $\int_{Z^\varepsilon}(z - \frac{z_0+z_1^\varepsilon}{2})\, dz = 0$, we have:

$$\int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)|\, d\xi\, dz = 4\int_{Z^\varepsilon} \left(z - \frac{z_0 + z_1^\varepsilon}{2}\right) (\bar{u}(z) - \bar{x}^\varepsilon)\, dz$$
$$= -4\int_{Z^\varepsilon} \left(z - \frac{z_0 + z_1^\varepsilon}{2}\right) v^\varepsilon(z)\, dz. \qquad (38)$$

Finally, thanks to (38), (37) becomes:

$$\|v^\varepsilon\|_{L^2}^2 = \frac{W'(0^+) + O(\varepsilon)}{-2\omega^\varepsilon + o_\varepsilon(1)} \int\int_{(Z^\varepsilon)^2} |\bar{u}(\xi) - \bar{u}(z)|\, d\xi\, dz. \qquad (39)$$

Step 4: We estimate ω^ε.

See (32) for the definition of ω^ε. In this step, we denote by $\bar{\rho}((a,b))$, with a, $b \in \mathbb{R}\cup\{-\infty, +\infty\}$, the $\bar{\rho}$-measure of the open interval (a,b). Since x_0, $x_0+\varepsilon \in$

supp $\bar\rho = \overline{\bar u([0,1])}$ and $\bar u$ is a steady-state of (24),

$$
\begin{aligned}
0 &= \Big(\int_{\{\xi\in[0,1];\,\bar u(\xi)\neq x_0+\varepsilon\}} W'(\bar u(\xi)-(x_0+\varepsilon))\,d\xi - V'(x_0+\varepsilon)\Big)\\
&\quad - \Big(\int_{\{\xi\in[0,1];\,\bar u(\xi)\neq x_0\}} W'(\bar u(\xi)-x_0)\,d\xi - V'(x_0)\Big)\\
&= \Big(\int_0^1 \Big(W'(0^+)\mathrm{sign}(\bar u(\xi)-(x_0+\varepsilon)) + \tilde W'(\bar u(\xi)-(x_0+\varepsilon))\Big)\,d\xi - V'(x_0+\varepsilon)\Big)\\
&\quad - \Big(\int_0^1 \Big(W'(0^+)\mathrm{sign}(\bar u(\xi)-x_0) + \tilde W'(\bar u(\xi)-x_0)\Big)\,d\xi - V'(x_0)\Big)\\
&= \Bigg(W'(0^+)\Big(\bar\rho\big((x_0+\varepsilon,+\infty)\big) - \bar\rho\big((-\infty,x_0+\varepsilon)\big)\Big)\\
&\qquad + \int_0^1 \tilde W'(\bar u(\xi)-(x_0+\varepsilon))\,d\xi - V'(x_0+\varepsilon)\Bigg)\\
&\quad - \Big(W'(0^+)\Big(\bar\rho\big((x_0,+\infty)\big) - \bar\rho\big((-\infty,x_0)\big)\Big) + \int_0^1 \tilde W'(\bar u(\xi)-x_0)\,d\xi - V'(x_0)\Big)\\
&= -W'(0^+)\left[\bar\rho(\{x_0,x_0+\varepsilon\}) + 2\bar\rho\big((x_0,x_0+\varepsilon)\big)\right]\\
&\quad - \left[\int_0^1 \tilde W''(\bar u(\xi)-\bar x^\varepsilon)\,d\xi + V''(\bar x^\varepsilon)\right]\varepsilon + o(\varepsilon),
\end{aligned}
$$

where we applied a Taylor expansion to the regular terms $x\mapsto \tilde W'(\bar u(\xi)-x)$ and $x\mapsto V'(x)$ at point $x=\bar x^\varepsilon$ (the rest term is estimated as in (31)). We notice that

$$
\begin{aligned}
\int_0^1 \tilde W''(\bar u(\xi)-\bar x^\varepsilon)\,d\xi + V''(\bar x^\varepsilon) &= \omega^\varepsilon + \int_{Z^\varepsilon} \tilde W''(\bar u(\xi)-\bar x^\varepsilon)\,d\xi\\
&= \omega^\varepsilon + O(|Z^\varepsilon|),
\end{aligned}
$$

and then,

$$
-\varepsilon(\omega^\varepsilon + O(|Z^\varepsilon|)) = W'(0^+)\left[\bar\rho(\{x_0,x_0+\varepsilon\}) + 2\bar\rho\big((x_0,x_0+\varepsilon)\big)\right] + o(\varepsilon). \quad (40)
$$

Since $|\omega^\varepsilon| \le \|W''\|_{L^\infty(\mathrm{supp}\ \bar\rho-\mathrm{supp}\ \bar\rho)} + \|V''\|_{L^\infty(\mathrm{supp}\ \bar\rho)}$, we have in particular that $|Z^\varepsilon|$ is of order ε:

$$
|Z^\varepsilon| = \bar\rho([x_0,x_0+\varepsilon]) = O(\varepsilon), \quad (41)
$$

and then, using again (40), we get that for ε small enough,

$$
\begin{aligned}
-\omega^\varepsilon &= \frac{W'(0^+)}{\varepsilon}\left[\bar\rho(\{x_0,x_0+\varepsilon\}) + 2\bar\rho\big((x_0,x_0+\varepsilon)\big)\right] + o_\varepsilon(1)\\
&\ge W'(0^+)\,\frac{1}{\varepsilon}\bar\rho\big([x_0,x_0+\varepsilon]\big) + o_\varepsilon(1).
\end{aligned}
$$

We assumed (see (25)) that $\frac{1}{\varepsilon}\int_{[x_0,x_0+\varepsilon]}\bar\rho(x)\,dx > C > 0$ for ε small enough. Then, for $\varepsilon > 0$ small enough,

$$-\omega^\varepsilon \geq C > 0. \tag{42}$$

Step 5: We conclude.

Thanks to (39), (33) becomes:

$$\begin{aligned}
E(\rho^\varepsilon) - E(\bar\rho) &= -\frac{W'(0^+) + O(\varepsilon)}{2}\int\int_{(Z^\varepsilon)^2}|\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\
&\quad +\frac{1}{2}\left(-\omega^\varepsilon + o_\varepsilon(1)\right)\frac{W'(0^+) + O(\varepsilon)}{-2\omega^\varepsilon + o_\varepsilon(1)}\int\int_{(Z^\varepsilon)^2}|\bar u(\xi) - \bar u(z)|\,d\xi\,dz \\
&= -\left[\frac{W'(0^+)}{4} + o_\varepsilon(1)\right]\int\int_{(Z^\varepsilon)^2}|\bar u(\xi) - \bar u(z)|\,d\xi\,dz,
\end{aligned}$$

thanks to (42). Finally, we assumed that x_0 is an accumulation point of $\operatorname{supp}\rho^0\cap[x_0,\infty)$, ε can thus be chosen small enough for $o_\varepsilon(1) \leq \frac{W'(0^+)}{8}$ to hold, and then,

$$E(\rho^\varepsilon) - E(\bar\rho) \leq -\frac{W'(0^+)}{8}\int\int_{(Z^\varepsilon)^2}|\bar u(\xi) - \bar u(z)|\,d\xi\,dz. \tag{43}$$

Since x_0 is an accumulation point of $\operatorname{supp}\bar\rho\cap[x_0,x_0+\varepsilon] = \bar u(Z^\varepsilon)$, $\bar u$ cannot be constant on Z^ε, and then:

$$E(\rho^\varepsilon) - E(\bar\rho) < 0. \tag{44}$$

□

3.2 Potentials having a repulsive singularity at $x = 0$

In this section, we shall consider potentials having a repulsive singularity at $x = 0$, that is interaction potentials W such that $W'(0) < 0$:

Assumption 6

$$V \in C^2(\mathbb{R}), \quad W \in C^0(\mathbb{R}),$$

and there exists $W'(0^+) < 0$ such that

$$(x \mapsto \tilde W(x) := W(x) - W'(0^+)|x|) \in C^2(\mathbb{R}).$$

For such potentials, we don't know any existence theory, we thus prove in Prop. 5 that if Assumptions 1, 2, 3 and 6 are satisfied, and if $\rho^0 \in W^{2,\infty}(\mathbb{R})$, then there exists a unique solution $\rho \in L^\infty(\mathbb{R}_+\times\mathbb{R})\cap\mathrm{Lip}_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$.

Proposition 5. *Let ρ^0, V, W satisfy Assumptions 1, 2, 3 and 6. Assume moreover that $\rho^0 \in W^{2,\infty}(\mathbb{R})$. Then there exists a unique solution*

$$\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$$

to (1).

If $\rho^0 \in W^{N,\infty}(\mathbb{R})$ and $V \in W^{N+2,\infty}(\mathbb{R})$ (for $N \in \mathbb{N}$), then $\rho \in Lip_{loc}(\mathbb{R}_+, W^{N,\infty}(\mathbb{R}))$

Remark 5. *The uniform bound $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ ensures that the solution does not converge to any singular measure. The behavior of the solution in this case is then very different from the two other cases (Assumptions 4 or 5) studied in this paper, where the solution generically converges to a sum of Dirac masses. For a short investigation on the transition from the situation of regular kernels to the situation where W has a singularity at $x = 0$ and is locally repulsive, see [17].*

Proof of Prop. 5

Step 1: We show some a priori estimates on ρ, using maximum principle arguments:

We consider first $x \in \mathbb{R}$ such that $\rho(t,x) = \|\rho(t,\cdot)\|_\infty$. Then $\partial_x\rho(t,x) = 0$, and

$$\begin{aligned}
\partial_t\rho(t,x) &= \partial_x\rho(t,x)(W' * \rho)(t,x) + \rho(t,x)\left((\tilde{W}'' * \rho)(t,x) + V''(x)\right) \\
&\quad -2W'(0^+)\rho(t,x)^2 \\
&= \left((\tilde{W}'' * \rho)(t,x) + V''(x) - 2W'(0^+)\rho(t,x)\right)\rho(t,x) \\
&\leq \left(\|\tilde{W}''\|_{L^\infty} + \|V''\|_\infty - 2W'(0^+)\|\rho(t,\cdot)\|_\infty\right)\|\rho(t,\cdot)\|_\infty.
\end{aligned}$$

Then,

$$\|\rho(t,\cdot)\|_\infty \leq \max\left(\|\rho^0\|_\infty, \frac{1}{2|W'(0^+)|}\left(\|\tilde{W}''\|_{L^\infty} + \|V''\|_\infty\right)\right). \tag{45}$$

Let now $N \in \mathbb{N}$ and $x \in \mathbb{R}$ be such that $|\partial_x^N\rho(t,x)| = \|\partial_x^N\rho(t,\cdot)\|_\infty$. W.l.o.g.,$\partial_x^N\rho(t,x) \geq$

0, then,

$$\begin{aligned}
\partial_t\partial_x^N\rho(t,x) &= \partial_x^{N+1}\left(\rho(W'*\rho+V')\right)(t,x) \\
&= \sum_{n=0}^{N+1}\left(\begin{array}{c}N\\ n\end{array}\right)\partial_x^n\rho(t,x)\partial_x^{N+1-n}(W'*\rho+V')(t,x) \\
&= \sum_{n=1}^{N}\left(\begin{array}{c}N\\ n\end{array}\right)\partial_x^n\rho(t,x)(\tilde{W}''*\partial_x^{N-n}\rho-2W'(0^+)\partial_x^{N-n}\rho+\partial_x^{N+2-n}V)(t,x) \\
&\quad +\partial_x(\partial_x^N\rho)(t,x)(W'*\rho+V')(t,x) \\
&\quad +\rho(t,x)\Big[-2W'(0^+)\partial_x^N\rho(t,x)+\tilde{W}''*\partial_x^N\rho+\partial_x^{N+2}V\Big] \\
&\le \sum_{n=1}^{N}\left(\begin{array}{c}N\\ n\end{array}\right)\Big[\Big(\|\tilde{W}''\|_{L^1([-2C,2C])}+2W'(0^+)\Big)\|\partial_x^n\rho(t,\cdot)\|_\infty\|\partial_x^{N-n}\rho(t,\cdot)\|_\infty \\
&\qquad +\|\rho(t,\cdot)\|_{W^{N,\infty}}\|V\|_{W^{N+2,\infty}([-C,C])}\Big] \\
&\quad +0+\|\rho(t,\cdot)\|_\infty\Big[\|\tilde{W}''\|_{L^1([-2C,2C])}\|\rho(t,\cdot)\|_{W^{N,\infty}}+\|V\|_{W^{N+2,\infty}([-C,C])}\Big] \\
&\le C\left(1+\|\rho(t,\cdot)\|_{W^{N-1,\infty}}\right)\|\rho(t,\cdot)\|_{W^{N,\infty}},
\end{aligned}$$

where we used the assumption on x to get $\partial_x(\partial_x^N\rho)(t,x)=0$, the assumption $\partial_x^N\rho(t,x)\ge 0$ to get $\rho(t,x)\Big[-2W'(0^+)\partial_x^N\rho(t,x)\Big]\le 0$, and (6) to get that $\text{supp}\,\rho(t,\cdot)\subset[-C,C]$ (uniformly in time).

Since this inequality holds for any $N\ge 1$, and $\|\rho(t,\cdot)\|_{L^\infty}<Cst$ by (45), an induction argument shows that if $\rho^0\in W^{N,\infty}$, there exists $C=C(N,\,\|\rho^0\|_{W^{N,\infty}})$ such that

$$\|\rho(t,\cdot)\|_{W^{N,\infty}}\le\|\rho^0\|_{W^{N,\infty}}e^{Ct}. \tag{46}$$

Step 2: We build the solution using the above a priori estimates:

In order to prove the existence of a solution $\rho\in L^\infty(\mathbb{R}_+\times\mathbb{R})\cap\text{Lip}_{loc}(\mathbb{R}_+,W^{2,\infty}(\mathbb{R}))$ to (1), we use the inductive scheme: $\rho_0(t,x):=\rho^0(x)$, and

$$\left\{\begin{array}{l}\rho_{n+1}(0,\cdot)=\rho^0,\\ \partial_t\rho_{n+1}(t,x)=\partial_x\left(\rho_{n+1}W'*\rho_n+V'\right).\end{array}\right.$$

Thanks to estimates similar to the a priori estimates done in the first part of this proof, one gets the following (uniform in n) estimates:

$$\|\rho_{n+1}(t,\cdot)\|_\infty\le\|\rho^0\|_\infty e^{C\,t},$$

and ther exist C, $T>0$ such that $\forall t\le T$,

$$\|\partial_x\rho_{n+1}(t,\cdot)\|_\infty\le C\|\partial_x\rho^0\|_\infty,\quad \|\partial_t\rho_{n+1}(t,\cdot)\|_\infty\le C\left(\|\partial_x\rho^0\|_\infty+\|\rho^0\|_\infty\right).$$

Those estimates show that (ρ_n) converges in $L^\infty([0,T]\times\mathbb{R})$ up to an extraction. A further study of $(\rho_{n+1}-\rho_n)$ shows that the whole sequence converges to the unique strong solution ρ of (1).

Finally, estimate (46) shows the propagation of regularity anounced in Prop. 5.

□

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