Macroscopic limit from a structured population model to the Kirkpatrick-Barton model

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Abstract

We consider an ecology model where the population is structured by a spatial variable and a phenotypic trait. The model combines a parabolic operator on the space variable to a kinetic operator on the trait variable. The kinetic operator, that represents the effect of sexual reproduction, satisfies a Tanaka-type inequality: it implies a contraction for the Wasserstein distance in the phenotypic trait space. We combine this contraction argument to parabolic estimates controlling the spatial regularity of solutions to derive a macroscopic limit of the equation. More precisely when the reproduction rate is large, the moments of solutions of the kinetic model converge to the solution of the Kirkpatrick-Barton model.

1 Introduction

We are interested in a structured population model that describes the dynamics of a biological population (typically a species of trees submitted to climate change). At each time $t \geq 0$ the population is structured by a phenotypic trait $y \in \mathbb{R}$ and a spatial variable $x \in \mathbb{T}^d$ (the $d \in \mathbb{N}^*$ dimensional torus, typically $d \in \{1, 2, 3\}$). The population is then represented by a density $n = n(t, x, y)$, and the dynamics of this population is given by the *Spatially structured Infinitesimal Model* (see [30]):

$$\textbf{(SIM)} \quad \begin{aligned} \partial_t n(t,x,y) = \Delta_x n(t,x,y) + \left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t,x))^2 - \int n(t,x,z)\,dz\right) n(t,x,y) \\ + \gamma \left(\int\int \Gamma_{A/2}\left(y - \frac{y_* + y_*'}{2}\right) \frac{n(t,x,y_*)n(t,x,y_*')}{\int n(t,x,z)\,dz}\,dy_*\,dy_*' - n(t,x,y)\right), \end{aligned}$$

where $A > 0$ is the *phenotypic variance at linkage equilibrium* of the population (see [22, 14]), $y_{opt} : \mathbb{T}^d \to \mathbb{R}$ is a description of the environment (typically, $y_{opt}(t, x)$ is the temperature at time t and location x), and $\Gamma_{A/2}$ designates the Gaussian distribution with variance $A/2$:

$$\Gamma_{A/2}(y) := \frac{1}{\sqrt{2\pi A}} e^{\frac{-|y|^2}{A}}.$$

The SIM is composed of parabolic terms, that are usual in ecology models (see the Fisher-KPP equation [25], or more recently [3]), and a kinetic term, with a factor $\gamma > 0$, that represents the effect of sexual reproductions. Beyond the importance of this model for applications, the SIM is an opportunity to develop the analysis methods developed for other kinetic models (in particular the Boltzmann equation), with the help of an unusual diffusion term in the space variable. This diffusive term allows us to develop a new method to derive a macroscopic limit: starting from Wasserstein estimates on the collision operator, we are able to show that when $\gamma > 0$ is large, the dynamics of n can be described through a closed equation on its two first moments.

More precisely, in this article, we provide a rigorous proof of the connection between the SIM and the Kirkpatrick-Barton Model: we show that if $\gamma > 0$ is large, the solutions of the SIM satisfy $n(t, x, y) \sim N(t, x)\Gamma_A (y - Z(t, x))$, where the macroscopic quantities N and Z satisfy asymptotically the *Kirkpatrick-Barton Model* (this model, introduced in [34], is widely used in ecology):

$$
(\mathbf{KBM}) \qquad \begin{cases} \partial_t N(t,x) - \Delta_x N(t,x) = \left[1 - \frac{1}{2}(Z(t,x) - y_{opt}(t,x))^2 - N(t,x)\right] N(t,x), \\ \\ \partial_t Z(t,x) - \Delta_x Z(t,x) = 2\frac{\nabla_x N \cdot \nabla_x Z}{N}(t,x) - A(Z(t,x) - y_{opt}(t,x)). \end{cases}
$$

The SIM and KBM have received little attention from the mathematical community. To our knowledge, the only mathematical studies are [33], where the local existence of solutions for SIM-type models is discussed, [30], where the propagation fronts for a simplified model are built (this article also contains non-rigorous asymptotics related to the present study), and [29] which investigates the long time dynamics of a simplified model related to the KBM (this simplified model is different from the one considered in [30]). Several groups are currently working on the KBM, and we can expect some progress on the mathematical understanding of this model in the near future. We refer to Section 2.2 for a discussion of the biological aspects of the SIM, KBM, and the biological implications of our result.

In the case of asexual populations, the last term of the SIM simplifies considerably: it is then replaced by a local term plus a diffusive part (that represents mutations). Those asexual population models have received considerable attention recently, and the propagation phenomena that they exhibit are now well understood. The main idea is the asexual case is to consider the model as a semi-linear parabolic equation, to control the non-local competition term thanks to a Harnack inequality, and to use topological fixed-point arguments to build propagation fronts [3, 7, 11]. Additional difficulties appear when the phenotypic trait y has an impact on the spatial diffusion of individuals in space (see [11, 39, 8]), and those models may lead to accelerating fronts [8, 12]. Finally, when the mutation rate is small, those asexual models can be related to constrained Hamilton-Jacobi equations [13, 39, 10]. Note that in the asexual case, the propagation speed of the population (which plays an important role for biology) is given by a linearisation of the model, and is then explicit in terms of a certain principal eigenvalue problem. This simple characterisation of the propagation speed no longer holds in the case of sexual populations, and the macroscopic limit described here may provide a way to describe the propagation phenomena for the SIM (we refer to [16, 37] for a related idea in mathematical physics).

The macroscopic limit we present here is based on the Wasserstein contraction induced by the reproduction operator (see Theorem 4.1). This contraction property exists for a range of operators appearing in physics or econometry [6, 9, 42], and was originally obtained by Tanaka [38]. To our knowledge, few rigorous macroscopic/hydrodynamic results have been established using those results (see [35] for a spatially homogeneous result). Note that the strategy here is to combine Wasserstein estimates (for the reproduction term) to estimates of a different nature (parabolic estimates for the spatial dimension). This strategy is related to the work of Carlen and Gangbo [17] (see also [1]), who are interested in a kinetic Fokker-Planck equation which combines a hyperbolic transport term in space to a kinetic operator in the velocity space. This kinetic operator implies a contraction for the Wasserstein distance. The authors show the long time convergence of solutions to the set of local Maxwellians, but this large-time convergence is not quantitative, due to the lack of regularity estimates in the spatial variable. In the present study, the presence of a diffusive term in the space variable allows us to push the analysis further. Finally, we are also able to cope with the selection/competition term to justify the macroscopic limit of the SIM described above.

2 Main result and organisation of the paper

2.1 Main result

Throughout this manuscript, we will consider an optimal phenotypic trait $(t,x) \mapsto y_{opt}(t,x)$ and an initial population $(x,y) \mapsto n^0(x,y)$ satisfying

Assumption 2.1. *(i)* $y_{opt} \in C^1(\mathbb{R}_+ \times \mathbb{T}^d, \mathbb{R})$ *such that* $\|y_{opt}\|_{W^{1,\infty}(\mathbb{R}_+\times\mathbb{T}^d,\mathbb{R})} < \infty$.

(ii) $n^0 \in L^1(\mathbb{T}^d \times \mathbb{R}, \mathbb{R}_+)$*, such that*

$$\left\| \int (1+|y|^4) \frac{n^0(x,y)}{\int n^0(x,z)\,dz}\,dy \right\|_{L^\infty(\mathbb{T}^d)} < \infty, \qquad \left\| \int n^0(\cdot,y)\,dy \right\|_{W^{2,\infty}(\mathbb{T}^d)} < \infty,$$

$$\left\| \int y \frac{n^0(\cdot,y)}{\int n^0(\cdot,z)\,dz}\,dy \right\|_{W^{2,\infty}(\mathbb{T}^d)} < \infty,$$

and $\min_{\mathbb{T}^d} \int n^0(\cdot,y)\,dy > 0$.

Let n a solution of the SIM. Let N and Z the two first moments of n in the y variable:

$$N(t,x) = \int n(t,x,y)\,dy, \quad Z(t,x) = \int y \frac{n(t,x,y)}{\int n(t,x,z)\,dz}\,dy. \tag{1}$$

N and Z have a biological interpretation: they represent respectively the population size and the mean phenotypic trait. Our main result, stated below, shows that when $\gamma > 0$ is large, n satisfies:

$$n(t,x,y) \underset{\gamma \gg 1}{\sim} N(t,x)\Gamma_A(y - Z(t,x)),$$

and (N,Z) is close to the solution of the KBM with initial data

$$\Big(N(0,\cdot), Z(0,\cdot)\Big) = \left(\int n^0(x,y)\,dy, \int y \frac{n^0(x,y)}{\int n^0(x,z)\,dz}\,dy\right).$$

In (2), W_2 stands for the Wasserstein distance, and we refer to Section 4.1 in the Appendix for a description of those distances.

Theorem 2.2. *Let* y_{opt}, n^0 *satisfying Assumption 2.1 and* $A > 0$. *There exist* $\bar\gamma > 0$, $C > 0$ *and* $\theta \in (0,1)$ *such that for any* $\gamma > \bar\gamma$, *there exists a global solution* $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1((1 + |y|^4)\,dy))$ *of the SIM with initial data* n^0. *N and Z, defined by* (1)*, are Hölder continuous, and more precisely*

For $s, t \in \mathbb{R}_+$ and $x, y \in \mathbb{T}^d$, the functions N and Z defined by (1) *are Hölder continuous, and more precisely they satisfy*

$$\forall (t,x),(s,z) \in \mathbb{R}_+ \times \mathbb{T}^d, \quad \frac{|Z(t,x)-Z(s,z)|}{(|t-s|+|x-z|)^\theta} + \frac{|N(t,x)-N(s,z)|}{(|t-s|+|x-z|)^\theta} \leq C. \tag{2}$$

There exist φ_N, $\varphi_Z : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ satisfying

$$\|\varphi_N(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|\varphi_Z(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \frac{C}{\gamma^\theta} + C\mathbb{1}_{[0,C\ln\gamma/\gamma]}(t) \tag{3}$$

such that the following equations hold in the sense of distributions:

$$\begin{cases} \partial_t N(t,x) - \Delta_x N(t,x) = \left[1 - \frac{1}{2}(Z(t,x) - y_{opt}(t,x))^2 - N(t,x) + \varphi_N(t,x)\right] N(t,x), \\ \partial_t Z(t,x) - \Delta_x Z(t,x) = 2\frac{\nabla_x N \cdot \nabla_x Z}{N}(t,x) - A(Z(t,x) - y_{opt}(t,x)) + \varphi_Z(t,x), \end{cases} \tag{4}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^d$. Moreover,

$$\max_{(t,x)\in[\theta\ln\gamma/\gamma,\infty)\times\mathbb{T}^d} W_2\left(\frac{n(t,x,\cdot)}{N(t,x)}, \Gamma_A(\cdot - Z(t,x))\right) \leq \frac{C}{\gamma^\theta}. \tag{5}$$

Remark 2.3. *Theorem 2.2 implies in particular that the macroscopic quantities (N,Z) converge to the unique solution of the KBM with initial condition $(N(0,\cdot), Z(0,\cdot))$ when $\gamma \to \infty$. We show this implication in Section 4.4 of the Appendix.*

Theorem 2.2 combined to Proposition 3.1) implies the existence of a constant $C > 0$ independent from $\gamma > \bar{\gamma}$ such that for any $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^d$,

$$\int |y|^4 \frac{n(t,x,y)}{\int n(t,x,z)\,dz}\,dy \leq C.$$

The estimates given by Theorem 2.2 are global in time, even though $N(t,\cdot)$ may converge to 0 when $t \to \infty$. This is possible because the last term of the SIM (ie the "kinetic" operator) scales linearly with n. This is important for applications: those models are often used to investigate the possible extinction of species.

In a preliminary section (Section 3.1) we derive equations satisfied by various quantities such as N or Z. In Section 3.2, we show that an $L^\infty([0,\tau]\times\mathbb{T}^d)$ bound on Z (with $\tau \geq 0$) implies an estimate on the fourth moment of $y \mapsto n(t,x,\cdot)$ for $t \in [0,\tau]$. This implies in particular the existence of solutions for the SIM for a slightly longer time interval $[0,\tau+\sigma)$, with $\sigma > 0$ independent of the parameter $\gamma > \bar{\gamma} > 0$. In Section 3.3 we show that Z is Hölder continuous, provided we have a bound on $\|Z\|_{L^\infty}$. This regularity is used in Section 3.4 together with a Tanaka-type inequality (see Theorem 4.1 in the Appendix) to show that $\frac{n(t,x,\cdot)}{N(t,x)}$ is close to $\Gamma_A(\cdot - Z(t,x))$ for the Wasserstein distance when $\gamma \gg 1$. Finally in Section 3.5 we use the estimates mentioned above to obtain a uniform bound on $\|Z\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}$. This estimate implies both the existence of global solutions of the SIM when $\gamma > 0$ is large enough and the macroscopic limit described in Theorem 2.2.

2.2 Biological interpretation of the model and impact for ecology

The first term on the right-hand side of the SIM, $\Delta_x n' t, x, y)$, represents the diffusion of individuals in space. The term $\left(1 + \frac{A}{2} - \frac{1}{2}(y - y_{opt}(t,x))^2\right) n(t,x,y)$ represents the effect of natural selection: the individuals with a phenotypic trait y far from the optimal trait $y_{opt}(t,x)$ have a high mortality rate. The function y_{opt} should then be seen as a description of the environment, and is a given parameter of the model. For instance the trait y is could the temperature to which an individual is best adapted to, and y_{opt} is then the predicted map of temperatures. The term $-\left(\int n(t,x,z)\,dz\right) n(t,x,y)$ in the SIM represents competition: all individuals present at a given time in the same location are competing for e.g. resources. The last term describes the effect of sexual reproductions: when parents give birth to an offspring, the phenotypic trait of the offspring is drawn from a normal distribution with a fixed variance $A/2$ centered in the average of the traits of the parents. This model for the effect of sexual reproduction on a continuous phenotypic trait is known as the *Infinitesimal Model*. It was introduced by Fisher in 1919 [21], and is employed in population genetics either for theoretical purpose [14, 40, 5] or for practical applications [28, 41]. The limit $\gamma \gg 1$ corresponds to a short generation time and it can be seen as the implicit assumption behind the classical *Linkage Equilibrium* assumption used in population genetics (see e.g. [14]): in the framework of the *Infinitesimal Model* the Linkage Equilibrium assumption implies that the distribution of the population $\tilde{n}(t,x,\cdot)$ is Gaussian with fixed variance. Numerical simulations (see [30]) suggest the macroscopic limit model KBM provides a good description of the dynamics of solutions of the SIM for γ as small as 2.

We expect the SIM to be related to a well chosen Individual Based Model through a large number of individuals argument, but to our knowledge, this asymptotic doesn't exist at the moment. This type of derivation exists for asexual model [18], but here an additional difficulty arises: describing the SIM as a large population limit of an Individual Based Model will require a precise understanding of the connection between explicit genetic models and the *Infinitesimal Model* (which is at the root of the reproduction operator appearing in the SIM). In spite of some recent developments (see [5]), additional work on this connection is necessary.

The KBM was introduced by Kirkpatrick and Barton in 1997 [34], and is widely used to model the dynamics of populations' ranges, in particular when those populations are submitted to climate change, see e.g. [15, 2]. The success of the KBM comes from to the complex dynamics it exhibits [34, 30]: even for a very simple environment described by $y_{opt}(t,x) = Bx$ (and $x \in \mathbb{R}$), the population can either go extinct, survive without propagating, or propagate (see [34]). Mathematically, these dynamics raise a number of challenging questions. Several simplified models exist (see [32, 30]), and we refer to [29, 30] for the analysis some of those simplified model.

A good understanding of connections between the SIM and the KBM (and further connections to stochastic models) has practical implications: the different scales (such as the mesoscopic scale of the SIM and macroscopic scale of the KBM) are not clearly distinct in most biological systems, and an easy navigation between different scales of description is an essential feature of the theory. This was illustrated recently by [2] where the macroscopic limit from the SIM to the KBM plays an important role. We believe these models will play an important role in forthcoming years: the KBM provides a precise description of the effect of climate change on species and is a valuable complement to Species Distribution Models (see e.g. [24]) that are currently prevailing.

3 Proof of the main result

Throughout the manuscript, $C > 0$ designates a constant depending only on y_{opt}, n^0 and A, while $C_\kappa > 0$ is a constant that additionally depends on $\kappa > 0$.

3.1 Preliminary: equations satisfied by solutions of the SIM

If we integrate the SIM along the variable y, we get that the population size N (see (1) for its definition) satisfies

$$\partial_t N - \Delta_x N = \left[1 + \frac{A}{2} - N(t,x)\right] N(t,x) - \frac{1}{2}\int (y - y_{opt}(t,x))^2 n(t,x,y)\, dy. \tag{6}$$

We define the normalized profile of the population,

$$\tilde{n}(t,x,y) = \frac{n(t,x,y)}{N(t,x)}, \tag{7}$$

which satisfies

$$\begin{aligned}
&\partial_t \tilde{n}(t,x,y) - \Delta_x \tilde{n}(t,x,y) \\
&\quad = 2\frac{\nabla_x N(t,x)}{N(t,x)} \cdot \nabla_x \tilde{n}(t,x,y) + \gamma\left(T(\tilde{n}(t,x,\cdot)) - \tilde{n}(t,x,y)\right) \\
&\qquad + \frac{1}{2}\tilde{n}(t,x,y)\left(\int (z - y_{opt}(t,x))^2 \tilde{n}(t,x,z)\, dz - (y - y_{opt}(t,x))^2\right),
\end{aligned} \tag{8}$$

where T, the Infinitesimal operator, is defined in the Appendix (see (47)). From this expression, we can deduce the following equation on the mean phenotypic trait of the population Z (see (1) for its definition):

$$\begin{aligned}
&\partial_t Z(t,x) - \Delta_x Z(t,x) \\
&\quad = 2\frac{\nabla_x N(t,x) \cdot \nabla_x Z(t,x)}{N(t,x)} - \frac{1}{2}\int (y - Z(t,x))\,(y - y_{opt}(t,x))^2 \tilde{n}(t,x,y)\, dy.
\end{aligned} \tag{9}$$

Finally, from (8), we can also derive the following equation satisfied by $V(t,x) := \int |y|^4 \tilde{n}(t,x,y)\, dy$:

$$\begin{aligned}
&\partial_t V(t,x) - \Delta_x V(t,x) \\
&\quad = 2\frac{\nabla_x N(t,x)}{N(t,x)} \cdot \nabla_x V(t,x) + \frac{1}{2}\int \left(V(t,x) - |y|^4\right)(y - y_{opt}(t,x))^2 \tilde{n}(t,x,y)\, dy \\
&\qquad + \gamma\left(\int |y|^4 T(\tilde{n}(t,x,\cdot))(y)\, dy - V(t,x)\right).
\end{aligned} \tag{10}$$

3.2 Estimates on the 4^{th} moment of solutions and short time existence

In this section, we show that a bound on $\|Z\|_{L^\infty([0,\tau)\times S^1)}$ implies a bound on $\|V\|_{L^\infty([0,\tau)\times \mathbb{T}^d)}$:

Proposition 3.1. *Let y_{opt}, n^0 satisfying Assumption 2.1, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$ and $C_\kappa > 0$ such that for any $\gamma > \bar{\gamma}$ and $\tau \in (0, +\infty]$, the following statement holds: if a solution $n \in L^\infty([0,\tau) \times \mathbb{T}^d, L^1(\mathbb{R}))$ of the SIM with initial condition n^0 satisfies $\|Z\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \le \kappa$, then*

$$\forall (t,x) \in [0,\tau) \times \mathbb{T}^d, \qquad \int |y|^4 \frac{n(t,x,y)}{\int n(t,x,z)\, dz}\, dy \le C_\kappa.$$

Remark 3.2. *Under the assumptions of the proposition above,* (11) *shows the following estimate, that will be useful on several occasions in the manuscript:*

$$\int |y|^4 T(\tilde{n}(t,x,\cdot))(y)\,dy \leq C_\kappa$$

Proof of Proposition 3.1. The dynamics of V is given by (10), and to estimate the last term of that equation, we take advantage of (48) and Theorem 4.1: for $(t,x) \in [0,\tau) \times \mathbb{T}^d$,

$$\begin{aligned}
&\int |y|^4 T(\tilde{n}(t,x,\cdot))(y)\,dy = W_4(T(\tilde{n}(t,x,\cdot)),\delta_0)^4 \\
&\quad \leq [W_4\left(T(\tilde{n}(t,x,\cdot)), T(\Gamma_A(Z(t,x)-\cdot))\right) + W_4\left(\Gamma_A(Z(t,x)-\cdot),\delta_0\right)]^4 \\
&\quad \leq \left[\frac{1}{2^{1/4}} W_4(\tilde{n}(t,x,\cdot),\Gamma_A(Z(t,x)-\cdot)) + W_4\left(\Gamma_A(Z(t,x)-\cdot),\delta_0\right)\right]^4 \\
&\quad \leq \left(\frac{1}{2^{1/4}} W_4(\tilde{n}(t,x,\cdot),\delta_0) + 2W_4(\delta_0,\Gamma_A(Z(t,x)-\cdot))\right)^4 \leq \left(\frac{1}{2^{1/4}} W_4(\tilde{n}(t,x,\cdot),\delta_0) + 2Z(t,x) + C\right)^4 \\
&\quad \leq \frac{2}{3} W_4^4(\tilde{n}(t,x,\cdot),\delta_0) + C(Z(t,x)^4+1), \qquad (11)
\end{aligned}$$

for some constant $C > 0$, thanks to a Young inequality. The last term of (10) then satisfies

$$\gamma\left(\int |y|^4 T(\tilde{n}(t,x,\cdot))(y)\,dy - V(t,x)\right) \leq \gamma\left(C\left(\|Z\|^4_{L^\infty([0,\tau)\times\mathbb{T}^d)}+1\right) - \frac{1}{3}V(t,x)\right).$$

To estimate the second term on the right hand side of (10), we use a Cauchy-Schwarz inequality as follows

$$\begin{aligned}
&\int \left(V(t,x)-|y|^4\right)(y-y_{opt}(t,x))^2 \tilde{n}(t,x,y)\,dy \leq V(t,x)\int (y-y_{opt}(t,x))^2\tilde{n}(t,x,y)\,dy \\
&\quad \leq CV(t,x)\int \left(|y|^2+1\right)\tilde{n}(t,x,y)\,dy \leq C\left(1+\sqrt{V(t,x)}\right)V(t,x).
\end{aligned}$$

We use both estimates to obtain that on $[0,\tau] \times \mathbb{T}^d$,

$$\begin{aligned}
\partial_t V(t,x) - \Delta_x V(t,x) \quad &\leq \quad 2\frac{\nabla_x N(t,x)}{N(t,x)}\cdot\nabla_x V(t,x) + C\left(1+\sqrt{V(t,x)}\right)V(t,x) \\
&\quad + \gamma\left(C(\|Z\|^4_{L^\infty([0,\tau)\times\mathbb{T}^d)}+1) - \frac{1}{3}V(t,x)\right). \qquad (12)
\end{aligned}$$

Let

$$\bar{V} := \max\left(\|V(0,\cdot)\|_{L^\infty(\mathbb{T}^d)}, 7C\left(\|Z\|^4_{L^\infty([0,\tau)\times\mathbb{T}^d)}+1\right)\right).$$

As soon as $\gamma \geq 6C\left(1+\sqrt{\bar{V}}\right)$, $\phi \equiv \bar{V}$ is a supersolution of (12). The parabolic comparison principle then implies that for $(t,x) \in [0,\tau) \times \mathbb{T}^d$, $V(t,x) \leq \bar{V}$. □

Proposition 3.3. *Let y_{opt}, n^0 satisfying Assumption 2.1 and $A > 0$. There exist $\sigma > 0$ and $\bar{\gamma} > 0$ such that the following statement holds: if $\tau \geq 0$ and $n \in L^\infty([0,\tau) \times \mathbb{T}^d, L^1((1+|y|^4)\,dy))$, a non-negative solution of the SIM with initial condition n^0, satisfies*

$$\|Z\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + 1, \qquad (13)$$

then the solution can be extended as $n \in L^\infty([0,\tau+\sigma]\times\mathbb{T}^d, L^1((1+|y|^4)\,dy))$*, which satisfies* $n \geq 0$ *on* $[0,\tau+\sigma]\times\mathbb{T}^d\times\mathbb{R}$*, and*

$$\|Z\|_{L^\infty([0,\tau+\sigma)\times\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + 2. \tag{14}$$

Proof of Proposition 3.3. Thanks to (6) and the comparison principle,

$$\|N\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq \max\left(1+\frac{A}{2}, \|N(0)\|_{L^\infty(\mathbb{T}^d)}\right), \tag{15}$$

and applying the comparison principle to the SIM shows that

$$\|n\|_{L^\infty([0,\tau)\times\mathbb{T}^d\times[-R,R])} \leq \max\left(\|n^0\|_{L^\infty(\mathbb{T}^d\times[-R,R])}, \|\Gamma_{A/2}\|_{L^\infty(\mathbb{R})}\right). \tag{16}$$

We introduce now a modified SIM: Let $R>0$, we consider solutions $n_R(\tau+\cdot,\cdot,\cdot)$ of the modified SIM where $\Gamma_{A/2}\left(y-\frac{y_*+y_*'}{2}\right)$ is replaced by $\Gamma_{A/2}\left(y-\frac{y_*+y_*'}{2}\right)1_{|y|\leq R}$, with initial condition $n_R(\tau,x,y) := n(\tau,x,y)1_{|y|\leq R}$.

The existence and uniqueness of the solution $n_R(\tau+\cdot,\cdot,\cdot)$ over a short time $[\tau,\tau+\sigma_{\gamma,R})$ follows from a classical Cauchy-Lipschitz argument in $L^\infty([\tau,\tau+\sigma_{\gamma,R})\times\mathbb{T}^d\times[-R,R])$, and $n_R(t,x,y)=0$ for $|y|\geq R$ (and $(t,x)\in[\tau,\tau+\sigma_{\gamma,R})\times\mathbb{T}^d$). If we repeat the comparison principle argument at the begining of the present proof, we can extend estimate (16) for times $t\in[\tau,\tau+\sigma_{\gamma,R})$, and obtain that $\|n_R\|_{L^\infty([0,\tau+\sigma_{\gamma,R})\times\mathbb{T}^d\times[-R,R])} \leq \max\left(\|n^0\|_{L^\infty(\mathbb{T}^d\times[-R,R])}, \|\Gamma_{A/2}\|_{L^\infty(\mathbb{R})}\right)$. This uniform estimate implies that maximal solutions of the modified SIM are indeed global solutions: $n_R \in L^\infty([\tau,\infty)\times\mathbb{T}^d\times\mathbb{R})$. $n_R(t,x,\cdot)$ is compactly supported in y, which implies $n_R(t,x,\cdot)\in L^1\left((1+|y|^4)\,dy\right)$, and if we denote by (N_R,Z_R,V_R) the moments corresponding to n_R (see (1) and (10)), we get from (9) (or rather the equation similar to (9) satisfied by Z_R) that for $(t,x)\in[\tau,\infty)\times\mathbb{T}^d$,

$$\partial_t Z_R(t,x) - \Delta_x Z_R(t,x) = 2\frac{\nabla_x N_R(t,x)\cdot\nabla_x Z_R(t,x)}{N_R(t,x)} + \mathcal{O}\left(1+\|V_R(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}\right).$$

Thanks to the comparison principle, for $t\geq\tau$,

$$\frac{d}{dt}\|Z_R(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq C\left(1+\|V_R(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}\right). \tag{17}$$

The estimate (12) can be repeated here, and provided $\gamma>0$ is large enough, for $t\geq\tau$,

$$\frac{d}{dt}\|V_R(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq C\left(1+\|V_R(t,\cdot)\|^{3/2}_{L^\infty(\mathbb{T}^d)}\right) + \|Z_R(t,\cdot)\|^4_{L^\infty(\mathbb{T}^d)}. \tag{18}$$

We recall that $n_R(\tau,x,y) = n(\tau,x,y)1_{|y|\leq R}$, and Proposition 3.1 implies

$$\|V_R(\tau,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \|V(\tau,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq C,$$

provided $\gamma>0$ is large enough, with a constant $C>0$ is independent of $\gamma>0$ and $\tau\geq0$. This estimate combined to (17) and (18) implies the existence of $\tilde\sigma>0$ independent of $R>0$, $\tau\geq0$ and $\gamma>\bar\gamma$ such that

$$\|V_R\|_{L^\infty([\tau,\tau+\tilde\sigma)\times\mathbb{T}^d)} + \|Z_R\|_{L^\infty([\tau,\tau+\tilde\sigma)\times\mathbb{T}^d)} \leq C.$$

Finally, this estimate and (17) show that for some $\sigma \in (0, \tilde{\sigma})$ independent of $R > 0$, $\tau \geq 0$ and $\gamma > \bar{\gamma}$, the following estimate holds:

$$\|Z_R\|_{L^\infty([\tau,\tau+\sigma)\times\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + 2, \tag{19}$$

(19) and estimate (18) implies a bound on $\|V_R\|_{L^\infty([0,\tau+\sigma/2]\times\mathbb{T}^d)}$ independent of $R > 0$. This bound, estimate (15) and the boundedness of $[0, \tau + \sigma/2] \times \mathbb{T}^d$ imply that $(n_R|_{\{t\leq\tau+\sigma/2\}})_R$ is a tight family of Borel measures over $[0, \tau + \sigma/2] \times \mathbb{T}^d \times \mathbb{R}$. We can then apply Prokhorov's theorem: up to an extraction, $(n_R|_{\{t\leq\tau+\sigma/2\}})_R$ converges weakly in $\mathcal{M}([0, \tau + \sigma/2] \times \mathbb{T}^d \times \mathbb{R})$ ($\mathcal{M}$ designates the set of Borel measures) to a limit n. Estimates (19) and (18) hold for the limit n, which implies that $n \in L^\infty(\mathbb{T}^d, L^1((1 + |y|^4))$. Finally, it is possible to pass to the limit on a weak form of the SIM to show that n is a solution of the SIM in the sense of Distributions. □

3.3 Regularity of N and Z

Proposition 3.4. *Let y_{opt}, n^0 satisfying Assumption 2.1, $A > 0$, $\kappa > 0$ and $\delta > 0$. There exist $\bar{\gamma} > 0$, $C_\kappa > 0$ and $\theta \in (0, 1)$ such that if $\gamma \geq \bar{\gamma}$ and $n \in L^\infty([0, \tau) \times \mathbb{T}^d, L^1((1+|y|^4)\, dy))$ is a solution of the SIM with initial condition n^0 satisfying $\|Z\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq \kappa$ for some $\tau \in (0, +\infty]$, then for any $s, t \in [0, \tau)$ and $x, y \in \mathbb{T}^d$,*

$$\frac{|Z(t, x) - Z(s, y)|}{(|t - s| + |x - y|)^\theta} + \frac{|N(t, x) - N(s, y)|}{(|t - s| + |x - y|)^\theta} \leq C_\kappa.$$

where N and Z are defined by (1).

Proof of Proposition 3.4. Let $\bar{\gamma} > 0$ as in Proposition 3.1.

Step 1: Lower bound on $N(t, x)$
Since $\|Z\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq \kappa$, Proposition 3.1 implies that $\int |y|^4\tilde{n}(t, x, y)\, dy$ is uniformly bounded on $[0, \tau) \times \mathbb{T}^d$, and there exist a constant $C_\kappa > 0$ such that for $(t, x) \in [0, \tau) \times \mathbb{T}^d$,

$$\left| \left[1 + \frac{A}{2} - N(t, x)\right] N(t, x) - \frac{1}{2}\int (y - y_{opt}(t, x))^2 n(t, x, y)\, dy \right| \leq C_\kappa N(t, x). \tag{20}$$

Let $t \in [0, 1] \cap [0, \tau)$. Thanks to (20) and the comparison principle,

$$N(t, x) \geq e^{-C_\kappa t} \inf_{\mathbb{T}^d} N(0, \cdot) \geq C_\kappa, \tag{21}$$

thanks to Assumption 2.1. Thanks to (20) also, we can apply the Harnack inequality for $t \in [0, \tau) \setminus [0, 1]$ (see [27], or Theorem 3 in [4]): there exists $C_\kappa > 0$ such that for any $t \in [0, \tau) \setminus [0, 1]$,

$$\max_{(s,x)\in[t-3/4,t-1/2]\times\mathbb{T}^d} N(s, x) \leq C_\kappa \min_{(s,x)\in[t-1/3,t]\times\mathbb{T}^d} N(t, x).$$

Since $\partial_t N - \Delta_x N \leq N$, we may consider the super-solution $(s, x) \mapsto (\max_{x\in\mathbb{T}^d} N(t - 1/2, x))\, e^{s-(t-1/2)}$, and the comparison principle implies, for $t \in [0, \tau) \setminus [0, 1]$,

$$\max_{(s,x)\in[t-3/4,t]\times\mathbb{T}^d} N \leq C_\kappa \min_{[t-1/3,t]\times\mathbb{T}^d} N. \tag{22}$$

Step 2: Estimate on $\frac{\nabla_x N(t,x)}{N(t,x)}$ for $t \in [0,1]$

We notice that for $(t,x) \in (-\infty,\tau) \times \mathbb{R}$, $N(t,x) = (N(0,x) + \mathcal{N}(t,x))\, 1_{t\geq 0}$, where $\mathcal{N}$ is a solution of

$$\partial_t \mathcal{N}(t,x) - \Delta_x \mathcal{N}(t,x) = \mu_N(t,\pi(x)) 1_{t\geq 0}, \quad (t,x) \in (-\infty,\tau) \times \mathbb{R}^d, \tag{23}$$

where $\pi(x)$ is the standard projection of $x \in \mathbb{R}^d$ on $\mathbb{T}^d$, and

$$\mu_N(t,x) = \Delta_x N^0(x) + \left(1 + \frac{A}{2} - \frac{1}{2}\int (y - y_{opt}(t,x))^2 \tilde{n}(t,x,y)\, dy - N(t,x)\right) N(t,x).$$

Note that $\mathcal{N}(t,\cdot) \equiv 0$ for $t \leq 0$. Thanks to (20) and Assumption 2.1, we have $\|\mu_N\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} < C_\kappa$, and we can apply Theorem 7.22 of [26] to obtain

$$\|\partial_x \mathcal{N}\|_{L^{d+3}([t-1/4,t]\times\mathbb{T}^d)} \leq C_\kappa \left(\|\mathcal{N}\|_{L^{d+3}([t-1/3,t]\times\mathbb{T}^d)} + 1\right), \tag{24}$$

for any $t \in \mathbb{R}$. For $t \in [0,1]$, we combine this estimate to (15) and (21) to obtain

$$\left\|\frac{\nabla_x N}{N}\right\|_{L^{d+3}([0,1]\times\mathbb{T}^d)} \leq C_\kappa. \tag{25}$$

Step 3: Estimate on $\frac{\nabla_x N(t,x)}{N(t,x)}$ for $t \in [0,\tau) \setminus [0,1]$

The argument here is similar to the one developed for step 2, but on equation (6) instead of (23). Theorem 7.22 of [26] applied to (6) implies that for $t \geq 1$,

$$\|\partial_x N\|_{L^{d+3}([t-1/4,t]\times\mathbb{T}^d)} \leq C_\kappa \|N\|_{L^{d+3}([t-1/3,t]\times\mathbb{T}^d)}, \tag{26}$$

which we combine to (22) to obtain, for $t \geq 1$,

$$\left\|\frac{\nabla_x N}{N}\right\|_{L^{d+3}(([t-1/4,t+1/4]\cap[0,\tau))\times\mathbb{T}^d)} \leq C_\kappa \frac{\|N\|_{L^{d+3}(([t-1/3,t])\times\mathbb{T}^d)}}{\|N\|_{L^\infty(([t-3/4,t])\times\mathbb{T}^d)}} \leq C_\kappa. \tag{27}$$

Step 4: Regularity of N and Z

Just as we have done for $\mathcal{N}(t,x) = N(t,x) - N^0(x)$ (see (23)), we can define $\mathcal{Z} = (Z(t,x) - Z(0,x))\, 1_{t\geq 0}$, solution of

$$\partial_t \mathcal{Z}(t,x) - \Delta_x \mathcal{Z}(t,x) = 2\frac{\nabla_x N(t,x)}{N(t,x)} \cdot \nabla_x \mathcal{Z}(t,x) + \mu_Z(t,\pi(x)) 1_{t\geq 0}, \quad (t,x) \in (-\infty,\tau) \times \mathbb{R}^d,$$

where $\|\mu_Z\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} < C_\kappa$ thanks to Proposition 3.1 and Assumption 2.1, and $\frac{\nabla_x N}{N}$ satisfies (25), (27). We can then apply Theorem 4 from [4] (a corollary of the Harnack inequality) to $\mathcal{N}$ and $\mathcal{Z}$, and obtain a Hölder estimate on both $\mathcal{N}$ and $\mathcal{Z}$, which concludes the proof of the proposition.

□

3.4 Distance of solutions of the SIM to a local Maxwellian

Proposition 3.5. *Let y_{opt}, n^0 satisfying Assumption 2.1, $A > 0$, $\kappa > 0$. There exist $\bar\gamma > 0$, $C_\kappa > 0$ and $\theta \in (0,1)$ such that for any $\gamma > \bar\gamma$, and $\tau \in (0,+\infty]$, the following statement holds: if a solution $n \in L^\infty([0,\tau) \times \mathbb{T}^d, L^1((1+|y|^4)\,dy))$ of the SIM with initial condition n^0 satisfies $\|Z\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq \kappa$, then*

$$\forall t \in \left[C_\kappa \frac{\ln\gamma}{\gamma}, \tau\right), \quad \max_{x\in\mathbb{T}^d} W_2^2\Big(\tilde n(t,x,\cdot), \Gamma_A(\cdot - Z(t,x))\Big) \leq \frac{C_\kappa}{\gamma^\theta}, \tag{28}$$

where $\tilde n$ is given by (7), Z is defined by (1) and Γ_A is defined by (49).

Proof of Proposition 3.5. In this proof, we will use the linear problems and estimates presented in Section 4.3 of the Appendix. Let in particular $(t,x) \mapsto \phi_{s,z,y}(t,x)$ defined by (51). For $t \geq 0$, we can use a Duhamel formula to write $\tilde n$ (we recall that $\tilde n$ satisfies (8)) as follows

$$\begin{aligned}
\tilde n(t,x,y) &= e^{-\gamma t}\int \tilde n(0,z,y)\phi_{0,z,y}(t,x)\,dz \\
&+ \frac12\int_0^t e^{-\gamma(t-s)}\int \phi_{s,z,y}(t,x)\tilde n(s,z,y)\left(\int (w - y_{opt}(s,z))^2\tilde n(s,z,w)\,dw\right)dz\,ds \\
&+ \gamma\int_0^t e^{-\gamma(t-s)}\int \phi_{s,z,y}(t,x)T(\tilde n(s,z,\cdot))(y)\,dz\,ds.
\end{aligned}$$

Since $\tilde n(t,x,\cdot)$ is a probability measure, the $y-$integral of the right hand size of the equation above sums up to one. This and the convexity of the squared Wasserstein distance W_2^2 (see Section 4.1 in the Appendix) implies

$$\begin{aligned}
&W_2^2\left(\tilde n(t,x,\cdot), \Gamma_A(\cdot - Z(t,x))\right) \leq e^{-\gamma t}\int\left(\int \tilde n(0,z,y)\phi_{0,z,y}(t,x)\,dy\right) \\
&\quad W_2^2\left(\frac{\tilde n(0,z,y)\phi_{0,z,y}(t,x)}{\int \tilde n(0,z,y)\phi_{0,z,y}(t,x)\,dy}, \Gamma_A(\cdot - Z(t,x))\right)dz \\
&\quad + \frac12\int_0^t e^{-\gamma(t-s)}\int\left(\int \phi_{s,z,y}(t,x)\tilde n(s,z,y)\left(\int (w-y_{opt}(s,z))^2\tilde n(s,z,w)\,dw\right)dy\right) \\
&\qquad W_2^2\left(\frac{\phi_{s,z,\cdot}(t,x)\tilde n(s,z,\cdot)}{\int \phi_{s,z,y}(t,x)\tilde n(s,z,y)\,dy}, \Gamma_A(\cdot - Z(t,x))\right)dz\,ds \\
&\quad + \gamma\int_0^t e^{-\gamma(t-s)}\int\left(\int \phi_{s,z,y}(t,x)T(\tilde n(s,z,\cdot))(y)\,dy\right) \\
&\quad W_2^2\left(\frac{\phi_{s,z,\cdot}(t,x)T(\tilde n(s,z,\cdot))}{\int \phi_{s,z,y}(t,x)T(\tilde n(s,z,\cdot))(y)\,dy}, T\left(\Gamma_A(\cdot - Z(t,x))\right)\right)dz\,ds.
\end{aligned} \tag{29}$$

Note that we have used that $\Gamma_A(\cdot - Z(t,x))$ is a fixed point for T (see (48)). To estimate the first two terms on the right hand side of (29), a rough estimate is sufficient: for any $(s,z) \in [0,\infty) \times \mathbb{T}^d$ and $(t,x) \in [s,\infty) \times \mathbb{T}^d$,

$$\begin{aligned}
&W_2^2\left(\frac{\phi_{s,z,\cdot}(t,x)\tilde n(s,z,\cdot)}{\int \phi_{s,z,y}(t,x)\tilde n(s,z,y)\,dy}, \Gamma_A(\cdot - Z(t,x))\right) \\
&\leq \left(W_2^2\left(\frac{\phi_{s,z,\cdot}(t,x)\tilde n(s,z,\cdot)}{\int \phi_{s,z,y}(t,x)\tilde n(s,z,y)\,dy}, \delta_0\right) + W_2\left(\delta_0, \Gamma_A(\cdot - Z(t,x))\right)\right)^2 \\
&\quad \leq 2\int |y|^2\frac{\phi_{s,z,y}(t,x)\tilde n(s,z,y)}{\int \phi_{s,z,y'}(t,x)\tilde n(s,z,y')\,dy'}\,dy + 2\int |y|^2\Gamma_A(y - Z(t,x))\,dy \leq C_\kappa,
\end{aligned} \tag{30}$$

where the final estimate follows from Section 4.3 in the Appendix: if we define R by (53) and R' as in (55) (note that $|R'| \leq C_\kappa$), then (54), (56) and Proposition 3.1 imply

$$\begin{aligned}\int |y|^2 \frac{\phi_{s,z,y}(t,x)\tilde{n}(s,z,y)}{\int \phi_{s,z,y'}(t,x)\tilde{n}(s,z,y')\,dy'}\,dy &\leq \int_{[-R',R']^c} |y|^2 \frac{\left(\min_{|\tilde{y}|\leq R}\phi_{s,z,\tilde{y}}(t,x)\right)\tilde{n}(s,z,y)}{\int_{|y'|\leq R}\phi_{s,z,y'}(t,x)\tilde{n}(s,z,y')\,dy}\,dy \\ &+C_\kappa \int_{-R'}^{R'} \frac{\phi_{s,z,y}(t,x)\tilde{n}(s,z,y)}{\int \phi_{s,z,y'}(t,x)\tilde{n}(s,z,y')\,dy'}\,dy \leq C_\kappa \quad (31)\end{aligned}$$

We repeat the estimate (30) (using additionally the estimate of Remark 3.2) to control the last term of (29) for $s \leq t-\varepsilon$, for some $\varepsilon > 0$ that we will define later on. We obtain then, for $s \leq t-\varepsilon$,

$$W_2^2\left(\frac{\phi_{s,z,\cdot}(t,x)T(\tilde{n}(s,z,\cdot))}{\int \phi_{s,z,y}(t,x)T(\tilde{n}(s,z,\cdot))(y)\,dy}, T\left(\Gamma_A(\cdot - Z(t,x))\right)\right) \leq C_\kappa. \quad (32)$$

For $s \in [t-\varepsilon, t]$, a more precise estimate is necessary. Let $\bar{\phi}_{s,z}(t,x)$ defined by (52), and we define

$$\begin{aligned}\pi(y_1,y_2) &= \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}T(\tilde{n}(s,z,\cdot))(y_1)\delta_{y_1=y_2} \\ &+\left(1-\frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right)T(\tilde{n}(s,z,\cdot))(y_1)\frac{\phi_{s,z,y_2}(t,x)T(\tilde{n}(s,z,\cdot))(y_2)}{\int \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y')\,dy'}.\end{aligned}$$

π is then a probability measure on $\mathbb{R}\times\mathbb{R}$ (note that $\phi_{s,z,y_1}(t,x) \leq \bar{\phi}_{s,z}(t,x)$, thanks to (57)), with marginals

$$\pi|_1(y_1) = T(\tilde{n}(s,z,\cdot))(y_1) \quad \text{and} \quad \pi|_2(y_2) = \frac{\phi_{s,z,y_2}(t,x)T(\tilde{n}(s,z,\cdot))(y_2)}{\int \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y')\,dy'}.$$

Then,

$$\begin{aligned}&W_2^2\left(\frac{\phi_{s,z,y}(t,x)T(\tilde{n}(s,z,\cdot))(y)}{\int \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y')\,dy'}, T(\tilde{n}(s,z,\cdot))\right) \leq \int |y_1-y_2|^2\,d\pi(y_1,y_2) \\ &\leq 2\int\int \left(y_1^2+y_2^2\right)\left(1-\frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right)T(\tilde{n}(s,z,\cdot))(y_1)\frac{\phi_{s,z,y_2}(t,x)T(\tilde{n}(s,z,\cdot))(y_2)}{\int \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y')\,dy'}\,dy_1\,dy_2 \\ &\leq 2\int y_1^2\left(1-\frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right)T(\tilde{n}(s,z,\cdot))(y_1)\,dy_1 \\ &\quad +2\left(1-\frac{\int \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y')\,dy'}{\bar{\phi}_{s,z}(t,x)}\right)\int y_2^2\frac{\phi_{s,z,y_2}(t,x)T(\tilde{n}(s,z,\cdot))(y_2)}{\int \phi_{s,z,y'}(t,x)T(\tilde{n}(s,z,\cdot))(y')\,dy'}\,dy_2.\end{aligned} \quad (33)$$

We estimate the first integral term of (33) by breaking the integral into two integral terms. The first integral term can then be controlled thanks to a Chebyshev's inequality (we recall Remark 3.2), while we use the estimate (57), derived in the Appendix, to estimate the second integral term:

$$\int y_1^2 \left(1 - \frac{\phi_{s,z,y_1}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y_1)\,dy_1 \leq \int_{|y_1|\geq (t-s)^{-1/3}} y_1^2 T(\tilde{n}(s,z,\cdot))(y_1)\,dy_1$$
$$+ \int_{|y_1|\leq (t-s)^{-1/3}} y_1^2 \left(1 - e^{-(t-s)\frac{1}{2}(y_1+\mathcal{O}(1))^2}\right) T(\tilde{n}(s,z,\cdot))(y_1)\,dy_1$$
$$\leq C_\kappa |t-s|^{2/3} + \left(1 - e^{\frac{-1}{3}(t-s)^{1/3}}\right) \int_{|y_1|\leq (t-s)^{-1/3}} y_1^2 T(\tilde{n}(s,z,\cdot))(y_1)\,dy_1$$
$$\leq C_\kappa |t-s|^{2/3} + C_\kappa \left(1 - e^{\frac{-1}{3}(t-s)^{1/3}}\right) \leq C_\kappa |t-s|^{1/3}, \tag{34}$$

provided $|t-s|$ is small enough. The last term of (33) is a factor of two terms. We reproduce the argument (34) (with 1 instead of y_1^2) to estimate the first factor, and use (31) for the second factor:

$$2\left(1 - \frac{\int \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y')\,dy'}{\bar{\phi}_{s,z}(t,x)}\right) \int y_2^2 \frac{\phi_{s,z,y_2}(t,x) T(\tilde{n}(s,z,\cdot))(y_2)}{\int \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y')\,dy'}\,dy_2$$
$$\leq \left(\int \left(1 - \frac{\phi_{s,z,y'}(t,x)}{\bar{\phi}_{s,z}(t,x)}\right) T(\tilde{n}(s,z,\cdot))(y')\,dy'\right) C_\kappa \leq C_\kappa |t-s|^{1/3}, \tag{35}$$

provided $|t-s|$ is small enough. Thanks to (34) and (35), the estimate (33) becomes

$$W_2^2\left(\frac{\phi_{s,z,y}(t,x) T(\tilde{n}(s,z,\cdot))(y)}{\int \phi_{s,z,y'}(t,x) T(\tilde{n}(s,z,\cdot))(y')\,dy'}, T(\tilde{n}(s,z,\cdot))\right) \leq C_\kappa |t-s|^{1/3}.$$

This estimate combined to the regularity estimates on N and Z obtained in Proposition 3.4 lead to

$$W_2\left(\frac{\phi_{s,z,\cdot}(t,x) T(\tilde{n}(s,z,\cdot))}{\int \phi_{s,z,y}(t,x) \tilde{n}(s,z,y)\,dy}, T(\Gamma_A(\cdot - Z(t,x)))\right)$$
$$\leq W_2\left(\frac{\phi_{s,z,\cdot}(t,x) T(\tilde{n}(s,z,\cdot))}{\int \phi_{s,z,y}(t,x) \tilde{n}(s,z,y)\,dy}, T(\tilde{n}(s,z,\cdot))\right)$$
$$+ W_2\left(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))\right) + |Z(t,x) - Z(s,z)|$$
$$\leq W_2^2\Big(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))\Big) + C_\kappa |t-s|^\theta + C_\kappa |x-z|^\theta, \tag{36}$$

for some $\theta \in (0,1)$, provided $\gamma > 0$ is large enough. We are now ready to consider the original estimate (29): thanks to (30), (32) and (36), the estimate (29) implies

$$W_2^2\left(\tilde{n}(t,x,\cdot), \Gamma_A(\cdot - Z(t,x))\right) \leq e^{-\gamma t} \int \left(\int \tilde{n}(0,z,y) \phi_{0,z,y}(t,x)\,dy\right) C_\kappa\,dz$$
$$+ \frac{C_\kappa}{2} \int_0^t e^{-\gamma(t-s)} \int \left(\int \phi_{s,z,y}(t,x) \tilde{n}(s,z,y) \left(\int (w - y_{opt}(s,z))^2 \tilde{n}(s,z,w)\,dw\right) dy\right) dz\,ds$$
$$+ \gamma \int_0^{t-\varepsilon} e^{-\gamma(t-s)} \int \left(\int \phi_{s,z,y}(t,x) T(\tilde{n}(s,z,\cdot))(y)\,dy\right) C_\kappa\,dz\,ds$$
$$+ \gamma \int_{t-\varepsilon}^t e^{-\gamma(t-s)} \int \left(\int \phi_{s,z,y}(t,x) T(\tilde{n}(s,z,\cdot))(y)\,dy\right)$$
$$\left(W_2^2\Big(T(\tilde{n}(s,z,\cdot)), T(\Gamma_A(\cdot - Z(s,z)))\Big) + C_\kappa |t-s|^\theta + C_\kappa |x-z|^\theta\right) dz\,ds.$$

We can now use (57), an estimate derived in Section 4.3 of the Appendix to obtain

$$\begin{aligned}
&W_2^2\left(\tilde n(t,x,\cdot),\Gamma_A(\cdot - Z(t,x))\right) \le e^{-\gamma t}\left(\int \bar\phi_{0,z}(t,x)\,dz\right) C_\kappa \\
&\quad + \frac{1}{2}\int_0^t e^{-\gamma(t-s)}\left(\int \bar\phi_{s,z}(t,x)\,dz\right) C_\kappa\,ds + \gamma\int_0^{t-\varepsilon} e^{-\gamma(t-s)}\left(\int \bar\phi_{s,z}(t,x)\,dz\right) C_\kappa\,dz\,ds \\
&\quad + \gamma\int_{t-\varepsilon}^t e^{-\gamma(t-s)}\left(\int \bar\phi_{s,z}(t,x)\,dz\right)\max_{z\in\mathbb{T}^d} W_2^2\Big(T(\tilde n(s,z,\cdot)),T(\Gamma_A(\cdot - Z(s,z)))\Big)\,ds \\
&\quad + \gamma\int_{t-\varepsilon}^t e^{-\gamma(t-s)}\int \bar\phi_{s,z}(t,x)\left(C_\kappa|t-s|^\theta + C_\kappa|x-z|^\theta\right)\,dz\,ds,
\end{aligned}$$

Another estimate from Section 4.3 of the Appendix, (60) states $\int \bar\phi_{0,z}(t,x)\,dz = 1$, while (58) shows that $\int \bar\phi_{s,z}(t,x)|x-z|^\theta \le C_\kappa|t-s|^{\frac{\theta}{2}}$. Then,

$$\begin{aligned}
W_2^2\Big(\tilde n(t,x,\cdot),\Gamma_A(\cdot - Z(t,x))\Big) &\le C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \\
&\quad + \gamma\int_{t-\varepsilon}^t e^{-\gamma(t-s)}\max_{z\in\mathbb{T}^d} W_2^2\Big(T(\tilde n(s,z,\cdot)),T(\Gamma_A(\cdot - Z(s,z)))\Big)\,ds.
\end{aligned}$$

Since the right hand side of the estimate above is independent of $x\in\mathbb{T}^d$, we can consider the maximum over that variable. If moreover we apply the Tanaka inequality (see Theorem 4.1), we obtain

$$I(t) \le C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} + \frac{\gamma}{2}\int_{t-\varepsilon}^t e^{-\gamma(t-s)} I(s)\,ds,$$

where $I(s) := \max_{x\in\mathbb{T}^d} W_2^2\Big(\tilde n(s,x,\cdot),\Gamma_A(\cdot - Z(s,x))\Big)$. Thanks to a Grönwall inequality (see e.g. [19]),

$$\begin{aligned}
I(t) &\le C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \\
&\quad + \frac{\gamma}{2}e^{-\gamma t}\int_{t-\varepsilon}^t \left(C_\kappa e^{-\gamma s} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}}\right) e^{\gamma s} e^{\frac{\gamma}{2}(t-s)}\,ds \\
&\le C_\kappa e^{-\gamma t} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}} \\
&\quad + \left(C_\kappa e^{-\gamma(t-\varepsilon/2)} + \frac{C_\kappa}{\gamma} + C_\kappa e^{-\gamma\varepsilon} + \frac{C_\kappa}{\gamma^{\theta/2}}\right).
\end{aligned}$$

We can chose $\varepsilon := \frac{\theta\ln\gamma}{2\gamma}$ to obtain

$$I(t) \le C_\kappa e^{-\gamma(t-\varepsilon/2)} + \frac{C_\kappa}{\gamma^{\theta/2}},$$

so that finally, for any $\gamma > 0$ large enough,

$$\max_{t\in[\theta\ln\gamma/\gamma,\tau)} I(t) \le \frac{C_\kappa}{\gamma^{\theta/2}}.$$

The result follows (note that we need to define a slightly different parameter θ: $\tilde\theta := \frac{\theta}{2} > 0$). $\square$

3.5 Existence of global solutions for the SIM and proof of the main result

Proposition 3.6. *Let y_{opt}, n^0 satisfying Assumption 2.1, $A > 0$ and $\kappa > 0$. There exist $\bar{\gamma} > 0$ and $\theta \in (0,1)$ such that for any $\gamma > \bar{\gamma}$, there exists a solution $n \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d, L^1((1+|y|^4)\,dy))$ of the SIM with initial condition n^0 such that*

$$\|Z\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + 1,$$

where Z is defined by (1).

Proof of Proposition 3.6. We will prove this result through a recurrence argument: For $k \in \mathbb{N}$, we assume that a solution $n \in L^\infty([0,k\sigma]\times\mathbb{T}^d, L^1((1+|y|^4)\,dy))$ of the SIM exists and satisfies (13) for $\tau = k\sigma$ and $\gamma \geq \bar{\gamma}$. Our goal is to show that this solution can be extended into $n \in L^\infty([0,(k+1)\sigma]\times\mathbb{T}^d, L^1((1+|y|^4)\,dy))$ such that (13) folds for $\tau' = (k+1)\sigma$. This will hold for some $\sigma > 0$ chosen small enough and any γ large enough.

Case $k = 0$. Thanks to Proposition 3.3, for any $\gamma \geq \bar{\gamma}$, there exists a solution $n \in L^\infty([0,\sigma)\times\mathbb{T}^d, L^1((1+|y|^4)\,dy))$ of the SIM (we recall that in Proposition 3.3, the constant $\sigma > 0$ is independent from $\gamma > 0$, provided γ is large enough), and it safisfies (14) with $\tau = \sigma$. We can then apply Proposition 3.4, which ensures the Hölder regularity of Z (defined by (1)),

$$\|Z(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + Ct^\theta,$$

and then, provided we choose $\sigma > 0$ small enough, Z satisfies

$$\|Z\|_{L^\infty([0,(k+1)\sigma]\times\mathbb{T}^d)} < \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + 1. \tag{37}$$

In particular, n satisfies (13) for $\tau = \sigma$, which completes the initialisation step of this recurrence.

Case $k \geq 1$.

Since (13) holds for for $\tau = k\sigma$, we can apply Proposition 3.3 and there exists a solution $n \in L^\infty([0,(k+1)\sigma)\times\mathbb{T}^d, L^1((1+|y|^4)\,dy))$ of the SIM such that (14) is satisfied with $\tau = (k+1)\sigma$), that is

$$\|Z\|_{L^\infty([0,(k+1)\sigma]\times\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + 2. \tag{38}$$

From (9), we get for $(t,x) \in [0,(k+1)\sigma]\times\mathbb{T}^d$

$$\begin{aligned}
&\partial_t Z(t,x) - \Delta_x Z(t,x) - 2\frac{\nabla_x N(t,x)\cdot\nabla_x Z(t,x)}{N(t,x)} \\
&\quad= -\frac{1}{2}\int (y - Z(t,x))\,(y - y_{opt}(t,x))^2\Gamma_A(y - Z(t,x))\,dy \\
&\qquad+ \int (y - Z(t,x))\,(y - y_{opt}(t,x))^2\,(\Gamma_A(y - Z(t,x)) - \tilde{n}(t,x,y))\,dy.
\end{aligned} \tag{39}$$

The first term above can be simplified as follows

$$\begin{aligned}
&-\frac{1}{2}\int (y - Z(t,x))\,(y - y_{opt}(t,x))^2\Gamma_A(y - Z(t,x))\,dy \\
&\quad= -\left(Z(t,x) - y_{opt}(t,x)\right)\int |y|^2\Gamma_A(y)\,dy = -A\left(Z(t,x) - y_{opt}(t,x)\right),
\end{aligned} \tag{40}$$

and to estimate the last term of (39), we introduce for some $R>0$ and a Lipschitz function $\phi_R:\mathbb{R}\mapsto[0,1]$ such that $\phi_R|_{[-R,R]}=1$, $\phi_R|_{[-R-1,R+1]}=0$ and $\|\phi_R'\|_{L^\infty(\mathbb{R})}<2$. Then,

$$\begin{aligned}
&\left|\int (y-Z(t,x))\,(y-y_{opt}(t,x))^2\,(\Gamma_A(y-Z(t,x))-\tilde n(t,x,y))\,dy\right|\\
&\quad\le\left|\int \phi_R(y)\,(y-Z(t,x))\,(y-y_{opt}(t,x))^2\,(\Gamma_A(y-Z(t,x))-\tilde n(t,x,y))\,dy\right|\\
&\qquad+\left|\int (1-\phi_R(y))\,(y-Z(t,x))\,(y-y_{opt}(t,x))^2\,(\Gamma_A(y-Z(t,x))-\tilde n(t,x,y))\,dy\right|\\
&\quad\le\max_{y\in\mathbb{R}}\left|\frac{d}{dy}\left(\phi_R(y)\,(y-Z(t,x))\,(y-y_{opt}(t,x))^2\right)\right|W_1\left(\tilde n(t,x,\cdot),\Gamma_A(\cdot-Z(t,x))\right)\\
&\qquad+C_\kappa\int_{|y|\ge R}|y+\kappa|^3\tilde n(t,x,y)\,dy+C_\kappa\int_{|y|\ge R}|y+\kappa|^3\Gamma_A(y-Z(t,x))\,dy,
\end{aligned}$$

where $\kappa:=\|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)}+\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}+2$ is the bound on $\|Z\|_{L^\infty([0,(k+1)\sigma)\times\mathbb{T}^d)}$ provided by (38). Note that we have used the Kantorovich-Rubinstein estimate (see Section 4.1 in the Appendix) to obtain the first term on the right hand side of the estimate above. We use next the fact that ϕ_R is supported in $[-R-1,R+1]$ and the Chebyshev's inequality to obtain

$$\begin{aligned}
&\left|\int (y-Z(t,x))\,(y-y_{opt}(t,x))^2\,(\Gamma_A(y-Z(t,x))-\tilde n(t,x,y))\,dy\right|\\
&\quad\le C_\kappa(R+\kappa)^3W_2\left(\tilde n(t,x,\cdot),\Gamma_A(\cdot-Z(t,x))\right)+\frac{C_\kappa}{R}\int|y|^4\tilde n(t,x,y)\,dy\\
&\qquad+\frac{C_\kappa}{R}\int|y|^4\Gamma_A(y-Z(t,x))\,dy,
\end{aligned}$$

To estimate the three terms that appear in the estimate above, we use Proposition 3.5 (note that for $\gamma>0$ large enough, $C_\kappa\frac{\ln\gamma}{\gamma}<k\sigma$, so that estimate (28) holds for $t\in[k\sigma,(k+1)\sigma]$), the estimate on $\int|y|^4\tilde n(t,x,y)\,dy$ provided by Proposition 3.1 and the estimate (38) to obtain

$$\left|\int (y-Z(t,x))\,(y-y_{opt}(t,x))^2\,(\Gamma_A(y-Z(t,x))-\tilde n(t,x,y))\,dy\right|\le\frac{C_\kappa R^3}{\gamma^\theta}+\frac{C_\kappa}{R}\le\frac{C_\kappa}{\gamma^{\theta/4}},\tag{41}$$

for $(t,x)\in[\theta\ln\gamma/\gamma,\tau]\times\mathbb{T}^d$, provided we chose $R=\gamma^{\theta/4}$. Thanks to (40) and (41), we obtain that for $t\in[\theta\ln\gamma/\gamma,\tau],(k+1)\sigma)$ and $\gamma\ge\bar\gamma$ (this may require to increase the value of $\bar\gamma>0$, but this new value of $\bar\gamma$ remains independent of k),

$$\partial_tZ(t,x)-\Delta_xZ(t,x)=2\frac{\nabla_xN(t,x)\cdot\nabla_xZ(t,x)}{N(t,x)}-A(Z(t,x)-y_{opt}(t,x))+\mathcal{O}(1),$$

where $|\mathcal{O}(1)|\le A$. This estimate combined to (37) and the parabolic comparison principle imply that (13) is satisfied for $\tau=(k+1)\sigma$, which conclude the recurrence argument and the proof. □

We are now ready to prove Theorem 2.2:

Proof of Theorem 2.2. Thanks to Proposition 3.6, there exists a solution $n\in L^\infty(\mathbb{R}_+\times\mathbb{T}^d,L^1((1+|y|^4)\,dy))$ of the SIM with initial condition n^0 such that

$$\|Z\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}\le\kappa:=\|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)}+\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}+1.\tag{42}$$

Thanks to (6) and (9), we get the following expressions for the functions φ_N and φ_Z appearing in (4):

$$\varphi_N(t,x) = \left(-\frac{1}{2}\int (y - y_{opt}(t,x))^2 \tilde{n}(t,x,y)\,dy + \frac{A}{2} + \frac{1}{2}(Z(t,x) - y_{opt}(t,x))^2\right) N(t,x).$$

$$\varphi_Z(t,x) = -\frac{1}{2}\int (y - Z(t,x))(y - y_{opt}(t,x))^2 \tilde{n}(t,x,y)\,dy + A(Z(t,x) - y_{opt}(t,x)).$$

Thanks to (42), we can apply Proposition 3.1 with $[0,\tau) = [0,\infty)$, and there exists a constant $C > 0$ such that $\int |y|^4 \tilde{n}(t,x,y)\,dy \leq C$, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^d$. This combined to the boundedness of Z provided by (42) implies the existence of a constant $C > 0$ such that

$$\forall t \geq 0, \quad \|\varphi_N(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|\varphi_Z(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq C.$$

To show (3), we need to show that after an initial layer, this estimate can be improved. For φ_Z, we can use an estimate derived in the proof of Proposition3.6: (41) and (40) imply

$$\forall t \geq C\frac{\ln\gamma}{\gamma}, \quad \|\varphi_Z(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \frac{C}{\gamma^{\theta/4}}.$$

To estimate $\|\varphi_N(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}$, we note that

$$\int (y - y_{opt}(t,x))^2 \Gamma_A\left(y - Z(t,x)\right)\,dy = A + \left(Z(t,x) - y_{opt}(t,x)\right)^2,$$

and then

$$\varphi_N(t,x) = \frac{N(t,x)}{2}\int \left(y - y_{opt}(t,x)\right)^2 \left(\Gamma_A\left(y - Z(t,x)\right) - \tilde{n}(t,x,y)\right)\,dy.$$

We can repeat the argument developed in (40)-(41) to estimate the second term. Then,

$$\forall t \geq C\frac{\ln\gamma}{\gamma}, \quad \|\varphi_N(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \frac{C}{\gamma^{\theta/4}}.$$

To conclude the proof, we notice that (2) is a consequence of Proposition 3.4, and (5) is a consequence of Proposition 3.5. To obtain estimate (3), we define a slightly different parameter θ: $\tilde{\theta} := \frac{\theta}{4} > 0$. Finally, (N,Z) is a solution of (4) in the sense of distributions thanks to Remark 4.3. $\square$

4 Appendix

4.1 Wasserstein distances

In this section, we review the definition of the Wasserstein distance and several useful formula. We refer to [43] for more on this topic. Let $p \geq 1$, and $\mathcal{P}_p(\mathbb{R})$ the set of probability measures with finite $p-$moment, that is the set of probability measures μ over $\mathbb{R}$ such that

$$\int |y|^d\,d\mu(y) < \infty. \tag{43}$$

If π is a probability measure over $\mathbb{R}^2$, we call marginals the probability measures $\pi|_1$ and $\pi|_2$ such that for any Borelian $A \subset \mathbb{R}$,

$$\pi(A \times \mathbb{R}) = \pi|_1(A), \quad \pi(\mathbb{R} \times A) = \pi|_2(A).$$

For $\tilde{n}, \tilde{m} \in \mathcal{P}_2(\mathbb{R})$, we call transference plans the probability measures π over $\mathbb{R}^2$ such that $\pi|_1 = \tilde{n}$ and $\pi|_2 = \tilde{m}$, and $\Pi(\tilde{n}, \tilde{m})$ the set of such plans:

$$\Pi(\tilde{n}, \tilde{m}) := \left\{\pi \in \mathcal{P}(\mathbb{R}^2);\, \pi|_1 = \tilde{n},\, \pi|_2 = \tilde{m}\right\}. \tag{44}$$

We can now define the $p-$ Wasserstein distance between two measures $\tilde{n}, \tilde{m} \in \mathcal{P}_p(\mathbb{R})$ as follows

$$W_p(\tilde{n}, \tilde{m}) = \left(\inf_{\pi \in \Pi(\tilde{n}, \tilde{m})} \int |y_1 - y_2|^p \, d\pi(y_1, y_2)\right)^{\frac{1}{p}}.$$

Note that $W_p(\tilde{n}, \delta_{\bar{y}}) = \int |y - \bar{y}|^p \, d\tilde{n}(y)$, for any $\bar{y} \in \mathbb{R}$ and $\tilde{n} \in \mathcal{P}_p(\mathbb{R})$.

For $\tilde{n}, \tilde{m} \in \mathcal{P}_2(\mathbb{R})$ and $f \in W^{1,\infty}(\mathbb{R})$, the Kantorovich-Rubinstein is the following useful estimate:

$$\left| \int f(y) d\tilde{n}(y) - \int f(y) d\tilde{m}(y) \right| \le \|f'\|_{L^\infty(\mathbb{R})} W_1(\tilde{n}, \tilde{m}).$$

For $\tilde{n}, \tilde{m} \in \mathcal{P}_p(\mathbb{R})$ (with $p \ge 1$), the Kantorovich duality provides the following equality

$$W_p(\tilde{n}, \tilde{m}) = \left(\sup_{(\varphi, \psi) \in F} \int \varphi(y) \, d\tilde{n}(y) + \int \psi(Y) \, d\tilde{m}(Y)\right)^{\frac{1}{p}}, \tag{45}$$

where $F = \left\{(\varphi, \psi) \in (C_b^0(\mathbb{R}, \mathbb{R}))^2;\, \forall y, Y \in \mathbb{R}, \varphi(y) + \psi(Y) \le |y - Y|^p\right\}$.

Finally, we will also use the convexity of the squared Wasserstein distance W_2. Let $\tilde{n}_1, \tilde{m} \in \mathcal{P}_2(\mathbb{R}) \cap L^1(\mathbb{R})$ and, $\tilde{n}_2 \in L^\infty([0,t] \times \mathbb{T}^d, \mathcal{P}_2\left(\mathbb{R}\right) \cap L^1(\mathbb{R}))$, for some $t > 0$. For any $\alpha \in [0,1]$ and $\beta \in L^1([0,t] \times \mathbb{T}^d)$ such that $\int_{[0,t]\times\mathbb{T}^d} \beta = 1 - \alpha$, we have

$$\begin{aligned} &W_2^2\left(\alpha \tilde{n}_1 + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) \tilde{n}_2(t, x, \cdot) \, dx \, d\sigma, \tilde{m}\right) \\ &\qquad \le \alpha W_2^2(\tilde{n}_1, \tilde{m}) + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) W_2^2\left(\tilde{n}_2(\sigma, x, \cdot), \tilde{m}\right) \, dx \, d\sigma. \end{aligned} \tag{46}$$

To obtain this estimate, let $(\varphi, \psi) \in F$ with $p = 2$. Then,

$$\begin{aligned} &\int \varphi(y) \left(\alpha \tilde{n}_1(y) + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) \tilde{n}_2(\sigma, x, y) \, dx \, d\sigma\right) \, dy + \int \psi(Y) \tilde{m}(Y) \, dY \\ &\le \alpha \left(\int \varphi(y) \tilde{n}_1(y) \, dy + \psi(Y) \tilde{m}(Y) \, dY\right) \\ &\quad + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) \left(\int \varphi(y) \tilde{n}_2(\sigma, x, y) \, dy + \int \psi(Y) \tilde{m}(Y) \, dY\right) \, dx \, d\sigma \\ &\le \alpha W_2^2\left(\tilde{n}_1, \tilde{m}\right) + \int_0^t \int_{\mathbb{T}^d} \beta(\sigma, x) W_2^2\left(\tilde{n}_2(\sigma, x, \cdot), \tilde{m}\right) \, dx \, d\sigma, \end{aligned}$$

and (46) follows thanks to (45), if we consider the suppremum over $(\varphi, \psi) \in F$.

4.2 The Infinitesimal operator

We define the Infinitesimal operator T on the space $\mathcal{P}_2(\mathbb{R})$ (see Section 4.1):

$$T(\tilde{n})(y) := \int \Gamma_{A/2}\left(y - \frac{y_* + y_*'}{2}\right) \tilde{n}(t, y_*)\tilde{n}(t, y_*')\, dy_*\, dy_*'. \tag{47}$$

Then, for any $\tilde{n} \in \mathcal{P}_2(\mathbb{R})$,

$$\int T(\tilde{n})(y)\, dy = \int \tilde{n}(y)\, dy = 1, \qquad \int y\, T(\tilde{n})(y)\, dy = \int y\, \tilde{n}(y)\, dy,$$

and for any $Z \in \mathbb{R}$,

$$\forall y \in \mathbb{R}, \quad T\left(\Gamma_A(\cdot - Z)\right)(y) = \Gamma_A(y - Z). \tag{48}$$

where

$$\Gamma_A(y) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{|y|^2}{2A}}. \tag{49}$$

T induces a contraction for the Wasserstein distance W_2, which can be seen as a version of the Tanaka inequality [38] (see also [6, 9]):

Theorem 4.1 (A Tanaka inequality)**.** *Let $A > 0$, $\tilde{n}$, $\tilde{m} \in \mathcal{P}_2(\mathbb{R})$ such that $\int y\tilde{n}(y)\, dy = \int y\tilde{m}(y)\, dy$, and T defined by* (47)*. Then*

$$W_2(T(\tilde{n}), T(\tilde{m})) \leq \frac{1}{\sqrt{2}} W_2(\tilde{n}, \tilde{m}).$$

Proof of the Theorem 4.1. We consider φ, ψ such that for any $y, Y \in \mathbb{R}$, $\varphi(y) + \psi(Y) \leq |y - Y|^2$, and $\pi \in \Pi(\tilde{n}, \tilde{m})$. Then,

$$\begin{aligned}
&\int \varphi(y)T(\tilde{n})(y)\, dy + \int \psi(Y)T(\tilde{m})(Y)\, dY \\
&= \int\int\int \varphi(y)\Gamma_{A/2}\left(y - \frac{y_* + y_*'}{2}\right) \tilde{n}(y_*)\tilde{n}(y_*')\, dy_*\, dy_*'\, dy \\
&\quad + \int\int\int \psi(Y)\Gamma_{A/2}\left(Y - \frac{Y_* + Y_*'}{2}\right) \tilde{n}(Y_*)\tilde{n}(Y_*')\, dY_*\, dY_*'\, dY \\
&= \int\int\int \varphi\left(y + \frac{y_* + y_*'}{2}\right) \Gamma_{A/2}(y)\, \tilde{n}(y_*)\tilde{n}(y_*')\, dy_*\, dy_*'\, dy \\
&\quad + \int\int\int \psi\left(y + \frac{Y_* + Y_*'}{2}\right) \Gamma_{A/2}(Y)\, \tilde{n}(Y_*)\tilde{n}(Y_*')\, dY_*\, dY_*'\, dY \\
&= \int \Gamma_{A/2}(y) \int\int\int\int \varphi\left(y + \frac{y_* + y_*'}{2}\right) + \psi\left(y + \frac{Y_* + Y_*'}{2}\right) d\pi(y_*, Y_*)\, d\pi(y_*', Y_*')\, dy \\
&\leq \int \Gamma_{A/2}(y) \int\int\int\int \left|\left(y + \frac{y_* + y_*'}{2}\right) - \left(y + \frac{Y_* + Y_*'}{2}\right)\right|^2 d\pi(y_*, Y_*)\, d\pi(y_*', Y_*')\, dy \\
&\leq \frac{1}{4} \int\int\int\int \left|(y_* - Y_*) + (y_*' - Y_*')\right|^2 d\pi(y_*, Y_*)\, d\pi(y_*', Y_*').
\end{aligned} \tag{50}$$

We notice that

$$\int\int\int\int (y_* - Y_*)(y_*' - Y_*')\, d\pi(y_*, Y_*)\, d\pi(y_*', Y_*') = \left(\int y\tilde{n}(y)\, dy - \int y\tilde{m}(Y)\, dY\right)^2 = 0,$$

and then

$$\begin{aligned}
&\int \varphi(y) T(\tilde{n})(y)\, dy + \int \psi(Y) T(\tilde{m})(Y)\, dY \\
&\quad \leq \frac{1}{4} \int\int\int\int \left[(y_* - Y_*)^2 + 2(y_* - Y_*)(y'_* - Y'_*) + (y'_* - Y'_*)^2 \right] \, d\pi(y_*, Y_*)\, d\pi(y'_*, Y'_*) \\
&\quad = \frac{1}{2} \int\int (y - Y)^2 \, d\pi(y, Y).
\end{aligned}$$

Since this inequality holds for any $\pi \in \Pi(\tilde{n}, \tilde{m})$, we can consider the infinum of over these, to obtain, thanks to the definition of the Wasserstein distance:

$$\int \varphi(y) T(\tilde{n})(y)\, dy + \int \psi(Y) T(\tilde{m})(Y)\, dY \leq \frac{1}{2} W_2^2(\tilde{n}, \tilde{m}).$$

We can now take the supremum of this inequality over the functions φ, ψ satisfying $\varphi(y) + \psi(Y) \leq |y - Y|^2$ and conclude, thanks to the Kantorovich duality formula (45). □

Corollary 4.2 (A Tanaka inequality for W_4)**.** *Let $A > 0$, $\tilde{n}$, $\tilde{m} \in \mathcal{P}_4(\mathbb{R})$ such that $\int y\tilde{n}(y)\, dy = \int y\tilde{m}(y)\, dy$, and T defined by* (47)*. Then*

$$W_4(T(\tilde{n}), T(\tilde{m})) \leq \frac{1}{2^{1/4}} W_4(\tilde{n}, \tilde{m}).$$

Proof of the Corollary 4.2. We can reproduce the proof of Theorem 4.1 until (50), and obtain that for any φ, ψ satisfying $\varphi(y) + \psi(Y) \leq |y - Y|^4$ and $\pi \in \Pi(\tilde{n}, \tilde{m})$,

$$\begin{aligned}
&\int \varphi(y) T(\tilde{n})(y)\, dy + \int \psi(Y) T(\tilde{m})(Y)\, dY \\
&\quad \leq \frac{1}{16} \int\int\int\int \left| (y_* - Y_*) + (y'_* - Y'_*) \right|^4 \, d\pi(y_*, Y_*)\, d\pi(y'_*, Y'_*) \\
&\quad = \frac{1}{16} \int\int\int\int \Big[(y_* - Y_*)^4 + 4(y_* - Y_*)^3 (y'_* - Y'_*) + 6(y_* - Y_*)^2 (y'_* - Y'_*)^2 \\
&\qquad\qquad + 4(y_* - Y_*)(y'_* - Y'_*)^3 + (y'_* - Y'_*)^4 \Big] \, d\pi(y_*, Y_*)\, d\pi(y'_*, Y'_*) \\
&\quad = \frac{1}{8} \left(\int\int (y - Y)^4 \, d\pi(y, Y) \right) + \frac{3}{8} \left(\int (y - Y)^2 \, d\pi(y, Y) \right)^2 \\
&\quad \leq \frac{1}{2} \left(\int\int (y - Y)^4 \, d\pi(y, Y) \right).
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 4.1. □

4.3 Technical estimates for some linear problems

In this section, we derive estimates on solutions of linear parabolic problems that are used in Section 3.4 (proof of Proposition 3.5). We consider the assumption made in Proposition 3.5, and in particular: y_{opt}, n^0 satisfying Assumption 2.1, $\tau > 0$, $n \in L^\infty([0, \tau) \times \mathbb{T}^d, L^1((1 + |y|^4)\, dy))$ a solution of the SIM with initial condition n^0, and $\tilde{n}$, N, Z defined by (7) and (1), and we assume that $\|Z\|_{L^\infty([0,\tau) \times \mathbb{T}^d)} \leq \kappa$, for some $\kappa > 0$.

Some linear parabolic equations

For $(s,z,y)\in[0,\tau)\times\mathbb{T}^d\times\mathbb{R}$, let $\phi_{s,z,y}(t,x)$ the solution of

$$\begin{cases} \partial_t\phi_{s,z,y}(t,x)-\Delta_x\phi_{s,z,y}(t,x) \\ \qquad = 2\frac{\nabla_x N(t,x)}{N(t,x)}\cdot\nabla_x\phi_{s,z,y}(t,x)-\frac{1}{2}(y-y_{opt}(t,x))^2\phi_{s,z,y}(t,x),\ (t,x)\in[s,\tau)\times\mathbb{T}^d, \\ \phi_{s,z,y}(s,x)=\delta_z(x),\ x\in\mathbb{T}^d. \end{cases} \tag{51}$$

Let $(t,x)\mapsto\psi_{s,z,y}(t,x):=\phi_{s,z,y}(t,x)N(t,x)$, which satisfies

$$\begin{cases} \partial_t\psi_{s,z,y}(t,x)-\Delta_x\psi_{s,z,y}(t,x) \\ \qquad = \left(1+\frac{A}{2}-N(t,x)-\frac{1}{2}(y-y_{opt}(t,x))^2-\frac{1}{2}\int(y-y_{opt}(t,x))^2\tilde{n}(t,x,y)\,dy\right)\psi_{s,z,y}(t,x), \\ \qquad (t,x)\in[s,\tau)\times\mathbb{T}^d, \\ \psi_{s,z,y}(s,x)=N(s,z)\delta_z(x),\ x\in\mathbb{T}^d. \end{cases}$$

Since the factor on the right hand side of the equation satisfied by $\psi_{s,z,y}$ is bounded (see Proposition 3.1), the existence and uniqueness of $\psi_{s,z,y}$ derives from standards methods (see e.g. Theorem 7.3 and Theorem 7.4 in [20]), and this implies the existence and uniqueness of the solution $\phi_{s,z,y}$ of (51).

Let now $\bar\phi_{s,z}(t,x)$, the solution of

$$\begin{cases} \partial_t\bar\phi_{s,z}(t,x)-\Delta_x\bar\phi_{s,z}(t,x)=2\frac{\nabla_x N(t,x)}{N(t,x)}\cdot\nabla_x\bar\phi_{s,z}(t,x),\ (t,x)\in[s,\tau)\times\mathbb{T}^d, \\ \bar\phi_{s,z}(s,x)=\delta_z(x),\ x\in\mathbb{T}^d. \end{cases} \tag{52}$$

Just as for (51), the existence and uniqueness of $\bar\phi_{s,z}$ can be obtained through $\bar\psi_{s,z}(t,x):=\bar\phi_{s,z}N(t,x)$, which satisfies

$$\begin{cases} \partial_t\bar\psi_{s,z}(t,x)-\Delta_x\bar\psi_{s,z}(t,x) \\ \qquad = \left(1+\frac{A}{2}-N(t,x)-\frac{1}{2}\int(y-y_{opt}(t,x))^2\tilde{n}(t,x,y)\,dy\right)\bar\psi_{s,z}(t,x),\ (t,x)\in[s,\tau)\times\mathbb{T}^d, \\ \bar\psi_{s,z}(s,x)=N(s,z)\delta_z(x),\ x\in\mathbb{T}^d. \end{cases}$$

Estimate 1

Thanks to Proposition 3.1, there exists $C_\kappa>0$ such that $\int|y|^4 n(t,x,y)\,dy\le C_\kappa$ for any $(t,x)\in[0,\tau)\times\mathbb{T}^d$, and we can define

$$R=(2C_\kappa)^{1/4}. \tag{53}$$

Then, for any $(t,x)\in[0,\tau)\times\mathbb{T}^d$,

$$\int_{-R}^{R}\tilde{n}(t,x,y)\,dy=1-\int_{[-R,R]^c}\tilde{n}(t,x,y)\,dy\ge 1-\frac{1}{R^4}\int_{[-R,R]^c}|y|^4\tilde{n}(t,x,y)\,dy\ge\frac{1}{2}. \tag{54}$$

Let also

$$R'=R+\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}. \tag{55}$$

Then, for any $y\in[-R',R']^c$, we have $-\frac{1}{2}(y-y_{opt}(t,x))^2\le\min_{[-R,R]}\left(-\frac{1}{2}(y-y_{opt}(t,x))\right)^2$, and the parabolic comparison principle applied to (51) (comparing the case $y\in[-R',R']^c$ to the case where $\tilde y\in[-R,R]$) implies that for any $y\in[-R',R']^c$,

$$\forall(s,z)\in[0,\tau)\times\mathbb{T}^d,\ \forall(t,x)\in[s,\tau)\times\mathbb{T}^d,\quad \phi_{s,z,y}(t,x)\le\min_{\tilde y\in[-R,R]}\phi_{s,z,\tilde y}(t,x). \tag{56}$$

Estimate 2

For any $y \in \mathbb{R}$,

$$-\frac{1}{2}\left(y+\operatorname{sgn}(y)\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}\right)^2 \le -\frac{1}{2}(y-y_{opt}(t,x))^2 \le -\frac{1}{2}\left(y-\operatorname{sgn}(y)\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}\right)^2$$

Then $\phi_{s,z,y}(t,x)e^{(t-s)\frac{1}{2}\left(y+\operatorname{sgn}(y)\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}\right)^2}$ is a super-solution of (52), and thanks to the comparison principle, $\bar{\phi}_{s,z}(t,x) \le \phi_{s,z,y}(t,x)e^{(t-s)\frac{1}{2}\left(y+\operatorname{sgn}(y)\|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}\right)^2}$. The reverse estimate can be obtained similarily, and together, those estimates imply for any $(s,z,y) \in [0,\tau)\times\mathbb{T}^d\times\mathbb{R}$ and $(t,x)\in(s,\min(s+1,\tau))\times\mathbb{T}^d$,

$$\phi_{s,z,y}(t,x) = \bar{\phi}_{s,z}(t,x)e^{-(t-s)\frac{1}{2}(y+\mathcal{O}(1))^2}, \tag{57}$$

where $|\mathcal{O}(1)| \le \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}$.

Estimate 3

$\bar{\psi}_{s,z}$ satisfies $\bar{\psi}_{s,z}(s,\cdot) = N(s,z)\delta_z$ and

$$\partial_t\bar{\psi}_{s,z}(t,x) - \Delta_x\bar{\psi}_{s,z}(t,x) \le \left(1+\frac{A}{2}\right)\bar{\psi}_{s,z}(t,x).$$

Thanks to the comparison principle, $\bar{\psi}_{s,z}(t,x) \le N(s,z)e^{\left(1+\frac{A}{2}\right)(t-s)}\Gamma_{t-s}(x-z)$, and since $\bar{\psi}_{s,z}(t,x) = \bar{\phi}_{s,z}(t,x)N(t,x)$, we have

$$\int \bar{\phi}_{s,z}(t,x)|z-x|^\theta\,dz \le e^{\left(1+\frac{A}{2}\right)(t-s)}\int \Gamma_{t-s}(x-z)\frac{N(s,z)}{N(t,x)}|z-x|^\theta\,dz.$$

We can use the estimate (22) to show that $\left|\frac{N(s,z)}{N(t,x)}\right| \le C_\kappa$, as soon as $1 < s \le t \le \min(s+1,\tau)$. If $0 \le s \le t \le 2$, we can use the lower bound (21) and the upper bound $\|N\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} \le \max\left(1, \|N(0,\cdot)\|_{L^\infty(\mathbb{T}^d)}\right)$ to obtain a similar estimate. Then,

$$\int \bar{\phi}_{s,z}(t,x)|z-x|^\theta\,dz \le C_\kappa e^{\left(1+\frac{A}{2}\right)(t-s)}\int \Gamma_{t-s}(x-z)|z-x|^\theta\,dz \le C_\kappa(t-s)^{\frac{\theta}{2}}, \tag{58}$$

provided $0 < s \le t \le \min(s+1,\tau)$.

Estimate 4

For $(\bar{t},\bar{x}) \in (0,+\infty)\times\mathbb{T}^d$, let $u_{\bar{t},\bar{x}}$ the solution of the following problem (note that the *time variable* is here reversed compared to usual problems)

$$\begin{cases} -\frac{\partial u_{\bar{t},\bar{x}}}{\partial t}(t,x) - \Delta_x u_{\bar{t},\bar{x}}(t,x) = -2\nabla_x\cdot\left(\frac{\nabla_x N(t,x)}{N(t,x)}u_{\bar{t},\bar{x}}(t,x)\right), & (t,x)\in(-\infty,\bar{t}]\times\mathbb{T}^d \\ u_{\bar{t},\bar{x}}(\bar{t},x) = \delta_{\bar{x}}(x). \end{cases} \tag{59}$$

This problem is indeed the dual problem of (52) in the sense that $\frac{d}{dt}\int \bar{\phi}_{s,z}(t,x)u_{\bar{t},\bar{x}}(t,x)\,dx = 0$ for $t\in[s,\bar{t}]$. It follows that for any $s<\bar{t}$ and $z\in\mathbb{T}^d$,

$$\int \bar{\phi}_{s,z}(s,x)u_{\bar{t},\bar{x}}(s,x)\,dx = \int \bar{\phi}_{s,z}(t,x)u_{\bar{t},\bar{x}}(\bar{t},x)\,dx,$$

which, given the initial conditions specified in (52) and (59) (note that the reversion of time in this dual problem implies that the initial condition holds for the largest time considered, ie $t = \bar{t}$), is equivalent to

$$u_{\bar{t},\bar{x}}(s,z) = \bar{\phi}_{s,z}(\bar{t},\bar{x}).$$

The divergence form of (59) implies that $\int u_{\bar{t},\bar{x}}(s,z)\,dz = \int u_{\bar{t},\bar{x}}(\bar{t},z)\,dz = 1$, and then for any $\bar{t} > s$ and $\bar{x} \in \mathbb{T}^d$,

$$\int \bar{\phi}_{s,z}(\bar{t},\bar{x})\,dz = 1. \tag{60}$$

4.4 Uniqueness and stability of solutions of the KBM

In this section, we show that the estimate (3) implies the convergence of (N,Z) to the solution of the KBM when $\gamma > 0$ is large. The arguments we present also imply the uniqueness of solutions of the KBM.

Let $N, Z \in C^\theta([0,\tau) \times \mathbb{T}^d)$ satisfying (4), with $y_{opt} \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d)$ and φ_N, φ_Z satisfying (3). The maximum principle shows that

$$\|Z(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \|Z(0,\cdot)\|_{L^\infty(\mathbb{T}^d)} + \|y_{opt}\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)} + C\tau.$$

This estimate implies that

$$\min_{[0,\tau)\times\mathbb{T}^d} N \geq \inf_{\mathbb{T}^d} N(0,\cdot)e^{-C\tau} > 0.$$

We can now define $Y := NZ$, and note that $\|N\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} + \|Y\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq C$. Moreover, (N,Y) satisfies a close system of equations where only the 0^{th} order terms are non-linear:

$$\begin{cases} \partial_t N(t,x) - \Delta_x N(t,x) = \left(1 - \frac{1}{2N(t,x)^2}\left(Y - y_{opt}(t,x)N(t,x)\right)^2 - N(t,x) + \varphi_N(t,x)\right) N(t,x), \\ \partial_t Y(t,x) - \Delta_x Y(t,x) = \left(1 - \frac{1}{2N(t,x)^2}\left(Y - y_{opt}(t,x)N(t,x)\right)^2 - N(t,x) + \varphi_N(t,x)\right) Y(t,x)r \\ \qquad + \left(-\frac{A}{N(t,x)}\left(Y(t,x) - y_{opt}(t,x)N(t,x)\right) + \varphi_Z(t,x)\right) N(t,x). \end{cases} \tag{61}$$

Let $\bar{N}, \bar{Z} \in C^\theta([0,\tau) \times \mathbb{T}^d)$ a solution of (4) with $\varphi_N \equiv \varphi_Z \equiv 0$ (that is $(\bar{N},\bar{Z})$ solution of the KBM), and initial data $(\bar{N},\bar{Z})(0,\cdot) = (N,Z)(0,\cdot)$. We can define $\bar{Y} = \bar{N}\bar{Z}$, and the argument above show that $\|\bar{N}\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} + \|\bar{Y}\|_{L^\infty([0,\tau)\times\mathbb{T}^d)} \leq C$ (where the constant C depends only on the initial condition, A, and y_{opt}), and

$$\min_{[0,\tau)\times\mathbb{T}^d} \bar{N} \geq \inf_{\mathbb{T}^d} N(0,\cdot)e^{-C\tau} > 0.$$

We can then estimate

$$\partial_t(N-\bar{N})(t,x) - \Delta_x(N-\bar{N})(t,x) = \mathcal{O}(1)(N-\bar{N})(t,x) + \mathcal{O}(1)(Y-\bar{Y})(t,x) + \mathcal{O}(1)\varphi_N(t,x),$$

$$\begin{aligned}\partial_t(Y-\bar{Y})(t,x) - \Delta_x(Y-\bar{Y})(t,x) = \mathcal{O}(1)(N-\bar{N})(t,x) + \mathcal{O}(1)(Y-\bar{Y})(t,x) + \mathcal{O}(1)\varphi_N(t,x) \\ + \mathcal{O}(1)\varphi_Z(t,x).\end{aligned}$$

The parabolic maximum principle with functions independent of x then implies

$$\begin{aligned}&\frac{d}{dt}\left(\|(N-\bar N)(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}+\|(Y-\bar Y)(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}\right)\\&\qquad\leq C\left(\|(N-\bar N)(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}+\|(Y-\bar Y)(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}\right)+\frac{C}{\gamma^\theta}+C\mathbb{1}_{[0,1/\gamma^\theta]}(t),\end{aligned}$$

and thus

$$\|(N-\bar N)(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}+\|(Y-\bar Y)(t,\cdot)\|_{L^\infty(\mathbb{T}^d)}\leq \frac{C}{\gamma^\theta}e^{Ct}.$$

This estimate shows the uniqueness of solutions of the KBM (provided the initial condition satisfies Assumption 2.1). It also shows the convergence of solutions (N,Z) of (4) to the solution $(\bar N,\bar Z)$ of the KBM when $\gamma\to\infty$, in the sense that

$$N\underset{\gamma\to\infty}{\longrightarrow}\bar N\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+,L^\infty(\mathbb{T}^d)),$$

$$Z\underset{\gamma\to\infty}{\longrightarrow}\bar Z\quad\text{in}\quad L^\infty_{loc}(\mathbb{R}_+,L^\infty(\mathbb{T}^d)).$$

Remark 4.3. *Note that L^2 estimates on* (61) *imply that N, $\nabla_x N$, Y, $\nabla_x Y\in L^2_{loc}(\mathbb{R}_+\times\mathbb{T}^d)$. Since additionally $N(t,x)\geq Ce^{-Ct}$ and $N,Z\in L^\infty(\mathbb{R}_+\times\mathbb{T}^d)$ (see above), we have*

$$\nabla_x Y=\frac{\nabla_x Y N-Y\nabla_x N}{N^2}\in L^2_{loc}(\mathbb{R}_+\times\mathbb{T}^d).$$

Thanks to those estimates, $\frac{\nabla_x N\cdot\nabla_x Z}{N}\in L^1_{loc}(\mathbb{R}_+\times\mathbb{T}^d)$. Then, the KBM and (4) *are well defined in the sense of distributions.*

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