

Optimal make-take fees for market making regulation

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May 8, 2018

Abstract

We consider an exchange who wishes to set suitable make-take fees to attract liquidity on its platform. Using a principal-agent approach, we are able to describe in quasi-explicit form the optimal contract to propose to a market maker. This contract depends essentially on the market maker inventory trajectory and on the volatility of the asset. We also provide the optimal quotes that should be displayed by the market maker. The simplicity of our formulas allows us to analyze in details the effects of optimal contracting with an exchange, compared to a situation without contract. We show in particular that it leads to higher quality liquidity and lower trading costs for investors.

Keywords: Make-take fees, market making, financial regulation, high-frequency trading, principal-agent problem, stochastic control.

1 Introduction

With the fragmentation of financial markets, exchanges are nowadays in competition. Indeed the traditional international exchanges are now challenged by alternative trading venues, see [16]. Consequently, they have to find innovative ways to attract liquidity on their platforms. One solution is to use a make-taker fees system, that is a rule enabling them to charge in an asymmetric way liquidity provision and liquidity consumption. The most classical setting, used by many exchanges (such as Nasdaq, Euronext, BATS Chi-X...), is of course to subsidize the former while taxing the latter. In practice, this means associating a fee rebate to executed limit orders and applying a transaction cost for market orders.

In the recent years, the topic of make-take fees has been quite controversial. Indeed make-take fees policies are seen as a major facilitating factor to the emergence of a new type of market makers aiming at collecting fee rebates: the high frequency traders. As stated by the Securities and Exchanges commission in [26]: “Highly automated exchange systems and liquidity rebates have helped establish a business model for a new type of professional liquidity provider that is distinct from the more traditional exchange specialist and over-the-counter market maker.” The concern with high frequency traders becoming the new liquidity providers is two-fold. First, their presence implies that slower traders no longer have access to the limit order book, or only

*This work benefits from the financial support of the Chaires *Financial Risk* and *Finance and Sustainable Development*. Omar El Euch and Mathieu Rosenbaum gratefully acknowledge the financial support of the *ERC Grant 679836 Staqamof* and the *Chaire Analytics and Models for Regulation*. Nizar Touzi gratefully acknowledge the financial support *ERC Advanced Grant 321111 RoFiRM*.

in unfavorable situations when high frequency traders do not wish to support liquidity. This leads to the second classical criticism against high frequency market makers: they tend to leave the market in time of stress, see [3, 20, 21, 24] for detailed investigations about high frequency market making activity.

From an academic viewpoint, studies of make-take fees structures and their impact on the welfare of the markets have been mostly empirical, or carried out in rather stylized models. An interesting theory, suggested in [1] and developed in [5] is that make-take fees have actually no impact on trading costs in the sense that the *cum fee* bid-ask spread should not depend on the make-take fees policy. This result is consistent with the empirical findings in [17, 19]. Nevertheless, it is clearly shown in these works that many important trading parameters such as depths, volumes or price impact do depend on the make-take fees structure, see also [12]. Furthermore, the idea of the neutrality of the make-take fees schedule is also tempered in [10] where the authors show theoretically that make-take fees may increase welfare of markets provided the tick size is not equal to zero, see also [4].

In this work, our aim is to provide a quantitative and operational answer to the question of relevant make-take fees. To do so, we take the position of an exchange (or of the regulator) wishing to attract liquidity. The exchange is looking for the best make-take fees policy to offer to market makers in order to maximize its utility. In other words, it aims at designing an optimal contract with the market maker to create an incentive to increase liquidity. For simplicity, we consider a single market maker in a non-fragmented market.

Incentive theory has emerged in the 1970s in economics to model how an financial agent can delegate the management of an output process to another agent. Let us recall the formalism of principal-agent problems from the seminal works of Mirrlees [22] and Holmström [13]. A principal aims at contracting with an agent who provides efforts to manage an output process impacting the wealth of the principal. The principal is not able to control directly the output process since he cannot decide the efforts made by the agent. In our case, the principal is the exchange, the agent is the market maker, the efforts correspond to the quality of the liquidity provided by the market maker (essentially the size of the bid-ask spread proposed by the market maker) and the output process is the transactions flow on the platform. Several economics papers have investigated this kind of problems by identifying it with a Stackelberg equilibrium between the two parties. More precisely, since the principal cannot control the work of the agent, he anticipates his best-reaction effort for a given compensation. Knowing that, the principal aims at finding the best contract.

In our work, we deal with a continuous-time principal-agent problem. Indeed, the exchange monitors the spread set by the market maker around a Brownian-type efficient price and the transactions flow in continuous-time. Our paper follows the stream of literature initiated in [14]. Then in [25], the author recasts such issue into a stochastic control problem which has been further developed using backward stochastic differential equation theory in [7]. See also [8] for related literature.

In this paper, although we work in a quite general and realistic setting, we are able to solve our principal-agent problem. More precisely, we provide a quasi-explicit expression for the optimal contract the exchange should propose to the market maker, and also for the quotes the market

maker should set. The optimal contract depends essentially on the market maker inventory trajectory and on the volatility of the market. These simple formulas enable us to analyze in details the effects for the welfare of the market of optimal contracting with an exchange, compared to a situation without contract as in [2, 11]. We notably show that using such contracts leads to reduced spreads and lower trading costs for investors.

The paper is organized as follows. Our modeling approach is presented in Section 2. In particular, we define the market maker's as well as the exchange's optimization frameworks. In Section 3, we compute the optimal reaction of the market maker for a given contract. Optimal contracts are designed in Section 4 where we solve the exchange's problem. Then, in Section 5, we assess the benefits for market quality of the presence of an exchange contracting optimally with a market maker. Finally, useful technical results are gathered in an appendix.

2 The model

The framework considered throughout this paper is inspired by the seminal work [2] where the authors consider the problem of optimal market making, but without the intervention of an exchange. Let $T > 0$ be a final horizon time and (Ω, \mathcal{F}) be a measurable space such that $\Omega = \Omega_c \times (\Omega_d)^2$ with Ω_c the set of continuous functions from $[0, T]$ into \mathbb{R} , Ω_d the set of piecewise constant càdlàg functions from $[0, T]$ into \mathbb{N} and \mathcal{F} the Borel algebra on Ω . We consider the following canonical process $(\chi_t)_{t \in [0, T]} = (S_t, N_t^a, N_t^b)_{t \in [0, T]}$

$$\forall \omega = (s, n^a, n^b) \in \Omega \quad S_t(\omega) = s(t), \quad N_t^a(\omega) = n^a(t), \quad N_t^b(\omega) = n^b(t).$$

We endow the space (Ω, \mathcal{F}) with $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]} = (\mathcal{F}_t^c \otimes (\mathcal{F}_t^d)^{\otimes 2})_{t \in [0, T]}$ where $(\mathcal{F}_t^c)_{t \in [0, T]}$ and $(\mathcal{F}_t^d)_{t \in [0, T]}$ are the right-continuous completed filtrations associated with the components of $(\chi_t)_{t \in [0, T]}$.

We consider a market where there is only one market maker. This market maker has a view on the efficient price of the asset given by S_t . We assume that

$$S_t = S_0 + \sigma W_t, \tag{1}$$

with $S_0 > 0$, W a Brownian motion and $\sigma > 0$ the volatility of the price¹. The market maker fixes the bid and ask prices

$$P_t^b = S_t - \delta_t^b, \quad \text{and} \quad P_t^a = S_t + \delta_t^a.$$

We assume that the arrival of ask (resp. bid) market orders is modeled by a point process $(N_t^a)_{t \in [0, T]}$ (resp. $(N_t^b)_{t \in [0, T]}$) with intensity $(\lambda_t^a)_{t \in [0, T]}$ (resp. $(\lambda_t^b)_{t \in [0, T]}$). We also suppose that the volume of market orders is constant and equal to unity. Hence, the inventory process of the market maker is given by

$$Q_t = N_t^b - N_t^a, \quad t \geq 0. \tag{2}$$

As in [11], we impose a critical absolute inventory $\bar{q} \in \mathbb{N}$ above which the market maker stops quoting on the ask or bid side, i.e.

$$\lambda_t^a = \lambda_t^a \mathbb{I}_{\{Q_t > -\bar{q}\}}, \quad \text{and} \quad \lambda_t^b = \lambda_t^b \mathbb{I}_{\{Q_t < \bar{q}\}}.$$

¹In practice, the efficient price can be thought of as the mid-price of the asset.

Moreover, we recall that from classical financial economics results, see [9, 18, 28], the average number of trades per unit of time is essentially a decreasing function of the ratio between the spread and the volatility. Hence, we assume that

$$\lambda_t^a = \lambda(\delta_t^a)1_{\{Q_t > -\bar{q}\}}, \quad \text{and} \quad \lambda_t^b = \lambda(\delta_t^b)\mathbb{1}_{\{Q_t < \bar{q}\}}, \quad \text{with} \quad \lambda(x) = Ae^{-k\frac{x}{\sigma}}, \quad (3)$$

for fixed positive constants A and k .

2.1 Admissible controls and market maker's problem

We work with the following set \mathcal{A} of admissible controls $(\delta_t)_{t \in [0, T]} = (\delta_t^a, \delta_t^b)_{t \in [0, T]}$ such that δ is predictable and for any $t \in [0, T]$

$$|\delta_t^a| \vee |\delta_t^b| \leq \delta_\infty,$$

for some positive constant δ_∞ which will be fixed later to a sufficiently large value. For each control process $\delta = (\delta^a, \delta^b)$ of the market maker, we denote by \mathbb{P}^δ the associated probability measure under which S follows (1) and

$$\tilde{N}_t^{\delta, a} = N_t^a - \int_0^t \lambda(\delta_r^a)1_{\{Q_r > -\bar{q}\}} dr, \quad \tilde{N}_t^{\delta, b} = N_t^b - \int_0^t \lambda(\delta_r^b)1_{\{Q_r < \bar{q}\}} dr, \quad (4)$$

are martingales. In that case, the profit and loss process of the market maker is defined by

$$\text{PL}_t^\delta = X_t^\delta + Q_t S_t, \quad \text{where} \quad X_t^\delta = \int_0^t P_r^a dN_r^a - \int_0^t P_r^b dN_r^b, \quad t \in [0, T]. \quad (5)$$

Here, X^δ is the cash flow process, and QS represents the inventory risk process².

Next, we introduce the Doléans-Dade exponential

$$\begin{aligned} L_t^\delta = & \exp \left(\int_0^t \log \left(\frac{\lambda(\delta_r^a)}{A} \right) \mathbb{1}_{\{Q_r > -\bar{q}\}} dN_r^a + \log \left(\frac{\lambda(\delta_r^b)}{A} \right) \mathbb{1}_{\{Q_r < \bar{q}\}} dN_r^b \right. \\ & \left. - (\lambda(\delta_r^a) - A)1_{\{Q_r > -\bar{q}\}} dr - (\lambda(\delta_r^b) - A)1_{\{Q_r < \bar{q}\}} dr \right), \end{aligned} \quad (6)$$

which is a \mathbb{P}^0 -local martingale as it can be verified by direct application of Itô's formula:

$$dL_t^\delta = L_{t-}^\delta \left(\frac{\lambda(\delta_t^a) - A}{A} \mathbb{1}_{\{Q_t > -\bar{q}\}} d\tilde{N}_t^{0, a} + \frac{\lambda(\delta_t^b) - A}{A} \mathbb{1}_{\{Q_t < \bar{q}\}} d\tilde{N}_t^{0, b} \right),$$

Since δ^a and δ^b are uniformly bounded, this local martingale satisfies the Novikov-type criterion in [27] and thus is a martingale. From Theorem III.3.11 in [15], it follows that

$$\frac{d\mathbb{P}^\delta}{d\mathbb{P}^0} \Big|_{\mathcal{F}_t} = L_t^\delta. \quad (7)$$

In particular, all the probability measures \mathbb{P}^δ indexed by $\delta \in \mathcal{A}$ are equivalent. We therefore use the notation *a.s* for almost surely without ambiguity. We shall write \mathbb{E}_t^δ for the conditional expectation with respect to \mathcal{F}_t with probability measure \mathbb{P}^δ .

²As in [2], for sake of simplicity, we assume that the market maker **estimates his inventory risk using the efficient price S** .

We consider that the exchange is compensated for each market order arrival and so aims at keeping the market liquid. Thus, we assume that it proposes to the market maker a contract, defined by an \mathcal{F}_T -measurable random variable ξ , and representing a compensation for the market making activity as part of the market flow. In addition to the realized profit and loss (5) on $[0, T]$, the market maker receives a compensation ξ from the exchange at the final time T , thus leading to the maximization problem:

$$\begin{aligned} V_{\text{MM}}(\xi) &= \sup_{\delta \in \mathcal{A}} J_{\text{MM}}(\delta, \xi) \quad \text{where} \quad J_{\text{MM}}(\delta, \xi) = \mathbb{E}^\delta \left[-e^{-\gamma(\xi + \text{PL}_T^\delta - \text{PL}_0^\delta)} \right] \\ &= \mathbb{E}^\delta \left[-e^{-\gamma(\xi + \int_0^T \delta_t^a dN_t^a + \delta_t^b dN_t^b + Q_t dS_t)} \right]. \end{aligned} \quad (8)$$

Here, $\gamma > 0$ is the absolute risk aversion parameter of the CARA market maker. For each compensation ξ , we show that there exists a unique best reaction control $\hat{\delta}(\xi) = (\hat{\delta}^a(\xi), \hat{\delta}^b(\xi))$ of the market marker.

Remark 2.1. *The case $\xi = 0$ corresponds to the problem without exchange intervention treated in [2, 11].*

2.2 The exchange optimal contracting problem

We assume that the exchange is compensated by a fixed amount $c > 0$ for each market order that occurs in the market. In practice, some exchanges add to this fixed fee a component which is proportional to the traded amount in currency value. However, since we are anyway working on a short time interval, we take c independent of the price of the asset. Note that the fee schedule considered here for the taker side is simple. Indeed, in practice, complex fee policies are rather dedicated to market makers. Furthermore, we will in fact see that when acting optimally, the exchange is somehow indifferent to the value of c , see Section 4.3.

The exchange aims at maximizing the total number of market orders $N_T^a + N_T^b$, whose arrival intensities are controlled exclusively by the market maker. The role of the contract ξ proposed by the exchange to the market maker is to encourage the latter to increase the liquidity of the market. In this case, the profit and loss of the exchange is given by

$$c(N_T^a + N_T^b) - \xi.$$

Thus the exchange optimally chooses the contract to maximize its CARA utility function with absolute risk aversion parameter $\eta > 0$:

$$V_0^E = \sup_{\xi \in \mathcal{C}} \mathbb{E}^{\hat{\delta}(\xi)} \left[-e^{-\eta(c(N_T^a + N_T^b) - \xi)} \right]. \quad (9)$$

We now define the set of admissible contracts \mathcal{C} . Concerning the problem of the exchange, we need to ensure that $\mathbb{E}^{\hat{\delta}(\xi)} \left[-e^{-\eta(c(N_T^a + N_T^b) - \xi)} \right]$ is not degenerated. The natural condition that we need is then to assume that

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta \left[e^{\eta' \xi} \right] < +\infty, \quad \text{for some } \eta' > \eta. \quad (10)$$

Since N^a and N^b are point processes with bounded intensities, this condition together with an Hölder inequality ensure that the problem of the exchange (9) is well defined. In the same way, we will assume that

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta \left[e^{-\gamma' \xi} \right] < +\infty, \quad \text{for some } \gamma' > \gamma, \quad (11)$$

to ensure that $\mathbb{E}^\delta[-e^{-\gamma(\xi + \int_0^T \delta_t^a dN_t^a + \delta_t^b dN_t^b + Q_t dS_t)}]$ is not degenerate and hence the well-definition of the market maker problem (8). We will also assume that the latter only accepts contracts ξ such that the maximal utility $V_{MM}(\xi)$ is above a threshold value $R < 0$.

Hence, we denote by \mathcal{C} the space of admissible contracts defined by

$$\mathcal{C} = \left\{ \xi \text{ } \mathcal{F}_T\text{-measurable such that } V_{MM}(\xi) \geq R \text{ and (10) and (11) are satisfied} \right\}.$$

We will take $-R$ large enough so that \mathcal{C} contains the zero contract $\xi = 0$ and thus is nonempty.

3 Solving the market maker's problem

We start by solving the problem (8) of the market maker faced to an arbitrary contract $\xi \in \mathcal{C}$ proposed by the exchange.

3.1 Market maker's optimal response

For $(\delta, z, q) \in [-\delta_\infty, \delta_\infty]^2 \times \mathbb{R}^3 \times \mathbb{Z}$, with $\delta = (\delta^a, \delta^b)$ and $z = (z^S, z^a, z^b)$, we define

$$h(\delta, z, q) = \frac{1 - e^{-\gamma(z^a + \delta^a)}}{\gamma} \lambda(\delta^a) \mathbb{1}_{\{q > -\bar{q}\}} + \frac{1 - e^{-\gamma(z^b + \delta^b)}}{\gamma} \lambda(\delta^b) \mathbb{1}_{\{q < \bar{q}\}},$$

and

$$H(z, q) = \sup_{|\delta^a| \vee |\delta^b| \leq \delta_\infty} h(\delta, z, q),$$

For arbitrary constant $Y_0 \in \mathbb{R}$, and predictable processes $Z = (Z^S, Z^a, Z^b)$, with $\int_0^T |Z^S|^2 dt < \infty$, and (Z^a, Z^b) locally bounded, a.s., we introduce the process

$$Y_t^{Y_0, Z} = Y_0 + \int_0^t Z_r^a dN_r^a + Z_r^b dN_r^b + Z_r^S dS_r + \left(\frac{1}{2} \gamma \sigma^2 (Z_r^S + Q_r)^2 - H(Z_r, Q_r) \right) dr, \quad (12)$$

and we denote by \mathcal{Z} the collection of all such processes Z satisfying in addition Condition (10) with $\xi = Y_T^{0, Z}$ and

$$\sup_{\delta \in \mathcal{A}} \sup_{t \in [0, T]} \mathbb{E}^\delta [e^{-\gamma Y_t^{0, Z}}] < \infty. \quad (13)$$

Clearly, $\mathcal{Z} \neq \emptyset$ as it contains all bounded predictable processes and

$$\mathcal{C} \supset \Xi = \{ Y_T^{Y_0, Z} : Y_0 \in \mathbb{R}, Z \in \mathcal{Z}, \text{ and } V_{MM}(Y_T^{Y_0, Z}) \geq R \}. \quad (14)$$

The next result shows that these sets are in fact equal, and identifies the market maker utility value and the corresponding optimal response. To prove equality of these sets, we are reduced to the problem of representing any contract $\xi \in \mathcal{C}$ as $\xi = Y_T^{Y_0, Z}$ for some $(Y_0, Z) \in \mathbb{R} \times \mathcal{Z}$, which is known in the literature as a problem of backward stochastic differential equation. We refrain from using this terminology, as our analysis does not require any result from this literature.

Theorem 3.1. (i) *Any contract $\xi \in \mathcal{C}$ has a unique representation as $\xi = Y_T^{Y_0, Z}$, for some $(Y_0, Z) \in \mathbb{R} \times \mathcal{Z}$. In particular, $\mathcal{C} = \Xi$.*

(ii) *Under this representation, the market maker utility value is*

$$V_{MM}(\xi) = -e^{-\gamma Y_0}, \text{ so that } \Xi = \left\{ Y_T^{Y_0, Z} : Z \in \mathcal{Z}, \text{ and } Y_0 \geq \frac{-1}{\gamma} \log(-R) \right\},$$

with optimal bid-ask policy:

$$\hat{\delta}_t^a(\xi) = \Delta(Z_t^a), \quad \hat{\delta}_t^b(\xi) = \Delta(Z_t^b), \quad \text{where } \Delta(z) = (-\delta_\infty) \vee \left\{ -z + \frac{1}{\gamma} \log \left(1 + \frac{\sigma\gamma}{k} \right) \right\} \wedge \delta_\infty. \quad (15)$$

The proof of Part (i) of the previous result will be reported in Section A.2. This representation is obtained by using the dynamic continuation utility process of the market maker, following the approach of Sannikov [25]. In the present case, we shall prove that the continuation utility process satisfies the dynamic programming principle, so that the required representation follows from the Doob-Meyer decomposition of supermartingales together with the martingale representation theorem.

Proof of Theorem 3.1 (ii) Let $\xi = Y_T^{Y_0, Z}$ with $(Y_0, Z) \in \mathbb{R} \times \mathcal{Z}$. We first prove that for an arbitrary bid-ask policy $\delta \in \mathcal{A}$, we have $J_{\text{MM}}(\xi, \delta) \leq -e^{-\gamma Y_0}$. Denote $\bar{Y}_t = Y_t^{Y_0, Z} + \int_0^t \delta_t^a dN_t^a + \delta_t^b dN_t^b + Q_t dS_t$, $t \in [0, T]$. By direct application of Itô's formula, we see that

$$\begin{aligned} de^{-\gamma \bar{Y}_t} &= \gamma e^{-\gamma \bar{Y}_t} \left[- (Q_t + Z_t^S) dS_t - \frac{1}{\gamma} (1 - e^{-\gamma(Z_t^a + \delta_t^a)}) d\tilde{N}_t^{\delta, a} - \frac{1}{\gamma} (1 - e^{-\gamma(Z_t^b + \delta_t^b)}) d\tilde{N}_t^{\delta, b} \right. \\ &\quad \left. + (H(Z_t, Q_t) - h(\delta_t, Z_t, Q_t)) dt \right]. \end{aligned}$$

Hence $e^{-\gamma \bar{Y}}$ is a \mathbb{P}^δ -local submartingale. Thanks to Condition (13), the uniform boundedness of the intensities of N^a and N^b and the Hölder inequality, $(e^{-\gamma \bar{Y}_t})_{t \in [0, T]}$ is uniformly integrable and hence is a true submartingale. By Doob-Meyer decomposition theorem, we conclude that

$$\int_0^\cdot \gamma e^{-\gamma \bar{Y}_t} \left(- (Q_t + Z_t^S) dS_t - \frac{1}{\gamma} (1 - e^{-\gamma(Z_t^a + \delta_t^a)}) d\tilde{N}_t^{\delta, a} - \frac{1}{\gamma} (1 - e^{-\gamma(Z_t^b + \delta_t^b)}) d\tilde{N}_t^{\delta, b} \right),$$

is a true martingale. It follows that

$$J_{\text{MM}}(\xi, \delta) = \mathbb{E}^\delta \left[-e^{-\gamma \bar{Y}_T} \right] = -e^{-\gamma Y_0} - \mathbb{E}^\delta \left[\int_0^T \gamma e^{-\gamma \bar{Y}_t} (H(Z_t, Q_t) - h(\delta_t, Z_t, Q_t)) dt \right] \leq -e^{-\gamma Y_0}.$$

On the other hand, equality holds in the last inequality if and only if δ is chosen as the maximizer of the Hamiltonian H ($dt \times d\mathbb{P}$ -a.e.), thus leading to the unique maximizer $\hat{\delta}(\xi)$, which then induces $J_{\text{MM}}(\xi, \hat{\delta}(\xi)) = -e^{-\gamma Y_0}$. This completes the proof that $V_{\text{MM}}(\xi) = -e^{-\gamma Y_0}$ with optimal response $\hat{\delta}(Z)$.

4 Designing the optimal contract

Denote $\hat{Y}_0 = -\frac{1}{\gamma} \log(-R)$. By Theorem 3.1, the exchange problem (9) reduces to the control problem

$$V_0^E = \sup_{Y_0 \geq \hat{Y}_0} \sup_{Z \in \mathcal{Z}} \mathbb{E}^{\hat{\delta}(Y_T^Z)} \left[-e^{-\eta(c(N_T^a + N_T^b) - Y_T^{Y_0, Z})} \right], \quad (16)$$

where the continuation utility process of the market maker $Y^{Y_0, Z}$ is given by (12). In the present context, the objective function in (16) is clearly decreasing in Y_0 , implying that the maximization under the participation constraint is achieved at \hat{Y}_0 . Hence

$$V_0^E = e^{\eta \hat{Y}_0} \sup_{Z \in \mathcal{Z}} \mathbb{E}^{\hat{\delta}(Y_T^Z)} \left[-e^{-\eta(c(N_T^a + N_T^b) - Y_T^{0, Z})} \right]. \quad (17)$$

We will denote by $\tilde{N}^{\delta, b}$ and $\tilde{N}^{\delta, a}$ the $\mathbb{P}^{\hat{\delta}(Y_T^Z)}$ -compensated Poissons processes of N^a and N^b .

4.1 The HJB equation for the reduced exchange problem

In this section, we study the HJB equation corresponding to the stochastic control problem

$$v_0^E = \sup_{Z \in \mathcal{Z}} \mathbb{E}^{\delta(Y_T^Z)} \left[-e^{-\eta(c(N_T^a + N_T^b) - Y_T^{0,Z})} \right]. \quad (18)$$

Our approach is to derive a solution v of the corresponding HJB equation, and to proceed by the standard verification argument in stochastic control to prove that the proposed solution v coincides with the value function v^E .

Applying the standard dynamic programming approach (formally) to the last control problem, we are led to the HJB equation

$$\begin{cases} \partial_t v(t, q) + H_E(q, v(t, q), v(t, q+1), v(t, q-1)) = 0, & q \in \{-\bar{q}, \dots, \bar{q}\}, \quad t \in (0, T], \\ v(T, q) = -1, \end{cases} \quad (19)$$

where the Hamiltonian $H_E : [-\bar{q}, \bar{q}] \times (-\infty, 0]^3 \rightarrow \mathbb{R}$ is given by

$$H_E(q, \mathbf{y}, y_+, y_-) = H_E^1(q) + \mathbb{1}_{\{q > -\bar{q}\}} H_E^0(y, y_-) + \mathbb{1}_{\{q < \bar{q}\}} H_E^0(y, y_+), \quad (20)$$

with

$$\begin{aligned} H_E^1(q, \mathbf{y}) &= \sup_{z_s \in \mathbb{R}} h_E^1(q, \mathbf{y}, z_s), \quad \text{and} \quad h_E^1(q, \mathbf{y}, z_s) = \frac{\eta \sigma^2}{2} \mathbf{y} (\gamma(z_s + q)^2 + \eta z_s^2), \\ H_E^0(\mathbf{y}, y') &= \sup_{\zeta \in \mathbb{R}} h_E^0(\mathbf{y}, y', \zeta) \quad \text{and} \quad h_E^0(\mathbf{y}, y', \zeta) = \lambda(\Delta(\zeta)) \left[y' e^{\eta(\zeta - c)} - \mathbf{y} \left(1 + \eta \frac{1 - e^{-\gamma(\zeta + \Delta(\zeta))}}{\gamma} \right) \right]. \end{aligned}$$

A direct calculation reported in Lemma A.1 below reveals that the maximizers $\hat{z} = (\hat{z}^s, \hat{z}^a, \hat{z}^s)$ of H_E are

$$\hat{z}^s(t, q) = -\frac{\gamma}{\gamma + \eta} q, \quad \hat{z}^a(t, q) = \hat{\zeta}(v(t, q), v(t, q-1)), \quad \text{and} \quad \hat{z}^s(t, q) = \hat{\zeta}(v(t, q), v(t, q+1)), \quad (21)$$

where

$$\hat{\zeta}(\mathbf{y}, y') = \zeta_0 + \frac{1}{\eta} \log\left(\frac{\mathbf{y}}{y'}\right), \quad \zeta_0 = c + \frac{1}{\eta} \log\left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)}\right). \quad (22)$$

Here, we assume that δ_∞ is large enough so that the Condition (34) of Lemma A.1 is always met, namely

$$\delta_\infty \geq C_\infty + \frac{1}{\eta} \sup_{t \in [0, T]} \sup_{q \in [-\bar{q}, \bar{q}]} \left| \log\left(\frac{v(t, q)}{v(t, q+1)}\right) \right| \quad (23)$$

with the hope that our candidate solution of the HJB equation will verify it. This will be checked in our verification argument. Recall from Lemma A.1 that $C_\infty = |c| + \left(\frac{1}{\eta} + \frac{1}{\gamma}\right) \log\left(1 + \frac{\sigma \gamma}{k}\right) - \frac{1}{\eta} \log\left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)}\right)$.

Using again the calculation reported in Lemma A.1, we rewrite the HJB equation (19) as

$$\begin{cases} \partial_t v(t, q) + \frac{\gamma \eta^2 \sigma^2}{2(\gamma + \eta)} q^2 v(t, q) - C v(t, q) \left[\mathbb{1}_{\{q > -\bar{q}\}} \left(\frac{v(t, q)}{v(t, q-1)}\right)^{\frac{k}{\sigma \eta}} + \mathbb{1}_{\{q < \bar{q}\}} \left(\frac{v(t, q)}{v(t, q+1)}\right)^{\frac{k}{\sigma \eta}} \right] = 0, \\ v(T, q) = -1, \end{cases} \quad (24)$$

where the constant C is given by

$$C = A \frac{\sigma\eta}{k} \exp\left(\frac{kC}{\sigma} - \frac{k}{\sigma\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right) + \left(1 + \frac{k}{\sigma\eta}\right) \log\left(1 - \frac{\sigma^2\gamma\eta}{(k + \sigma\gamma)(k + \sigma\eta)}\right)\right).$$

We now make the key observation that this equation can be reduced to a linear equation by introducing $u = (-v)^{-\frac{k}{\sigma\eta}}$. Indeed, by direct substitution, we obtain the following linear differential equation:

$$\begin{cases} \partial_t u(t, q) - C_1 q^2 u(t, q) + C_2 (u(t, q + 1) \mathbb{1}_{\{q < \bar{q}\}} + u(t, q - 1) \mathbb{1}_{\{q > -\bar{q}\}}) = 0, \\ u(T, q) = 1, \end{cases} \quad (25)$$

with

$$C_1 = \frac{k\gamma\eta\sigma}{2(\gamma + \eta)} \quad \text{and} \quad C_2 = C \frac{k}{\sigma\eta}.$$

This equation can be written in terms of the $\mathbb{R}^{2\bar{q}+1}$ -valued function $\mathbf{u}(t) = (u(t, q))_{q \in \{-\bar{q}, \dots, \bar{q}\}}$, of the variable t only, as the linear ordinary differential equation:

$$\partial_t \mathbf{u} = -\mathbf{B}\mathbf{u}, \quad \text{where} \quad \mathbf{B} = \begin{pmatrix} -C_1 \bar{q}^2 & C_2 & & & \\ & \ddots & \ddots & \ddots & \\ & & C_2 & -C_1 \bar{q}^2 & C_2 \\ & & & \ddots & \ddots & \ddots \\ & & & & C_2 & -C_1 \bar{q}^2 \end{pmatrix} \leftarrow q\text{-th line},$$

is a tri-diagonal matrix with lines labelled $-\bar{q}, \dots, \bar{q}$. Denote by \mathbf{b}_q the vector of $\mathbb{R}^{2\bar{q}+1}$ with zeros everywhere except at the position q , i.e. $\mathbf{b}_{q,i} = \mathbb{1}_{\{i=q\}}$ for $i \in \{-\bar{q}, \dots, \bar{q}\}$, and $\mathbf{1} = \sum_{q=-\bar{q}}^{\bar{q}} \mathbf{b}_q$. Then, this ODE has a unique solution

$$\mathbf{u}(t) = e^{(T-t)\mathbf{B}} \mathbf{1}, \quad \text{so that} \quad u(t, q) = \mathbf{b}_q \cdot e^{(T-t)\mathbf{B}} \mathbf{1}, \quad \text{and} \quad v(t, q) = -(\mathbf{b}_q \cdot e^{(T-t)\mathbf{B}} \mathbf{1})^{\frac{k}{\sigma\eta}}. \quad (26)$$

In the next section, we shall prove that this solution v of the HJB equation (19) coincides with the value function of the reduced exchange problem (18), with optimal controls $\hat{z}(t, q)$ given in (21), thus inducing the optimal contract $Y_T^{\hat{Y}_0, \hat{Z}}$ with $\hat{Z}_t = \hat{z}(t, Q_t)$.

We conclude this section by an alternative representation of the function u .

Proposition 4.1. *The function u is $C^{1,1}$ and can be represented as*

$$u(t, q) = \mathbb{E}\left[e^{\int_t^T (-C_1(Q_s^{t,q})^2 + \bar{\lambda}_s + \underline{\lambda}_s) ds}\right],$$

where $Q_s^{t,q} = q + \int_t^s d(\bar{N}_s - \underline{N}_s)$, and (\bar{N}, \underline{N}) is a two-dimensional point process with intensity $(\bar{\lambda}_s, \underline{\lambda}_s) = C_2(\mathbb{1}_{\{Q_s^- < \bar{q}\}}, \mathbb{1}_{\{Q_s^- > -\bar{q}\}})$. In particular, we have the following bounds for the function u :

$$e^{-C_1 \bar{q}^2 T} \leq u \leq e^{2C_2 T}.$$

Moreover, Condition (23) is verified when

$$\delta_\infty \geq \Delta_\infty = C_\infty + \frac{\sigma}{k}(2C_2 + C_1 \bar{q}^2)T. \quad (27)$$

Proof. The regularity of u follows from the explicit expression in (26). Denote $f(q) = -C_1 q^2 + C_2(\mathbb{1}_{\{q > -\bar{q}\}} + \mathbb{1}_{\{q < \bar{q}\}})$, and $M_s = e^{\int_t^s f(Q_s^{t,x}) ds} u(t, Q_s^{t,x})$, $t \leq s \leq T$. We now show that M is a martingale, so that $u(t, q) = M_t = \mathbb{E}[M_T] = \mathbb{E}[e^{-\int_t^T f(Q_s^{t,x}) ds}]$, as $u(T, \cdot) = 1$. To see that M is a martingale, we compute by Itô's formula that

$$\begin{aligned} dM_s &= [u(s, Q_s^{t,x})f(Q_s^{t,x}) + \partial_t u(s, Q_s^{t,x})] ds \\ &\quad + C_2 [u(s, Q_{s^-}^{t,x} + 1) - u(s, Q_{s^-}^{t,x})] d\bar{N}_s + C_2 [u(s, Q_{s^-}^{t,x} - 1) - u(s, Q_{s^-}^{t,x})] d\underline{N}_s. \end{aligned}$$

Since u is solution of (25), we get

$$dM_s = C_2 [u(s, Q_{s^-}^{t,x} + 1) - u(s, Q_{s^-}^{t,x})] d\bar{M}_s + C_2 [u(s, Q_{s^-}^{t,x} - 1) - u(s, Q_{s^-}^{t,x})] d\underline{M}_s,$$

where $(\bar{M}, \underline{M}) = (\bar{N} - \int_0^\cdot \bar{\lambda}_s ds, \underline{N} - \int_0^\cdot \underline{\lambda}_s ds)$ is a martingale. The martingale property of M now follows from the boundedness of u as it can be verified from the expression (26).

Finally, the bound $|Q_s^{t,x}| \leq \bar{q}$ induces directly the announced bounds on u , which in turn imply Condition (23) when (27) is satisfied because $v = -u - \frac{\sigma\eta}{k}$ \square

4.2 The main result

We are now ready to verify that the function v introduced in the previous section is the value function of the exchange, with optimal feedback controls $(\hat{z}^s, \hat{z}^a, \hat{z}^b)$ as given in (21), thus identifying a unique optimal contract to be proposed by the exchange to the market maker. Recall that δ_∞ denotes the bound on the market maker bid and ask spreads. Our main explicit solution requires δ_∞ to be larger than the constant Δ_∞ introduced in (27).

Theorem 4.1. *Assume that $\delta_\infty \geq \Delta_\infty$. Then the optimal contract for the problem of the exchange (9) is given by*

$$\hat{\xi} = \hat{Y}_0 + \int_0^T \hat{Z}_r^a dN_r^a + \hat{Z}_r^b dN_r^b + \hat{Z}_r^S dS_r + \left(\frac{1}{2} \gamma \sigma^2 (\hat{Z}_r^S + Q_r)^2 - H(\hat{Z}_r, Q_r) \right) dr, \quad (28)$$

with $\hat{Z}_r^S = \hat{z}^s(r, Q_r)$, $\hat{Z}_r^a = \hat{z}^a(r, Q_r)$, and $\hat{Z}_r^b = \hat{z}^b(r, Q_r)$ as defined in (21). The market maker's optimal effort is given by

$$\hat{\delta}_t^a = \hat{\delta}_t^a(\hat{\xi}) = -\hat{Z}_t^a + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right), \quad \hat{\delta}_t^b = \hat{\delta}_t^b(\hat{\xi}) = -\hat{Z}_t^b + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right). \quad (29)$$

Proof. In order to prove this result, we verify that the function v introduced in (26) coincides at $(0, Q_0)$ with the value function of the reduced exchange problem (18), with maximum achieved at the optimal control \hat{Z} .

By Proposition 4.1, the function v is negative, $C^{1,1}$, bounded, and has bounded gradient. Moreover, since $\delta_\infty \geq \Delta_\infty$, it follows that v is a solution of the HJB equation (19) of the exchange reduced problem. For $Z \in \mathcal{Z}$, denote

$$K_t^Z = e^{-\eta(c(N_t^a + N_t^b) - Y_t^{0,Z})}, \quad t \in [0, T].$$

By direct application of Itô's formula, and substitution of $\partial_t v$ from the HJB equation satisfied by v , we see that

$$d[v(t, Q_t)K_t^Z] = K_{t^-}^Z \left((h_t^Z - \mathcal{H}_t) dt + \eta v(t, Q_t) Z_t^s dS_t + \sum_{i=a,b} [v(t, Q_{t^-} + \Delta Q_t) e^{-\eta(c - Z_t^i)} - v(t, Q_{t^-})] d\tilde{N}_t^{\delta, i} \right),$$

where, using the notations of (20) and the subsequent equations,

$$\mathcal{H}_t = H_E(Q_t, v(t, Q_t), v(t, Q_t + 1), v(t, Q_t - 1)),$$

and

$$h_t^Z = h_E^1(Q_t, Z_t^S) + \mathbb{I}_{\{Q_t > -\bar{q}\}} h_E^0(v(t, Q_t), v(t, Q_t - 1)) + \mathbb{I}_{\{Q_t < \bar{q}\}} h_E^0(v(t, Q_t), v(t, Q_t + 1)).$$

Exploiting the fact that v is bounded and that K^Z is uniformly integrable (see Lemma A.4), we get that

$$\begin{aligned} v(0, Q_0) &= \mathbb{E}^{\hat{\delta}(Y_T^Z)} \left[v(T, Q_T) K_T^Z + \int_0^T K_t^Z (\mathcal{H}_t - h_t^Z) dt \right] \\ &\geq \mathbb{E}^{\hat{\delta}(Y_T^Z)} [v(T, Q_T) K_T^Z] = \mathbb{E}^{\hat{\delta}(Y_T^Z)} [-K_T^Z], \end{aligned}$$

by the boundary condition $v(T, \cdot) = -1$. By arbitrariness of $Z \in \mathcal{Z}$, this provides the inequality $v(0, Q_0) \geq \sup_{Z \in \mathcal{Z}} \mathbb{E}^{\hat{\delta}(Y_T^Z)} [-K_T^Z] = v_0^E$.

On the other hand, consider the maximizer \hat{Z} of the reduced exchange problem, induced by the feedback controls \hat{z} in (21). As \hat{Z} is bounded, it follows that $\hat{Z} \in \mathcal{Z}$. Moreover, $h^{\hat{Z}} - \mathcal{H} = 0$, by definition, so that the last argument leads to the equality $v(0, Q_0) = \mathbb{E}^{\hat{\delta}(Y_T^{\hat{Z}})} [-K_T^{\hat{Z}}]$, instead of the inequality. This shows that $v(0, Q_0) = v_0^E$, the reduced exchange problem of (18), with optimal control \hat{Z} . \square

4.3 Discussion

The processes \hat{Z}^a , \hat{Z}^b and \hat{Z}^S allowing the exchange to build the optimal contract have actually quite natural interpretations. Indeed, using Lemma 4.1, we obtain that the quantities

$$-\log \left(\frac{u(T-t, q_{t-})}{u(T-t, q_{t-} - 1)} \right) \text{ and } -\log \left(\frac{u(T-t, q_{t-})}{u(T-t, q_{t-} + 1)} \right)$$

are roughly proportional respectively to Q_{t-} and $-Q_{t-}$. Thus, when the inventory is highly positive, the exchange provides incentives to the market-maker so that it attracts buy market orders and tries to dissuade him to accept more sell market orders, and conversely for a negative inventory. The integral

$$\int_0^T \hat{Z}_r^S dS_r$$

can be understood as a risk sharing term. Indeed, $\int_0^t Q_{r-} dS_r$ corresponds to the price driven component of the inventory risk $Q_t S_t$. Hence in the optimal contract, the exchange supports part of this risk so that the market maker maintains reasonable quotes despite some inventory. The proportion of risk handled by the platform is $\frac{\gamma}{\gamma + \eta}$.

Concerning the optimal bid and ask spreads, we recover the optimal bid/ask spread of [11] impacted by the exchange through the term $-c - \frac{1}{\eta} \log \left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)} \right)$.

Until now, we have focused on the maker part of the make-take fees problem since we have considered that the taker cost c is fixed. Nevertheless, our approach also enables us to suggest

the exchange a relevant value for c . Actually, we see that when acting optimally, the exchange transfers the totality of the fixed taker fee c to the market maker. It is therefore somehow neutral to the value of c . However, c plays an important role in the optimal spread offered by the market maker given by

$$-2c + \frac{\sigma}{k} \log \left(\frac{u(T-t, Q_{t-})^2}{u(T-t, Q_{t-}-1)u(T-t, Q_{t-}+1)} \right) - \frac{2}{\eta} \log \left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)} \right) + \frac{2}{\gamma} \log \left(1 + \frac{\sigma \gamma}{k} \right).$$

Furthermore, from numerical computations, we remark that

$$\frac{u(t, q)^2}{u(t, q-1)u(t, q+1)}$$

is close to unity for any t and q . Hence the exchange may fix in practice the transaction cost c so that the spread is close to one tick by setting

$$c \approx -\frac{1}{2} \text{Tick} - \frac{1}{\eta} \log \left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)} \right) + \frac{1}{\gamma} \log \left(1 + \frac{\sigma \gamma}{k} \right).$$

For $\sigma \gamma / k$ small enough, this equation reduces to

$$c \approx \frac{\sigma}{k} - \frac{1}{2} \text{Tick}. \quad (30)$$

Equation (30) is a particularly simple formula to fix the taker constant c . We see that the higher the volatility, the larger the taker cost should be. It is also quite natural that this cost is a decreasing function of k . Indeed, if k is large, the liquidity vanishes rapidly when the spread becomes wide, meaning that market takers are sensitive to extra costs relative to the efficient price. Therefore, the taker cost has to be small if the exchange wants to maintain a reasonable market order flow. Finally, note that the parameters σ and k can be easily estimated from market data. Therefore the formula (30) can be readily used in practice.

5 Impact of the presence of the exchange on market quality and comparison with [2, 11]

In this section, we compare our setting with the situation without incentive policy from an exchange towards market making activities. The latter is considered in [2, 11] where the authors deal with the issue of optimal market making without intervention of the exchange. The results in [2] are taken as benchmark for our investigation to emphasize the impact of the incentive policy on market quality. We will refer to this case as the neutral exchange case.

Let us first recall the results in [2, 11]. The optimal controls of the market maker denoted by $\tilde{\delta}^a$ and $\tilde{\delta}^b$ are given as a function of the inventory q_t by

$$\begin{aligned} \tilde{\delta}_t^a &= \frac{\sigma}{k} \log \left(\frac{\tilde{u}(t, Q_t)}{\tilde{u}(t, Q_t-1)} \right) + \frac{1}{\gamma} \log \left(1 + \frac{\sigma \gamma}{k} \right), \\ \tilde{\delta}_t^b &= \frac{\sigma}{k} \log \left(\frac{\tilde{u}(t, Q_t)}{\tilde{u}(t, Q_t+1)} \right) + \frac{1}{\gamma} \log \left(1 + \frac{\sigma \gamma}{k} \right), \end{aligned}$$

where \tilde{u} is the unique solution of the linear differential equation

$$\begin{cases} \partial_t \tilde{u}(t, q) + \widetilde{C}_1 q^2 \tilde{u}(t, q) - \widetilde{C}_2 (\tilde{u}(t, q+1) \mathbb{1}_{\{q < \bar{q}\}} + \tilde{u}(t, q-1) \mathbb{1}_{\{q > -\bar{q}\}}) = 0, (t, q) \in (0, T] \times [-\bar{q}, \bar{q}] \\ \tilde{u}(T, q) = 1, \end{cases} \quad (31)$$

with $\widetilde{C}_1 = \frac{\sigma\gamma k}{2}$ and $\widetilde{C}_2 = A \exp\left(-\left(1 + \frac{\sigma\gamma}{k}\right) \log\left(1 + \frac{\sigma\gamma}{k}\right)\right)$. In our case, the optimal quotes δ^{a*} and $\delta^{b,*}$ are obtained from Theorem 4.1 and satisfy

$$\begin{aligned} \delta_t^{a*} &= \frac{\sigma}{k} \log\left(\frac{u(t, Q_t)}{u(t, Q_t - 1)}\right) + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right) - c - \frac{1}{\eta} \log\left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma\gamma)(k + \sigma\eta)}\right), \\ \delta_t^{b*} &= \frac{\sigma}{k} \log\left(\frac{u(t, Q_t)}{u(t, Q_t + 1)}\right) + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right) - c - \frac{1}{\eta} \log\left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma\gamma)(k + \sigma\eta)}\right), \end{aligned}$$

where u is solution of the linear equation (25).

Numerical experiments show that u and \tilde{u} can decrease quickly to zero when q becomes large. Hence, the computation of the following crucial quantities appearing in the optimal quotes:

$$v_+(t, q) = \log\left(\frac{u(t, q+1)}{u(t, q)}\right), \quad \tilde{v}_+(t, q) = \log\left(\frac{\tilde{u}(t, q+1)}{\tilde{u}(t, q)}\right), \quad q \in \{-\bar{q}, \dots, \bar{q} - 1\}.$$

can be intricate in practice. To circumvent this numerical difficulty, we remark that v_+ and \tilde{v}_+ are solution of the following differential equations: **Attention: inversion du temps...**

$$\begin{cases} \partial_t v_+(t, q) + C_1(2q+1) - C_2(e^{v_+(t, q+1)} \mathbb{1}_{\{q < \bar{q}-1\}} + e^{-v_+(t, q)} - e^{v_+(t, q)} - e^{-v_+(t, q-1)} \mathbb{1}_{\{q > -\bar{q}\}}) = 0 \\ v_+(T, q) = 1, \end{cases} \quad (32)$$

and

$$\begin{cases} \partial_t \tilde{v}_+(t, q) + \widetilde{C}_1(2q+1) - \widetilde{C}_2(e^{\tilde{v}_+(t, q+1)} \mathbb{1}_{\{q < \bar{q}-1\}} + e^{-\tilde{v}_+(t, q)} - e^{\tilde{v}_+(t, q)} - e^{-\tilde{v}_+(t, q-1)} \mathbb{1}_{\{q > -\bar{q}\}}) = 0 \\ \tilde{v}_+(T, q) = 1. \end{cases} \quad (33)$$

We thus rather apply classical finite difference schemes to (32) and (33).

In the following numerical illustrations, in the spirit of [11, Section 6], we take $T = 600s$ for an asset with volatility $\sigma = 0.3 \text{ Tick} \cdot s^{-1/2}$ (unless specified differently). Market orders arrive according to the intensities (3) with $A = 0.9s^{-1}$ and $k = 0.3s^{-1/2}$. We assume that the threshold inventory of the market maker is $Q = 50$ unities and that his aversion of risk $\gamma = 0.01$. The exchange is taken more risk averse with $\eta = 1$. Finally, we assume that the taker cost $c = 0.5 \text{ Tick}$.

5.1 Impact of the exchange on the spread and market liquidity

We start by comparing the optimal spread $\hat{\delta}_0^a + \hat{\delta}_0^b$ at time 0 obtained when contracting optimally with the spread without incentives towards market making activities $\tilde{\delta}_0^a + \tilde{\delta}_0^b$. The optimal spreads are plotted in Figure 1 for different initial inventory values $q_0 \in \{-\bar{q}, \dots, \bar{q}\}$.

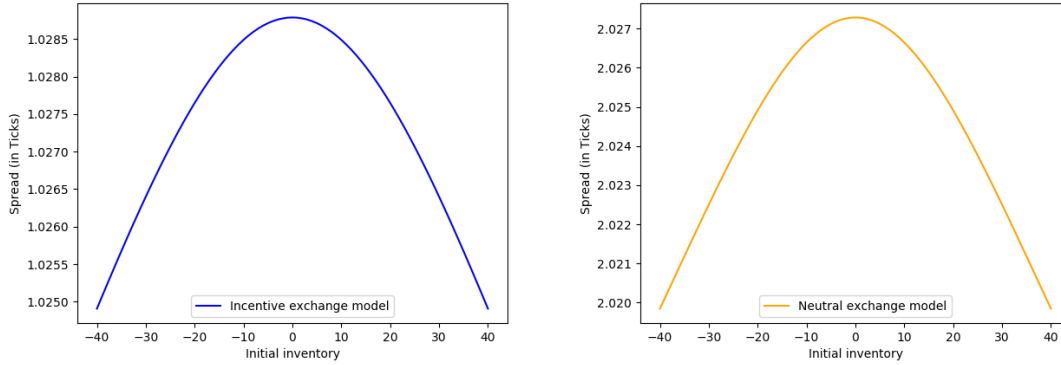


Figure 1: Comparison of optimal initial spreads with/without incentive policy from the exchange.

We observe in Figure 1 that the initial spread does not depend a lot on the initial inventory (because the considered time interval $[0, T]$ is not too small) and that it is reduced thanks to the optimal contract between the market maker and the exchange. This is not surprising since in our case the exchange aims at increasing the market order flow by proposing an incentive contract to the market maker inducing a spread reduction. Actually this phenomenon occurs over the whole trading period $[0, T]$. To see this, we generate 5000 paths of market scenarios and the average spread over $[0, T]$ for an initial inventory $q_0 = 0$. The results are given in Figure 2.

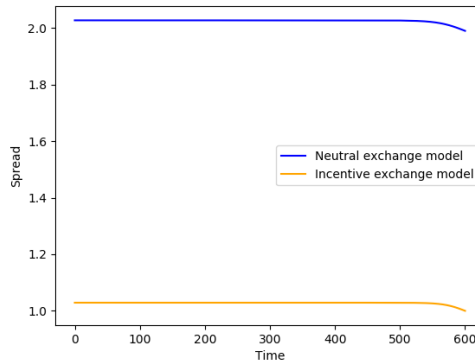


Figure 2: Average spread on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange toward the market maker.

Since the spread is tighter during the trading period under an incentive policy from the exchange, the arrival intensity of market orders is more important and hence the market more liquid as shown in Figure 3.

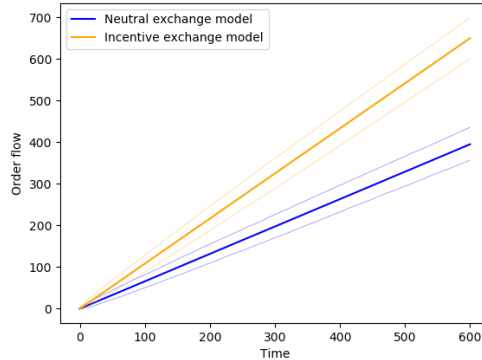


Figure 3: Average order flow on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange.

We now consider in Figure 4 the bid and ask sides separately. We see that when the inventory is positive and very large, $\hat{\delta}^a$ and $\tilde{\delta}^a$ are negative. It means the market maker is ready to sell at prices lower than the efficient price in order to attract market orders and reduce his inventory risk. On the contrary, if the inventory is negative and very large, in both situations, its ask quotes are well above the efficient price in order to repulse the arrival of buy market orders. However, since in our case the exchange remunerates the market maker for each arrival of market order, we get that the ask spread with contract $\hat{\delta}^a$ is smaller than $\tilde{\delta}^a$. A symmetric conclusion holds for the bid part of the spread.

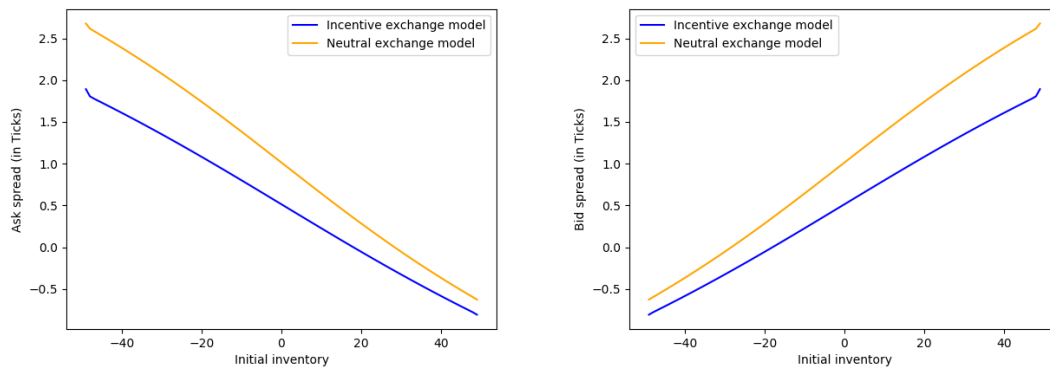


Figure 4: Optimal ask and bid spreads, with/without incentive policy from the exchange toward the market maker.

We now turn to the impact of the volatility on the spread. The optimal contract obtained in (28) induces an inventory risk sharing phenomenon through the term \hat{Z}^S . Hence, when the volatility increases, the spread difference between situations with/without incentive policy becomes less important, see Figure 5 in which we consider the optimal initial spread difference when the initial inventory is set to zero between both situations with/without incentive policy from the exchange to the market maker for different values of the volatility.

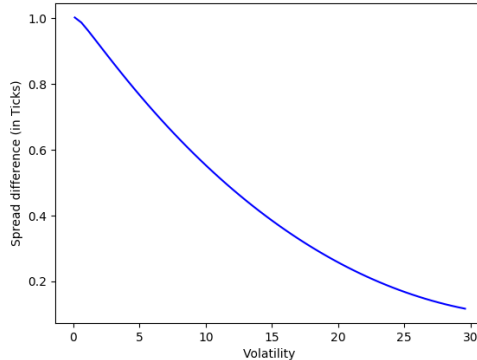


Figure 5: The initial optimal spread difference between both situation with/without incentive policy from the exchange toward the market maker as a decreasing function of the volatility σ .

5.2 Impact of the incentive policy on the profit and loss of the exchange and market maker

We assume that $q_0 = 0$. Recall that PL^δ defined in (5) denotes the trading part of the profit and loss (P&L) of the market maker for a given strategy δ . In our case, the underlying total P&L at time t of a market maker acting optimally, denoted by PL_t^* , is given by

$$PL_t^* = PL_t^{\delta^*} + Y_t^{Z^*},$$

where $Y_t^{Z^*}$ corresponds to the quantity on the right hand side of (28) with T replaced by t . We now investigate the behavior of this quantity, notably with respect to the benchmark $PL_t^{\tilde{\delta}^*}$ which corresponds to the optimal profit and loss without intervention of the exchange.

To make PL_t^* and $PL_t^{\tilde{\delta}^*}$ comparable, we choose Y^* in (28) so that the market maker gets the same utility in both situations, that is $\tilde{Y}_0 = \frac{k}{\sigma} \log(\tilde{u}(T, q_0))$. Thus, the market maker is indifferent between the situation with or without exchange intervention. We generate 5000 paths of market scenarios and compare the average of both P&L in Figure 6 with and without incentive policy.

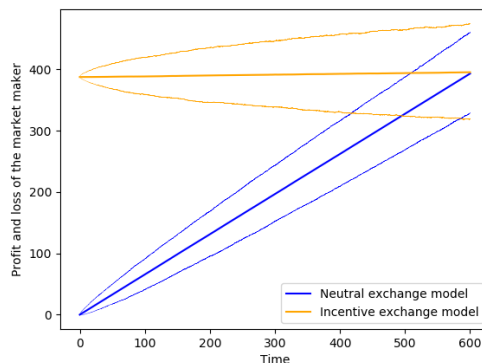


Figure 6: Average P&L of the market maker on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange.

Since \hat{Y}_0 is set to obtain the same utility in both cases, the two average P&L are very close at the end of the trading period. The variance of the P&L also seems to be the same in both situations. The only difference from the market maker viewpoint here is that in the case of a contract, the P&L is already made at time 0 thanks to the remuneration of the exchange and then fluctuates slightly. This is because he is earning the spread but paying a continuous “coupons” $H_t dt$ from the contract. In the case without exchange intervention, the market maker increases his P&L over the whole trading period thanks to the spread.

We now compare the profit and loss of the exchange in the two considered cases. When it applies an incentive policy towards the market maker, the P&L of the exchange is given by $c(N_t^a + N_t^b) - Y_t^{Z^*}$. When the exchange is neutral, its P&L is simply $c(N_t^a + N_t^b)$. We compare these two quantities in Figure 7.

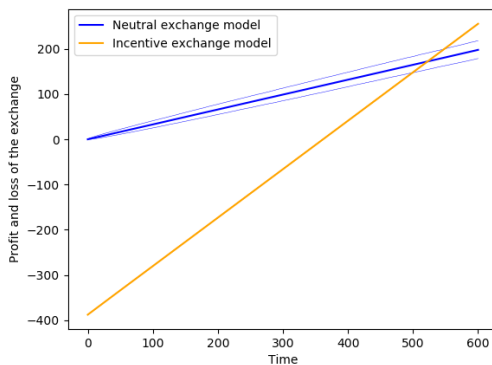


Figure 7: Average P&L of the exchange on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange.

We see that the initial P&L of the contracting exchange is negative because of the initial payment Y^* . However it finally exceeds, with a smaller standard deviation, the P&L in the situation without incentive policy from the exchange. Hence the incentive policy of the exchange proves to be successful. Indeed, both configurations are equivalent for market makers but the exchange obtains more revenues when contracting optimally. This is due to the fact that the contract triggers more market orders.

Finally, we plot the aggregated average P&L of the market maker and the exchange (independent of the choice of the initial payment). We observe that it is always greater in the optimal contract case.

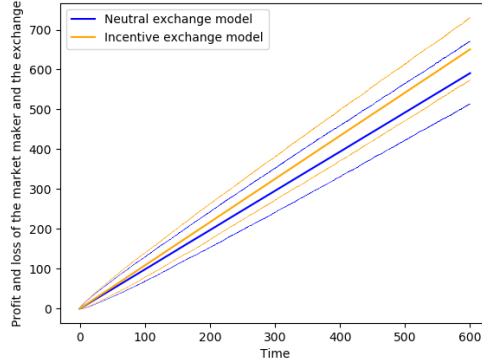


Figure 8: Average total P&L of the exchange and the market maker on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange.

5.3 Impact of the incentive policy on the trading cost

We now study the impact of the incentive policy on the investors, viewed as the market takers. We assume that there is only one market taker. In the case without exchange, with the specified parameters and under optimal reaction of the market maker, this investor buys on average 200 shares over $[0, T]$. To make the comparison with the case with exchange intervention, we modify the parameter A appearing in the intensity (3) when simulating a market with optimal contract. This new value is chosen so that the investor buys on average the same number of assets (200) over the time period. This amounts to take $A = 0.55s^{-1}$. We confirm in Figure 9 that the average ask order flows agree in both situations.

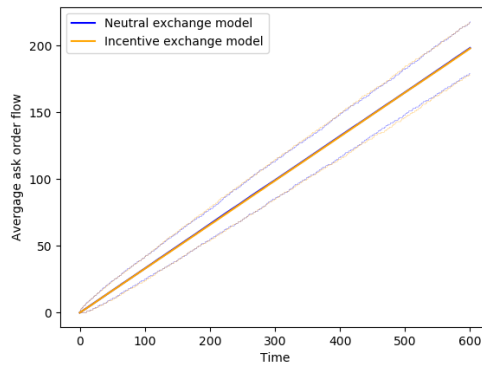


Figure 9: Setting similar average ask order flows on $[0, T]$ by taking different intensity basis A in the case with and in the case without incentive policy; 95% confidence interval.

We finally compare in Figure 10 the average cost of trading for the market taker:

$$\mathbb{E} \left[\int_0^T \delta_t^a dN_t^a \right],$$

with and without incentive.

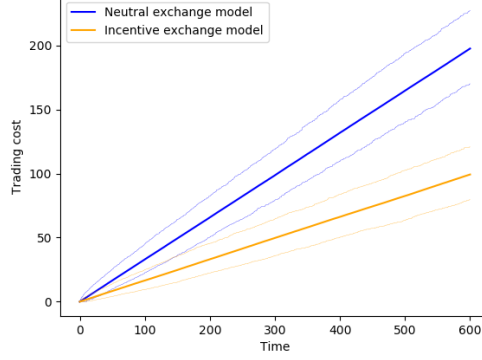


Figure 10: Average trading cost on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange.

We see that, thanks to the incentive policy of the exchange, the reduced spreads lead to significantly smaller trading costs for investors.

A Appendix

A.1 Exchange Hamiltonian maximization

Lemma A.1. *Let $c \in \mathbb{R}$, $\gamma, \eta, k, \sigma > 0$ and $v_1, v_2 < 0$. We define for $z \in \mathbb{R}$*

$$\varphi(z) = Ae^{-k\Delta(z)/\sigma} \left(v_1 e^{\eta(z-c)} - v_2 \left(\frac{\eta}{\gamma} (1 - e^{-\gamma(z+\Delta(z))}) + 1 \right) \right),$$

with $\Delta(z) = (-\delta_\infty) \vee (-z + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k})) \wedge \delta_\infty$ and $\delta_\infty > 0$. The function φ is nondecreasing on $(-\infty, -\delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k})]$ and non-increasing on $[\delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k}), \infty)$. Provided

$$\delta_\infty \geq C_\infty + \frac{1}{\eta} \left| \log\left(\frac{v_2}{v_1}\right) \right|, \quad (34)$$

with $C_\infty = |c| + (\frac{1}{\eta} + \frac{1}{\gamma}) \log(1 + \frac{\sigma\gamma}{k}) - \frac{1}{\eta} \log\left(1 - \frac{\sigma^2\gamma\eta}{(k+\sigma\gamma)(k+\sigma\eta)}\right)$. It admits a maximum on $[-\delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k}), \delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k})]$ attained in z^* given by

$$z^* = c + \frac{1}{\eta} \log(v_2/v_1) + \frac{1}{\eta} \log\left(1 - \frac{\sigma^2\gamma\eta}{(k+\sigma\gamma)(k+\sigma\eta)}\right).$$

In that case, we have

$$\varphi(z^*) = -Cv_2 \exp\left(\frac{k}{\sigma\eta} \log(v_2/v_1)\right),$$

where

$$C = A \frac{\sigma\eta}{k} \exp\left(\frac{kc}{\sigma} - \frac{k}{\sigma\gamma} \log(1 + \frac{\sigma\gamma}{k}) + (1 + \frac{k}{\sigma\eta}) \log\left(1 - \frac{\sigma^2\gamma\eta}{(k+\sigma\gamma)(k+\sigma\eta)}\right)\right).$$

Proof. Easy but tedious computations lead to prove that φ is non-decreasing on $(-\infty, -\delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k})]$ and non-increasing on $[\delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k}), \infty)$ if,

$$\delta_\infty \geq \left| c + \frac{1}{\eta} \log(v_2/v_1) - \left(\frac{1}{\eta} + \frac{1}{\gamma}\right) \log(1 + \frac{\sigma\gamma}{k}) \right|.$$

Moreover, we notice that φ admits a maximum on $[-\delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k}), \delta_\infty + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k})]$ attained in

$$z^* = c + \frac{1}{\eta} \log(v_2/v_1) + \frac{1}{\eta} \log\left(1 - \frac{\sigma^2\gamma\eta}{(k + \sigma\gamma)(k + \sigma\eta)}\right),$$

as soon as $\delta_\infty \geq | -z^* + \frac{1}{\gamma} \log(1 + \frac{\sigma\gamma}{k}) |$. By combining these two conditions, we get the result under Condition (34) on δ_∞ . \square

A.2 Dynamic programming principle and contract representation

For all \mathbb{F} -stopping time τ with values in $[t, T]$ and for any $\mu \in \mathcal{A}_\tau$, we define

$$J_T(\tau, \mu) = \mathbb{E}_\tau^\mu \left[-e^{-\gamma \int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + q_u dS_u)} e^{-\gamma\xi} \right], \quad \text{and} \quad \mathcal{J}_{\tau, T} = (J_T(\tau, \mu))_{\mu \in \mathcal{A}_\tau},$$

where \mathcal{A}_τ denotes the restriction of \mathcal{A} to controls on $[\tau, T]$. The continuation utility of the market maker is defined on all \mathbb{F} -stopping time τ by

$$V_\tau = \operatorname{ess\,sup}_{\mu \in \mathcal{A}_\tau} J_T(\tau, \mu).$$

Lemma A.2. *Let τ be an \mathbb{F} -stopping time with values in $[t, T]$. Then, there exists a non-decreasing sequence $(\mu^n)_{n \in \mathbb{N}}$ in \mathcal{A}_τ such that $V_\tau = \lim_{n \rightarrow +\infty} \uparrow J_T(\tau, \mu^n)$.*

Proof. For μ and μ' in \mathcal{A}_τ , define $\hat{\mu} = \mu \mathbf{1}_{J_T(\tau, \mu) \geq J_T(\tau, \mu')} + \mu' \mathbf{1}_{J_T(\tau, \mu) < J_T(\tau, \mu')}$. Then $\hat{\mu} \in \mathcal{A}_\tau$ and by definition of $\hat{\mu}$

$$J_T(\tau, \hat{\mu}) \geq \max(J_T(\tau, \mu), J_T(\tau, \mu')).$$

Hence $\mathcal{J}_{\tau, T}$ is directly upwards, and the required result follows from [23, Proposition VI.II p121]. \square

Lemma A.3. *Let $t \in [0, T]$, and τ an \mathbb{F} -stopping time with values in $[t, T]$. Then,*

$$V_t = \operatorname{ess\,sup}_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta \left[-e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} V_\tau \right]. \quad (35)$$

Proof. Let $0 < t < T$ and set an \mathbb{F} -stopping time τ with values in $[t, T]$. The proof is similar to [6, Proof of Proposition 6.2]. First, by the tower property,

$$V_t = \operatorname{ess\,sup}_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} \mathbb{E}_\tau^\delta \left[-e^{-\gamma \left(\int_\tau^T (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u) + \xi \right)} \right] \right].$$

For all $\delta \in \mathcal{A}$, the quotient $\frac{L_T^\delta}{L_\tau^\delta}$ does not depend on the values of δ before time τ . Then,

$$\begin{aligned} \mathbb{E}_\tau^\delta \left[-e^{-\gamma \left(\int_\tau^T (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u) + \xi \right)} \right] &= \mathbb{E}_\tau^0 \left[-\frac{L_T^\delta}{L_\tau^\delta} e^{-\gamma \left(\int_\tau^T (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u) + \xi \right)} \right] \\ &\leq \operatorname{ess\,sup}_{\mu \in \mathcal{A}_\tau} \mathbb{E}_\tau^\mu \left[-e^{-\gamma \left(\int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + q_u dS_u) + \xi \right)} \right] = V_\tau, \end{aligned}$$

Then,

$$V_t \leq \operatorname{ess\,sup}_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta \left[V_\tau e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} \right].$$

We next prove the reverse inequality. Let $\delta \in \mathcal{A}$ and $\mu \in \mathcal{A}_\tau$. We define $(\delta \otimes_\tau \mu)_u = \delta_u \mathbf{1}_{0 \leq u < \tau} + \mu_u \mathbf{1}_{\tau \leq u \leq T}$. Then, $\delta \otimes_\tau \mu \in \mathcal{A}$ and

$$\begin{aligned} V_t &\geq \mathbb{E}_t^{\delta \otimes_\tau \mu} \left[-e^{-\gamma \left(\int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u) + \int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + q_u dS_u) \right)} e^{-\gamma \xi} \right] \\ &= \mathbb{E}_t^{\delta \otimes_\tau \mu} \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} \mathbb{E}_\tau^{\delta \otimes_\tau \mu} \left[-e^{-\gamma \int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + q_u dS_u)} e^{-\gamma \xi} \right] \right]. \end{aligned} \quad (36)$$

From Bayes' Formula and by noticing that $\frac{L_T^{\delta \otimes_\tau \mu}}{L_\tau^{\delta \otimes_\tau \mu}} = \frac{L_T^\mu}{L_\tau^\mu}$, we get

$$\begin{aligned} \mathbb{E}_\tau^{\delta \otimes_\tau \mu} \left[-e^{-\gamma \int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + q_u dS_u)} e^{-\gamma \xi} \right] &= \mathbb{E}_\tau^0 \left[\frac{L_T^{\delta \otimes_\tau \mu}}{L_\tau^{\delta \otimes_\tau \mu}} \left(-e^{-\gamma \int_\tau^T (\mu_u^a dN_u^a + \mu_u^b dN_u^b + q_u dS_u)} e^{-\gamma \xi} \right) \right] \\ &= J_T(\tau, \mu). \end{aligned}$$

Thus, Inequality (36) becomes

$$V_t \geq \mathbb{E}_t^{\delta \otimes_\tau \mu} \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu) \right].$$

By using again Bayes' Formula and by noticing that $\frac{L_\tau^{\delta \otimes_\tau \mu}}{L_t^{\delta \otimes_\tau \mu}} = \frac{L_\tau^\delta}{L_t^\delta}$, we have

$$\begin{aligned} V_t &\geq \frac{\mathbb{E}_t^0 \left[L_T^{\delta \otimes_\tau \mu} e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu) \right]}{L_t^{\delta \otimes_\tau \mu}} \\ &= \mathbb{E}_t^0 \left[\mathbb{E}_\tau^0 \left[\frac{L_T^{\delta \otimes_\tau \mu}}{L_\tau^{\delta \otimes_\tau \mu}} \frac{L_\tau^{\delta \otimes_\tau \mu}}{L_t^{\delta \otimes_\tau \mu}} e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu) \right] \right] \\ &= \mathbb{E}_t^0 \left[\mathbb{E}_\tau^0 \left[\frac{L_T^{\delta \otimes_\tau \mu}}{L_\tau^{\delta \otimes_\tau \mu}} \right] \frac{L_\tau^{\delta \otimes_\tau \mu}}{L_t^{\delta \otimes_\tau \mu}} e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu) \right] \\ &= \mathbb{E}_t^0 \left[\frac{L_\tau^{\delta \otimes_\tau \mu}}{L_t^{\delta \otimes_\tau \mu}} e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu) \right] \\ &= \mathbb{E}_t^\delta \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu) \right]. \end{aligned}$$

Since the previous inequality holds for all $\mu \in \mathcal{A}_\tau$ we deduce from monotone convergence Theorem together with Lemma A.2 that there exists a sequence $(\mu^n)_{n \in \mathbb{N}}$ of control in \mathcal{A}_τ such that

$$\begin{aligned} V_t &\geq \lim_{n \rightarrow +\infty} \uparrow \mathbb{E}_t^\delta \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} J_T(\tau, \mu^n) \right] \\ &= \mathbb{E}_t^\delta \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} \lim_{n \rightarrow +\infty} \uparrow J_T(\tau, \mu^n) \right] = \mathbb{E}_t^\delta \left[e^{-\gamma \int_t^\tau (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)} V_\tau \right], \end{aligned}$$

thus concluding the proof. \square

Proof of Theorem 3.1 (i) We proceed in several steps.

Step 1. For $\delta \in \mathcal{A}$, it follows from the dynamic programming principle of Lemma A.3 that the process

$$U_t^\delta = V_t e^{-\gamma \int_0^t (\delta_u^a dN_u^a + \delta_u^b dN_u^b + q_u dS_u)}, \quad t \in [0, T], \quad (37)$$

defines a \mathbb{P}^δ -supermartingale³ for all $\delta \in \mathcal{A}$. By standard analysis, we may then consider it in its càdlàg version (by taking right limits along rationals). By the Doob-Meyer decomposition, we can write

$$U_t^\delta = M_t^\delta - A_t^{\delta,c} - A_t^{\delta,d}, \quad (38)$$

where M^δ is a \mathbb{P}^δ -martingale and $A^\delta = A^{\delta,c} + A^{\delta,d}$ is an integrable non-decreasing predictable process such that $A_0^{\delta,c} = A_0^{\delta,d} = 0$, with pathwise continuous component $A^{\delta,c}$, and a piecewise constant predictable process $A^{\delta,d}$.

By the martingale representation theorem under \mathbb{P}^δ , see Lemma A.5, there exists a predictable process $\tilde{Z}^\delta = (\tilde{Z}^{\delta,S}, \tilde{Z}^{\delta,a}, \tilde{Z}^{\delta,b})$ such that

$$M_t^\delta = V_0 + \int_0^t \tilde{Z}_r^\delta \cdot d\chi_r - \int_0^t \tilde{Z}_r^{\delta,a} \lambda(\delta_r^a) 1_{\{Q_r > -\bar{q}\}} dr - \int_0^t \tilde{Z}_r^{\delta,b} \lambda(\delta_r^b) 1_{\{Q_r < \bar{q}\}} dr, \quad (39)$$

where we recall that $\chi = (S, N^a, N^b)$.

Step 2. We show that V is a negative process. In fact, thanks to the uniform boundedness of δ , we show that

$$\frac{L_T^\delta}{L_t^\delta} \geq \alpha_{t,T} = e^{-\frac{k\delta_\infty}{\sigma}(N_T^a - N_t^a + N_T^b - N_t^b) - A(e^{\frac{k\delta_\infty}{\sigma}} + 1)(T-t)} > 0. \quad (40)$$

Therefore,

$$V_t \leq \mathbb{E}^0 \left[-\alpha_{t,T} e^{-\gamma\delta_\infty(N_T^a - N_t^a + N_T^b - N_t^b)e^{-\gamma\xi}} \right] < 0.$$

Step 3. Let Y be the process defined by $V_t = -e^{-\gamma Y_t}$ for all $t \in [0, T]$. As $A^{\delta,d}$ is a predictable point process and the jumps of (N^a, N^b) are totally inaccessible stopping times under \mathbb{P}^0 , we have $[N^a, A^{\delta,d}] = 0$ and $[N^b, A^{\delta,d}] = 0$ a.s, see Proposition I.2.24 in [15]. Using Itô's formula, we obtain from (38) and (39) that

$$Y_T = \xi, \quad \text{and} \quad dY_t = Z_t^a dN_t^a + Z_t^b dN_t^b + Z_t^S dS_t - dI_t - d\tilde{A}_t^d \quad (41)$$

where $Z^a, Z^b, Z^S, I, \tilde{A}^d$ are independent of δ , as they may be expressed as $Z_t^i dN_t^i = d[Y, N^i]_t$, $i \in \{a, b\}$, $Z_t^S \sigma^2 dt = d\langle Y_t, S_t \rangle_t$, \tilde{A}^d the predictable pure jumps of Y . **Moreover, Itô's Formula yields**

$$Z_t^a = -\frac{1}{\gamma} \log\left(1 + \frac{\tilde{Z}_t^{\delta,a}}{U_{t-}^\delta}\right) - \delta_t^a, \quad Z_t^b = -\frac{1}{\gamma} \log\left(1 + \frac{\tilde{Z}_t^{\delta,b}}{U_{t-}^\delta}\right) - \delta_t^b, \quad Z_t^S = -\frac{\tilde{Z}_t^{\delta,b}}{\gamma U_{t-}^\delta} - Q_t,$$

and

$$I_t = \int_0^t \bar{h}(\delta_r, Z_r, Q_r) dr - \frac{1}{\gamma U_r^\delta} dA_r^{\delta,c}, \quad \tilde{A}_t^d = \sum_{s \leq t} \log\left(1 - \frac{\Delta A_s^{\delta,d}}{U_{s-}^\delta}\right),$$

with $\bar{h}(\delta, z, q) = h(\delta, z, q) - \frac{1}{2}\gamma\sigma^2(z^s)^2$. In particular, the last relation between \tilde{A}^d and $A^{\delta,d}$ shows that $\Delta a_t = \frac{-\Delta A_t^{\delta,d}}{U_{t-}^\delta} \geq 0$ is independent of δ ; recall that $U^\delta < 0$.

In order to complete the proof, we argue in the subsequent steps that

$$A_t^{\delta,d} = -\sum_{s \leq t} U_{s-}^\delta \Delta a_s = 0, \quad (\text{so that } \tilde{A}_t^d = 0), \quad \text{and} \quad I_t = \int_0^t \bar{H}(Z_r, Q_r), \quad t \in [0, T], \quad (42)$$

³Note that $\mathbb{E}^\delta[U_T^\delta] = J_T(0, \delta) > -\infty$ using Hölder inequality together with (11) and the uniform boundedness of the intensities of N^a and N^b . Hence the process U^δ is integrable.

where $\bar{H}(z, q) = H(z, q) - \frac{1}{2}\gamma\sigma^2(z^s)^2$.

Step 4. Since $V_T = -1$, notice that

$$\begin{aligned} 0 &= \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta[U_T^\delta] - V_0 = \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta[U_T^\delta - M_T^\delta] \\ &= \gamma \sup_{\delta \in \mathcal{A}} \mathbb{E}^0 \left[L_T^\delta \int_0^T U_r^\delta (dI_r - \bar{h}(\delta_r, Z_r, Q_r) dr - \frac{da_r}{\gamma}) \right]. \end{aligned} \quad (43)$$

Moreover, since the controls are uniformly bounded, we have

$$U_t^\delta \leq -\beta_t = V_t e^{-\gamma \delta \infty (N_T^a + N_T^b) - \gamma \int_0^t q_r dS_r} < 0. \quad (44)$$

Then, since $A^{\delta, d} \geq 0$, $U^\delta \leq 0$, and $dI - \bar{h}(\delta, Z, Q) \leq 0$, it follows from (43) together with the inequalities (40) and (44),

$$0 \geq \sup_{\delta \in \mathcal{A}} \mathbb{E}^0 \left[\alpha_{0, T} \int_0^T \beta_r (-dI_r + \bar{h}(\delta_r, Z_r, Q_r) dr + da_r) \right] = \mathbb{E}^0 \left[\alpha_{0, T} \int_0^T \beta_r (-dI_r + \bar{H}(Z_r, Q_r) dr + da_r) \right],$$

which implies (42), \mathbb{P}^0 -a.s.

Step 5. We now prove that $Z \in \mathcal{Z}$ by showing that

$$\sup_{\delta \in \mathcal{A}} \sup_{t \in [0, T]} \mathbb{E}^\delta [e^{-\gamma(p+1)Y_t}] < \infty \quad \text{for some } p > 0 \quad (45)$$

Using Hölder inequality together with Condition (11) and the boundedness of the intensities of N^a and N^b , we have that $\sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta [|\int_0^T U_T^\delta|^{p'+1}] < \infty$ for some $p' > 0$. Hence

$$\sup_{\delta \in \mathcal{A}} \sup_{t \in [0, T]} \mathbb{E}^\delta [|\int_0^t U_t^\delta|^{p'+1}] = \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta [|\int_0^T U_T^\delta|^{p'+1}] < \infty,$$

because U^δ is a negative P^δ -supermartingale. This leads to (45) using Hölder inequality the uniform boundedness of the intensities of N^a and N^b and that $e^{-\gamma Y} = U^\delta e^{\gamma \int_0^\cdot \delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u}$.

Step 6. We finally prove uniqueness of the representation. Let $(Y_0, Z), (Y'_0, Z') \in \mathbb{R} \times \mathcal{Z}$ be such that $\xi = Y_T^{Y_0, Z} = Y_T^{Y'_0, Z'}$. By following the line of the verification argument in the proof of Theorem 3.1 (ii), we obtain the equality $Y_t^{Y_0, Z} = Y_t^{Y'_0, Z'}$ by considering the value of the continuation utility of the market maker

$$-\exp(-\gamma Y_t^{Y_0, Z}) = -\exp(-\gamma Y_t^{Y'_0, Z'}) = \operatorname{ess\,sup}_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta [-e^{-\gamma(\text{PL}_T^\delta - \text{PL}_t^\delta + \xi)}], \quad t \in [0, T].$$

This in turn implies that $Z_t^i dN_t^i = Z_t'^i dN_t^i = d[Y^{Y_0, Z}, \Delta N^i]_t$, $i \in \{a, b\}$, and $Z_t^S \sigma^2 dt = Z_t'^S \sigma^2 dt = d\langle Y, S \rangle_t$, $t \in [0, T]$. Hence $(Y_0, Z) = (Y'_0, Z')$.

A.3 On the verification argument for the exchange problem

The proof of the main result of Theorem 4.1 requires the following technical result. We observe that this is the place where Condition (10) is needed.

Lemma A.4. *Let $Z \in \mathcal{Z}$. There exists $C > 0$ and $\varepsilon > 0$ such that*

$$\sup_{t \in [0, T]} \mathbb{E}^{\delta(Y_T^Z)} [|\int_0^t K_t^Z|^{1+\varepsilon}] \leq C.$$

Proof. We recall the definition of K^Z for $Z \in \mathcal{Z}$

$$K_t^Z = e^{-\eta(c(N_t^a + N_t^b) - Y_t^{0,Z})}, \quad t \in [0, T].$$

Let $p > 1$. By using Hölder's Inequality and the uniform boundedness of the intensities of N^a and N^b , we deduce that there exists $C' > 0$ such that

$$\mathbb{E}^{\hat{\delta}(Y_T^Z)}[|K_t^Z|^p] \leq C' \mathbb{E}^0[(e^{-\gamma Y_t^{0,Z}})^{-\frac{p'}{\gamma}}]_{\frac{p'}{p}},$$

with any $p' > p$. Thus,

$$\begin{aligned} \mathbb{E}^{\hat{\delta}(Y_T^Z)}[|K_t^Z|^p] &\leq C' \left(1 + \mathbb{E}^0[(e^{-\gamma Y_t^{0,Z}})^{-\frac{p'}{\gamma}}] \right) \\ &= C' \left(1 + \mathbb{E}^0 \left[\left(-\sup_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta [-e^{-\gamma(Y_T^Z + PL_T^\delta - PL_t^\delta)}] \right)^{-\frac{p'}{\gamma}} \right] \right). \end{aligned}$$

From Jensen's Inequality then Hölder's inequality, we deduce that for any $p'' > p'$ we have

$$\begin{aligned} \mathbb{E}^{\hat{\delta}(Y_T^Z)}[|K_t^Z|^p] &\leq C' \left(1 + \mathbb{E}^0 \left[\sup_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta [e^{p'\eta(Y_T^Z + PL_T^\delta - PL_t^\delta)}] \right] \right) \\ &\leq C' \left(1 + \mathbb{E}^0 \left[\sup_{\delta \in \mathcal{A}} \mathbb{E}_t^\delta [e^{p''\eta Y_T^Z}] \right] \right). \end{aligned}$$

By using a dynamic programming principle, similarly to the proof of Lemma A.3 by noticing that the family $\left(\tilde{J}(\mu, \delta) = \mathbb{E}_\tau^\delta [e^{p''\eta Y_T^Z}] \right)_{\mu \in \mathcal{A}_\tau}$ is directly upwards, we get

$$\mathbb{E}^{\hat{\delta}(Y_T^Z)}[|K_t^Z|^p] \leq C' \left(1 + \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta [e^{p''\eta Y_T^Z}] \right).$$

By setting $\varepsilon = \frac{\eta' - \eta}{3}$, if we take $p = 1 + \varepsilon$, then $p' = p + \varepsilon$ and $p'' = p' + \varepsilon$, we obtain

$$\mathbb{E}^{\hat{\delta}(Y_T^Z)}[|K_t^Z|^{1+\varepsilon}] \leq C' \left(1 + \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta [e^{\eta' Y_T^Z}] \right).$$

From the definition of \mathcal{Z} (involving the condition (10)), we get for any $t \in [0, T]$

$$\mathbb{E}^{\hat{\delta}(Y_T^Z)}[|K_t^Z|^{1+\varepsilon}] \leq C,$$

with $C = C' \left(1 + \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta [e^{\eta' Y_T^Z}] \right) < +\infty$. □

A.4 Predictable representation

The following result is probably well-known, we report it here for completeness as we could not find a precise reference.

Lemma A.5. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space where $\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^N$ is the right continuous completed filtration of a Brownian motion W and a d -dimensional integrable point process $N = (N^1, \dots, N^d)$ with compensator $A = (A^1, \dots, A^d)$. Then, for any \mathbb{F} -martingale X there exists a predictable process $Z = (Z^W, Z^1, \dots, Z^d)$ such that*

$$X_t = X_0 + \int_0^t Z_s^W dW_s + \sum_{i=1}^d \int_0^t Z_s^i (dN_s^i - dA_s^i).$$

Proof. For sake of simplicity, we take $d = 1$. Let \mathbb{P} be a solution of the martingale problem associated to $M_t = N_t - A_t$ and W_t . By Theorem III.4.29 in [15], to prove Lemma A.5 we need to establish the uniqueness of \mathbb{P} .

We denote by \mathbb{P}^W the law \mathbb{P} conditional on W . We first show that M is still a martingale under \mathbb{P}^W . To do so we consider $B_s \in \mathcal{F}_s$ and want to prove that

$$\mathbb{E}^{\mathbb{P}^W} [\mathbb{1}_{B_s}(M_t - M_s)] = 0,$$

for $0 \leq s \leq t \leq T$. Let $C \in \mathcal{F}_T^W$. We aim at showing that

$$\mathbb{E} \left[\mathbb{1}_C \mathbb{E}^{\mathbb{P}^W} [\mathbb{1}_{B_s}(M_t - M_s)] \right] = \mathbb{E} [\mathbb{1}_C \mathbb{1}_{B_s}(M_t - M_s)] = 0.$$

Thanks to the martingale representation theorem for Brownian martingales, we can write

$$\mathbb{1}_C = \alpha_s + \int_s^T \phi_u dW_u,$$

where $\alpha_s = \mathbb{E}[\mathbb{1}_C | \mathcal{F}_s^W]$ and $(\phi_u)_{u \geq 0}$ is \mathbb{F}^W predictable process. Using the martingale property of M , we obtain

$$\mathbb{E} [\alpha_s \mathbb{1}_{B_s}(M_t - M_s)] = 0.$$

Then W and M being orthogonal martingales, we deduce

$$\mathbb{E} \left[\int_s^T \phi_u dW_u \mathbb{1}_{B_s}(M_t - M_s) \right] = 0,$$

Consequently, using Theorem III.1.21 in [15], \mathbb{P}^W is the unique probability measure such that M is an \mathbb{F} -martingale conditional on W . Finally, by integration, the uniqueness of \mathbb{P}^W implies that of \mathbb{P} . \square

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