On the monotonicity principle
of optimal Skorokhod embedding problem [*]

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Abstract

This is a continuation of our accompanying paper [18]. We provide an alternative proof of the monotonicity principle for the *optimal Skorokhod embedding problem* established in Beiglböck, Cox and Huesmann [2]. Our proof is based on the adaptation of the Monge-Kantorovich duality in our context, a delicate application of the optional cross-section theorem, and a clever conditioning argument introduced in [2].

Key words. Optimal Skorokhod embedding, stop-go pair, monotonicity principle.

Mathematics Subject Classification (2010). 60H30, 60G05.

1   Introduction

The Skorokhod embedding problem (SEP) consists in constructing a Brownian motion W and a stopping time τ so that W_τ has some given distribution. Among the numerous solutions of the SEP which appeared in the existing literature, some embeddings enjoy an optimality property w.r.t. some criterion. For instance, the Azéma-Yor solution [1] maximizes the expected running maximum, the Root solution [27] was shown by Rost [28] to minimize the expectation of the embedding stopping time.

Recently, Beiglböck, Cox & Huesmann [2] approached this problem by introducing the optimal SEP for some given general criterion. Their main result provides a dual formulation in the spirit of optimal transport theory, and a monotonicity principle characterizing optimal embedding stopping times. This remarkable result allows to recover all known embeddings which enjoy an optimality property, and provides a concrete method to derive new embeddings with such optimality property.

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Our main interest in this note is to provide an alternative proof of the last monotonicity principle, based on a duality result. Our argument follows the classical proof of the monotonicity principle for classical optimal transport problem, see Villani [31, Chapter 5] and the corresponding adaptation by Zaev [32, Theorem 3.6] for the derivation of the martingale monotonicity principle of Beiglböck & Juillet [5]. The present continuous-time setting raises however serious technical problems which we overcome in this paper by a crucial use of the optional cross section theorem.

In the recent literature, there is an important interest in the SEP and the corresponding optimality properties. This revival is mainly motivated by its connection to the model-free hedging problem in financial mathematics, as initiated by Hobson [22], and continued by many authors [9, 10, 11, 13, 18, 24, 25, 26], etc.

Finally, we emphasize that the connection between the model-free hedging problem and the optimal transport theory was introduced simultaneously by Beiglböck, Henry-Labordère & Penkner [4] in discrete-time, and Galichon, Henry-Labordère & Touzi [17] in continuous-time. We also refer to the subsequent literature on martingale optimal transport by [6, 7, 8, 14, 15, 19, 20, 21, 23, 24], etc.

In the rest of the paper, we formulate the monotonicity principle in Section 2, and then provide our proof in Section 3.

2 Monotonicity principle of optimal Skorokhod embedding problem

2.1 Preliminaries

Let $\Omega \subset C(\mathbb{R}_+, \mathbb{R})$ be the canonical space of all continuous functions ω on $\mathbb{R}_+$ such that $\omega_0 = 0$, B denote the canonical process and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the canonical filtration generated by B. Notice that Ω is a Polish space under the compact convergence topology, and its Borel σ−field is given by $\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$. Denote by $\mathcal{P}(\Omega)$ the space of all (Borel) probability measures on Ω and by $\mathbb{P}_0 \in \mathcal{P}(\Omega)$ the Wiener measure on Ω, under which B is a Brownian motion.

We next introduce an enlarged canonical space $\overline{\Omega} := \Omega \times \mathbb{R}_+$, equipped with canonical element $\overline{B} := (B, T)$ defined by

$$B(\bar{\omega}) := \omega \ \text{ and } \ T(\bar{\omega}) := \theta, \ \text{ for all } \ \bar{\omega} = (\omega, \theta) \in \overline{\Omega},$$

and the canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \geq 0}$ defined by

$$\overline{\mathcal{F}}_t \ := \ \sigma(B_u, u \leq t) \vee \sigma\big(\{T \leq u\}, u \leq t\big),$$

so that the canonical variable T is a $\overline{\mathbb{F}}$−stopping time. In particular, we have the σ−field $\overline{\mathcal{F}}_T$ on $\overline{\Omega}$. Define also $\overline{\mathcal{F}}^0 := \sigma(B_t, t \geq 0)$ as σ−field on $\overline{\Omega}$. Under the product topology, $\overline{\Omega}$ is still a Polish space, and its Borel σ−field is given by $\overline{\mathcal{F}} := \bigvee_{t \geq 0} \overline{\mathcal{F}}_t$. Similarly, we denote by $\overline{\mathcal{P}}(\overline{\Omega})$ the set of all (Borel) probability measures on $\overline{\Omega}$.

Next, for every $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$ and $t \in \mathbb{R}_+$, we define the stopped path by $\omega_{t \wedge \cdot} := \big(\omega_{t \wedge u}\big)_{u \geq 0}$ and $\bar{\omega}_{t \wedge \cdot} := (\omega_{t \wedge \cdot}, t \wedge \theta)$. For every $\bar{\omega} = (\omega, \theta), \bar{\omega}' = (\omega', \theta') \in \overline{\Omega}$, we define the concatenation $\bar{\omega} \otimes \bar{\omega}' \in \overline{\Omega}$ by

$$\bar{\omega} \otimes \bar{\omega}' \ := \ (\omega \otimes_\theta \omega', \theta + \theta'),$$

where

$$\left(\omega \otimes_\theta \omega'\right)_t \ := \ \omega_t \mathbf{1}_{[0,\theta)}(t) \ + \ \left(\omega_\theta + \omega'_{t-\theta}\right)\mathbf{1}_{[\theta,+\infty)}(t), \text{ for all } t \in \mathbb{R}_+.$$

Let $\xi : \overline{\Omega} \to \mathbb{R}$ be a random variable, and $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\overline{\Omega})$, we defined the expectation $\mathbb{E}[\xi] := \mathbb{E}[\xi^+] - \mathbb{E}[\xi^-]$, by the convention $\infty - \infty = -\infty$.

2.2 The optimal Skorokhod embedding problem

We now introduce an optimal Skorokhod embedding problem and its dual problem. Let μ be a centered probability measure on $\mathbb{R}$, i.e. admitting first order moment and with zero mean, we then introduce the set of all embeddings by

$$\overline{\mathcal{P}}^0(\mu) \ := \ \left\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^0 : B_T \sim^{\overline{\mathbb{P}}} \mu\right\},$$

with

$$\overline{\mathcal{P}}^0 \ := \ \left\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\overline{\Omega}) \ : \ B \text{ is } \overline{\mathbb{F}} - \text{Brownian motion and}\right.$$
$$\left. B_{T\wedge\cdot} \text{ is uniformly integrable under } \overline{\mathbb{P}}\right\}. \quad (2.1)$$

Let $\xi : \overline{\Omega} \longrightarrow \mathbb{R}$ be some (Borel) measurable map, we then define the optimal Skorokhod embedding problem (with respect to μ and ξ) by

$$P(\mu) \ := \ \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^0(\mu)} \mathbb{E}^{\overline{\mathbb{P}}}[\xi]. \quad (2.2)$$

We next introduce a dual formulation of the above Skorokhod embedding problem (2.2). Let Λ denote the space of all continuous functions $\lambda : \mathbb{R} \to \mathbb{R}$ of linear growth, and define for every $\lambda \in \Lambda$,

$$\mu(\lambda) \ := \ \int_{\mathbb{R}} \lambda(x)\mu(dx).$$

Define further

$$\mathcal{D} \ := \ \left\{(\lambda, S) \in \Lambda \times \mathcal{S} \ : \lambda(\omega_t) + S_t(\omega) \ \geq \ \xi(\omega, t), \text{ for all } t \geq 0, \ \mathbb{P}_0 - \text{a.s.}\right\},$$

where $\mathcal{S}$ denotes the collection of all $\mathbb{F}-$strong càdlàg supermartingales $S : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P}_0)$ such that $S_0 = 0$ and for some $L > 0$,

$$\left|S_t(\omega)\right| \ \leq \ L(1 + |\omega_t|), \text{ for all } \omega \in \Omega, t \in \mathbb{R}_+. \quad (2.3)$$

Then the dual problem is given by

$$D(\mu) \ := \ \inf_{(\lambda, S) \in \mathcal{D}} \mu(\lambda). \quad (2.4)$$

Remark 2.1. *By the Doob-Meyer decomposition together with the martingale representation w.r.t. the Brownian filtration, there is some $\mathbb{F}-$predictable process $H : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and non-increasing $\mathbb{F}-$predictable process A ($A_0 = 0$) such that $S_t = (H \cdot B)_t - A_t$ for all $t \geq 0$, $\mathbb{P}_0-$a.s., where $(H \cdot B)$ denotes the stochastic integral of H w.r.t B under $\mathbb{P}_0$. We then have another dual formulation, by replacing $\mathcal{D}$ with*

$$\mathcal{D}' \ := \ \left\{(\lambda, H) \ : \lambda(\omega_t) + (H \cdot B)_t(\omega) \ \geq \ \xi(\omega, t), \text{ for all } t \geq 0, \ \mathbb{P}_0 - \text{a.s.}\right\}.$$

Here, we use the formulation in terms of the set $\mathcal{D}$ for ease of presentation.

3

2.3 The monotonicity principle

We now introduce the monotonicity principle formulated and proved in Beiglböck, Cox and Huesmann [2], which provides a geometric characterization of the optimal embedding of problem (2.2) in terms of its support.

Let $\overline{\Gamma} \subseteq \overline{\Omega}$ be a subset, we define $\overline{\Gamma}^<$ by

$$\overline{\Gamma}^< := \left\{ \bar{\omega} = (\omega, \theta) \in \overline{\Omega} \; : \bar{\omega} = \bar{\omega}'_{\theta \wedge \cdot} \text{ for some } \bar{\omega}' \in \overline{\Gamma} \text{ with } \theta' > \theta \right\}.$$

Definition 2.2. *A pair $(\bar{\omega}, \bar{\omega}') \in \overline{\Omega} \times \overline{\Omega}$ is said to be a stop-go pair if $\omega_\theta = \omega'_{\theta'}$ and*

$$\xi(\bar{\omega}) + \xi(\bar{\omega}' \otimes \bar{\omega}'') \; > \; \xi(\bar{\omega} \otimes \bar{\omega}'') + \xi(\bar{\omega}') \quad \text{for all} \quad \bar{\omega}'' \in \overline{\Omega}^+,$$

where $\overline{\Omega}^+ := \left\{ \bar{\omega} = (\omega, \theta) \in \overline{\Omega} : \theta > 0 \right\}$. Denote by SG *the set of all stop-go pairs.*

The following monotonicity principle was introduced and proved in [2].

Theorem 2.3. *Let μ be a centered probability measure on $\mathbb{R}$. Suppose that the optimal Skorokhod embedding problem (2.2) admits an optimizer $\overline{\mathbb{P}}^* \in \overline{\mathcal{P}}^0(\mu)$, and the duality $P(\mu) = D(\mu)$ holds true. Then there exists a Borel subset $\overline{\Gamma}^* \subseteq \overline{\Omega}$ such that*

$$\overline{\mathbb{P}}^*\big[\overline{\Gamma}^*\big] = 1 \quad \text{and} \quad \text{SG} \, \cap \, \big(\overline{\Gamma}^{*<} \times \overline{\Gamma}^*\big) \; = \; \emptyset. \tag{2.5}$$

Remark 2.4. Suppose that ξ is bounded from above, and for all $\omega \in \Omega$, the map $t \mapsto \xi(\omega, t)$ is upper semicontinuous and right-continuous, then the conditions in Theorem 2.3 are satisfied (see e.g. Theorem 2.4 and Proposition 4.11 of [18], and also [2, 3] for a slightly different formulation).

3 Proof of Theorem 2.3

Throughout this section, we fix an optimizer $\overline{\mathbb{P}}^*$ of problem (2.2) in the context of Theorem 2.3.

3.1 An enlarged stop-go set

Notice that by Definition 2.2, the set SG is a universally measurable set (co-analytic set more precisely), but not a Borel set a priori. To overcome some measurability difficulty, we will consider another Borel set $\text{SG}^* \subset \overline{\Omega} \times \overline{\Omega}$ such that $\text{SG}^* \supset \text{SG}$, as in [2].

Recall that $\overline{\mathbb{P}}^*$ is a fixed optimizer of the problem (2.2), then it admits a family of regular probability probability distribution (r.c.p.d. see e.g. Stroock and Varadhan [29]) $(\overline{\mathbb{P}}^*_{\bar{\omega}})_{\bar{\omega} \in \overline{\Omega}}$ w.r.t. $\overline{\mathcal{F}}^0 := \sigma(B_t, t \geq 0)$ on $\overline{\Omega}$. Notice that for $\bar{\omega} = (\omega, \theta)$, the measure $\overline{\mathbb{P}}^*_{\bar{\omega}}$ is independent of θ, we will denote this family by $(\overline{\mathbb{P}}^*_\omega)_{\omega \in \Omega}$. In particular, one has $\overline{\mathbb{P}}^*_\omega[B_\cdot = \omega] = 1$ for all $\omega \in \Omega$. Next, for every $\bar{\omega} \in \overline{\Omega}$, define a probability $\overline{\mathbb{Q}}^1_{\bar{\omega}}$ on $(\overline{\Omega}, \overline{\mathcal{F}})$ by

$$\overline{\mathbb{Q}}^1_{\bar{\omega}}[\overline{A}] \; := \; \int_\Omega \overline{\mathbb{P}}^*_{\omega \otimes_\theta \omega'}(\overline{A}) \, \mathbb{P}_0(d\omega'), \quad \text{for all} \quad \overline{A} \in \overline{\mathcal{F}}. \tag{3.6}$$

Intuitively, $\overline{\mathbb{Q}}^1_{\bar\omega}$ is the conditional probability w.r.t the event $\{B_{\cdot\wedge\theta} = \omega_{\cdot\wedge\theta}\}$. We next define, for every $\bar\omega \in \overline{\Omega}$, a probability $\overline{\mathbb{Q}}^2_{\bar\omega}$ by

$$\overline{\mathbb{Q}}^2_{\bar\omega}[\overline{A}] := \overline{\mathbb{Q}}^1_{\bar\omega}[\overline{A}|T>\theta]\mathbf{1}_{\{\overline{\mathbb{Q}}^1_{\bar\omega}[T>\theta]>0\}} + \mathbb{P}^{\theta,\omega}_0 \otimes \delta_{\{\theta\}}[\overline{A}]\mathbf{1}_{\{\overline{\mathbb{Q}}^1_{\bar\omega}[T>\theta]=0\}}, \qquad (3.7)$$

for all $\overline{A} \in \overline{\mathcal{F}}$, where $\mathbb{P}^{t,\omega}_0$ is the shifted Wiener measure on $(\Omega,\mathcal{F})$ defined by

$$\mathbb{P}^{t,\omega}_0[A] := \mathbb{P}_0[\omega \otimes_t B \in A], \quad \text{for all } A \in \mathcal{F}.$$

We finally introduce a shifted probability $\overline{\mathbb{Q}}^*_{\bar\omega}$ by

$$\overline{\mathbb{Q}}^*_{\bar\omega}[\overline{A}] := \overline{\mathbb{Q}}^2_{\bar\omega}[\bar\omega \otimes B \in \overline{A}], \quad \text{for all } \overline{A} \in \overline{\mathcal{F}}.$$

and then define a new set $\mathrm{SG}^* \supset \mathrm{SG}$ by

$$\mathrm{SG}^* := \big\{ (\bar\omega, \bar\omega') \ : \omega_\theta = \omega'_{\theta'}, \xi(\bar\omega) + \mathbb{E}^{\overline{\mathbb{Q}}^*_{\bar\omega}}[\xi(\bar\omega' \otimes \cdot)] > \mathbb{E}^{\overline{\mathbb{Q}}^*_{\bar\omega}}[\xi(\bar\omega \otimes \cdot)] + \xi(\bar\omega') \big\}. \quad (3.8)$$

Lemma 3.1. (i) *The set $\mathrm{SG}^* \subset \overline{\Omega} \times \overline{\Omega}$ defined by (3.8) is $\overline{\mathcal{F}}_T \otimes \overline{\mathcal{F}}_T-$measurable.*
(ii) *Let $\tau \leq T$ be a $\overline{\mathbb{F}}-$stopping time, then the family $(\widehat{\mathbb{P}}_{\bar\omega})_{\bar\omega\in\overline{\Omega}}$ defined by*

$$\widehat{\mathbb{P}}_{\bar\omega} := \mathbf{1}_{\{\tau(\bar\omega)<\theta\}}\overline{\mathbb{Q}}^2_{(\omega,\tau(\bar\omega))} + \mathbf{1}_{\{\tau(\bar\omega)=\theta\}}\mathbb{P}^{\tau(\bar\omega),\omega}_0 \otimes \delta_{\{\theta\}}$$

is a family of conditional probability measures of $\overline{\mathbb{P}}^*$ w.r.t. $\overline{\mathcal{F}}_\tau$, i.e. $\bar\omega \mapsto \widehat{\mathbb{P}}_{\bar\omega}$ is $\overline{\mathcal{F}}_\tau-$measurable, and for all bounded $\overline{\mathcal{F}}-$measurable random variable ζ, one has $\mathbb{E}^{\overline{\mathbb{P}}^*}[\zeta|\overline{\mathcal{F}}_\tau](\bar\omega) = \widehat{\mathbb{P}}_{\bar\omega}[\zeta]$ for $\overline{\mathbb{P}}^*-a.e.$ $\bar\omega \in \overline{\Omega}$.

Proof. (i) Let us denote $[\omega]_t := \omega_{t\wedge\cdot}, [\theta]_t := \theta\mathbf{1}_{\{\theta\leq t\}}+\infty\mathbf{1}_{\{\theta>t\}}$ and $[\bar\omega]_t := ([\omega]_t, [\theta]_t)$. Then by Lemma A.2 of [18], a process $Y : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R}$ is $\overline{\mathbb{F}}-$optional if and only if it is $\mathcal{B}(\mathbb{R}_+) \otimes \overline{\mathcal{F}}-$measurable, and $Y_t(\bar\omega) = Y_t([\bar\omega]_t)$. Further, using Theorem IV-64 of Dellacherie and Meyer [12, Page 122], it follows that a random variable X is $\overline{\mathcal{F}}_T-$measurable if and only if it is $\overline{\mathcal{F}}-$measurable and $X(\bar\omega) = X([\omega]_\theta, \theta)$ for all $\bar\omega \in \overline{\Omega}$.

Next, by the definition of $\overline{\mathbb{Q}}^1_{\bar\omega}, \overline{\mathbb{Q}}^2_{\bar\omega}$ and $\overline{\mathbb{Q}}^*_{\bar\omega}$, it is easy to see that $\bar\omega \mapsto \big(\overline{\mathbb{Q}}^1_{\bar\omega}, \overline{\mathbb{Q}}^2_{\bar\omega}, \overline{\mathbb{Q}}^*_{\bar\omega}\big)$ are all $\overline{\mathcal{F}}-$measurable and satisfies $\overline{\mathbb{Q}}^*_{\bar\omega} = \overline{\mathbb{Q}}^*_{[\omega]_\theta,\theta}$ for all $\bar\omega \in \overline{\Omega}$. Then it follows that $\bar\omega \mapsto \overline{\mathbb{Q}}^*_{\bar\omega}$ is $\overline{\mathcal{F}}_T-$measurable, and hence by its definition in (3.8), SG^* is $\overline{\mathcal{F}}_T \otimes \overline{\mathcal{F}}_T-$measurable.

(ii) Let $\tau \leq T$ be a $\overline{\mathbb{F}}-$stopping time, we claim that $\mathbb{F}-$stopping time τ_0 on $(\Omega,\mathcal{F})$ such that

$$\text{there is some } \mathbb{F} - \text{stopping time } \tau_0 \text{ on } (\Omega,\mathcal{F}), \text{ s.t. } \tau(\bar\omega) = \tau_0(\omega) \wedge \theta. \quad (3.9)$$

Moreover, again by Theorem IV-64 of [12], we have

$$\overline{\mathcal{F}}_\tau = \sigma\big(B_{\tau\wedge t}, \ t \geq 0\big) \vee \sigma\big(T\mathbf{1}_{\{\tau=T\}}, \{\tau < T\}\big). \quad (3.10)$$

In the next of the proof, we will consider two sets $\{\tau = T\}$ and $\{\tau < T\}$ separately.

Let $(\widehat{\mathbb{P}}^0_{\bar\omega})_{\bar\omega\in\overline{\Omega}}$ be a family of regular conditional probability distribution (r.c.p.d. see e.g. Stroock and Varadhan [29]) of $\overline{\mathbb{P}}^*$ w.r.t. $\overline{\mathcal{F}}_\tau$, which implies that

$$\widehat{\mathbb{P}}^0_{\bar\omega}\big[B_{\tau\wedge\cdot} = \omega_{\tau(\bar\omega)\wedge\cdot}\big] = 1 \text{ for all } \bar\omega \in \overline{\Omega}; \quad \text{and } \widehat{\mathbb{P}}^0_{\bar\omega}[T=\theta] = 1 \text{ for all } \bar\omega \in \{\tau = T\}.$$

It follows that for $\overline{\mathbb{P}}^*-$a.e. $\bar\omega \in \{\tau = T\}$, one has $\widehat{\mathbb{P}}^0_{\bar\omega} = \widehat{\mathbb{P}}_{\bar\omega} := \mathbb{P}^{\tau(\bar\omega),\omega}_0 \otimes \delta_{\{\theta\}}$.

Next, recall that $\overline{\mathbb{P}}^*_\omega$ is a family of r.c.p.d of $\overline{\mathbb{P}}^*$ w.r.t. $\sigma(B_t, t \geq 0)$ and $\overline{\mathbb{Q}}^1_{\bar\omega}$ are defined by (3.6). Then $(\overline{\mathbb{Q}}^1_{\bar\omega})_{\bar\omega \in \overline{\Omega}}$ is a family of conditional probability measures of $\overline{\mathbb{P}}^*$ w.r.t. $\sigma(B_{\tau_0(\omega) \wedge t}, \ t \geq 0)$. Further, by the representation of $\overline{\mathcal{F}}_\tau$ in (3.10), it follows that for $\overline{\mathbb{P}}^*-$a.e. $\bar\omega \in \{\tau < T\}$, one has $\widehat{\mathbb{P}}^0_{\bar\omega} = \overline{\mathbb{Q}}^2_{\bar\omega}$.

We now prove the claim (3.9). For every $\omega \in \Omega$ and $t \in \mathbb{R}_+$, we denote $\overline{A}_{\omega,t} := \{\bar\omega' \in \overline{\Omega} \ : \ \omega'_{t\wedge\cdot} = \omega_{t\wedge\cdot}, \ \theta' > t\}$. Then it is clearly that $\overline{A}_{\omega,t}$ is an atom in $\overline{\mathcal{F}}_t$, i.e. for any set $C \in \overline{\mathcal{F}}_t$, one has either $\overline{A}_{\omega,t} \in C$ or $\overline{A}_{\omega,t} \cap C = \emptyset$. Let $\bar\omega \in \overline{\Omega}$ such that $\tau(\bar\omega) < \theta$, and $\theta' > \theta$, so that $\bar\omega \in A_{\omega,t}$ and $(\omega, \theta') \in A_{\omega,t}$ for every $t < \theta$. Let $t_0 := \tau(\bar\omega)$, then $\bar\omega \in A_{\omega,t_0}$, and $\bar\omega \in \{\tau = t_0\} \in \overline{\mathcal{F}}_{t_0}$, which implies that $(\omega, \theta') \in A_{\omega,t_0} \subset \{\tau = t_0\}$ since A_{ω,t_0} is an atom in $\overline{\mathcal{F}}_{t_0}$. It follows that $\tau(\omega, \theta') = \tau(\bar\omega)$ for all $\theta' > \theta$ and $\bar\omega \in \overline{\Omega}$ such that $\tau(\bar\omega) < \theta$. Notice that for each $t \in \mathbb{R}_+$, $\{\bar\omega \in \overline{\Omega} \ : \ \tau(\bar\omega) \leq t\}$ is $\overline{\mathcal{F}}_t-$measurable, then by Doob's functional representation Theorem, there is some Borel measurable function $f : \Omega \times (\mathbb{R}_+ \cup \{\infty\}) \to \mathbb{R}$ such that $\mathbf{1}_{\{\tau(\bar\omega) \leq t\}} = f([\omega]_t, [\theta]_t)$. It follows that for $\theta_0 \in \mathbb{R}_+$, $\{\omega \in \overline{\Omega} \ : \ \tau(\omega, \theta_0) \leq t\}$ is $\mathcal{F}_t-$measurable, and hence $\omega \mapsto \tau(\omega, \theta_0)$ is a $\mathbb{F}-$stopping time on $(\Omega, \mathcal{F})$. Then the random variable $\tau_0 : \Omega \to \mathbb{R}_+$ defined by $\tau_0(\omega) := \sup_{n\in\mathbb{N}} \tau(\omega, n)$ is the required $\mathbb{F}-$stopping time of claim (3.9).

Finally, we notice that by its definition, one has $\bar\omega \mapsto \widehat{\mathbb{P}}_{\bar\omega}$ is $\overline{\mathcal{F}}-$measurable and satisfies $\widehat{\mathbb{P}}_{\bar\omega} = \widehat{\mathbb{P}}_{[\bar\omega]_\theta}$ for all $\bar\omega \in \overline{\Omega}$. Moreover, we have proved that $\widehat{\mathbb{P}}^0_{\bar\omega} = \widehat{\mathbb{P}}_{\bar\omega}$ for $\overline{\mathbb{P}}^*-$a.e. $\bar\omega \in \overline{\Omega}$, where $(\widehat{\mathbb{P}}^0_{\bar\omega})_{\bar\omega \in \overline{\Omega}}$ is a family of r.c.p.d. of $\overline{\mathbb{P}}^*$ w.r.t. $\overline{\mathcal{F}}_\tau$. Therefore, $(\widehat{\mathbb{P}}_{\bar\omega})_{\bar\omega \in \overline{\Omega}}$ is a family of conditional probability measures of $\overline{\mathbb{P}}^*$ w.r.t. $\overline{\mathcal{F}}_\tau$. $\qquad\square$

Notice that $\mathrm{SG}^* \supset \mathrm{SG}$, then it is enough to show (2.5) for SG^* to prove Theorem 2.3.

3.2   Technical results

We first define a projection operator $\Pi_S : \overline{\Omega} \times \overline{\Omega} \to \overline{\Omega}$ by

$$\Pi_S\big[\overline{A}\big] \quad := \quad \big\{\bar\omega \ : \text{there exists some } \bar\omega' \in \overline{\Omega} \text{ such that } (\bar\omega, \bar\omega') \in \overline{A}\big\}.$$

Proposition 3.2. *Let the conditions in Theorem 2.3 holds true and $\overline{\mathbb{P}}^*$ be the fixed optimizer of the optimal SEP (2.2). Then there is some Borel set $\overline{\Gamma}^*_0 \subset \Omega$ such that $\overline{\mathbb{P}}^*[\overline{\Gamma}^*_0] = 1$ and for all $\overline{\mathbb{F}}-$stopping time $\tau \leq T$, one has*

$$\overline{\mathbb{P}}^*\big[\tau < T, \ \overline{B}_{\tau\wedge\cdot} \in \Pi_S\big(\mathrm{SG}^* \cap \big(\overline{\Omega} \times \overline{\Gamma}^*_0\big)\big)\big] \ = \ 0. \tag{3.11}$$

Proof. (i) Let us start with the duality result $P(\mu) = D(\mu)$ and the dual problem (2.4). By definition, we may find a minimizing sequence $\{(\lambda^n, S^n)\}_{n\geq 1} \subset \mathcal{D}$, so that $\mu(\lambda_n) \longrightarrow D(\mu) = P(\mu)$ as $n \longrightarrow \infty$. Then, there is some $\Gamma_0 \subset \Omega$ s.t. $\mathbb{P}_0(\Gamma_0) = 1$ and

$$\eta^n(\bar\omega) := \lambda^n(\omega_t) + S^n_t(\omega) - \xi(\bar\omega) \ \geq \ 0, \quad \text{for all } \bar\omega \in \Gamma_0 \times \mathbb{R}_+. \tag{3.12}$$

Notice that $(S^n_t)_{t\geq 0}$ are all strong supermartingales on $(\Omega, \mathcal{F}, \mathbb{P}_0)$ satisfying (2.3). It is then also a strong supermartingale on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}^*)$ w.r.t. $\overline{\mathbb{F}}$. It follows that

$$0 \ \leq \ \mathbb{E}^{\overline{\mathbb{P}}^*}\big[\eta^n\big] = \mathbb{E}^{\overline{\mathbb{P}}^*}\big[\lambda^n(B_T) + S^n_T - \xi\big] \leq \mu(\lambda^n) - P(\mu) \ \longrightarrow \ 0 \text{ as } n \longrightarrow \infty. \tag{3.13}$$

Therefore, we can find some $\overline{\Gamma}_0 \subseteq \overline{\Omega}$ such that $\overline{\mathbb{P}}^*(\overline{\Gamma}_0) = 1$, and after possibly passing to a subsequence,

$$\eta^n(\bar{\omega}) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \text{ for all } \bar{\omega} \in \overline{\Gamma}_0.$$

Moreover, since S^n can be viewed as a $\overline{\mathbb{F}}$−strong supermartingale on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}^*)$, then there is some Borel set $\overline{\Gamma}_1 \subset \overline{\Omega}$ such that $\overline{\mathbb{P}}^*[\overline{\Gamma}_1] = 1$, and for all $\bar{\omega} \in \overline{\Gamma}_1$, $\mathbb{P}_0[\omega \otimes_\theta B \in \Gamma_0] = 1$, and $(S^n_{\theta+t}(\omega \otimes_\theta \cdot))_{t \geq 0}$ is a $\mathbb{P}_0$−strong supermartingale. Set $\overline{\Gamma}_0^* := \overline{\Gamma}_0 \cap \overline{\Gamma}_1$, and we next show that $\overline{\Gamma}_0^*$ is the required Borel set.

(ii) Let us consider a fixed pair

$$(\bar{\omega}, \bar{\omega}') \in \text{ SG } \cap \left(\overline{\Omega} \times \overline{\Gamma}_0^*\right),$$

and define

$$\delta(\bar{\omega}'') := \xi(\bar{\omega}) + \xi(\bar{\omega}' \otimes \bar{\omega}'') - \xi(\bar{\omega} \otimes \bar{\omega}'') + \xi(\bar{\omega}'), \text{ for all } \bar{\omega}'' \in \overline{\Omega}.$$

By the definition of SG* (3.8), one has $\omega_\theta = \omega'_{\theta'}$. Then using the definition of η^n in (3.12), it follows that for all $\bar{\omega}'' \in \overline{\Omega}$,

$$\begin{aligned}
\delta(\bar{\omega}'') &= \lambda^n(\omega_\theta) + S^n_\theta(\omega) - \eta^n(\bar{\omega}) - \left(\lambda^n(\omega'_{\theta'}) + S^n_{\theta'}(\omega') - \eta^n(\bar{\omega}')\right) \\
&\quad + \lambda^n\left(\omega'_{\theta'} + \omega''_{\theta''}\right) + S^n_{\theta'+\theta''}(\omega' \otimes_{\theta'} \omega'') - \eta^n(\bar{\omega}' \otimes \bar{\omega}'') \\
&\quad - \left(\lambda^n\left(\omega_\theta + \omega''_{\theta''}\right) + S^n_{\theta+\theta''}(\omega \otimes_\theta \omega'') - \eta^n(\bar{\omega} \otimes \bar{\omega}'')\right) \\
&= S^n_\theta(\omega) - \eta^n(\bar{\omega}) + S^n_{\theta'+\theta''}(\omega' \otimes_{\theta'} \omega'') - \eta^n(\bar{\omega}' \otimes \bar{\omega}'') \\
&\quad - \left(S^n_{\theta'}(\omega') - \eta^n(\bar{\omega}') + S^n_{\theta+\theta''}(\omega \otimes_\theta \omega'') - \eta^n(\bar{\omega} \otimes \bar{\omega}'')\right) \\
&\leq \left(\eta^n(\bar{\omega} \otimes \bar{\omega}'') + \eta^n(\bar{\omega}')\right) - \eta^n(\bar{\omega}' \otimes \bar{\omega}'') \\
&\quad + \left(S^n_{\theta'+\theta''}(\omega' \otimes_{\theta'} \omega'') - S^n_{\theta'}(\omega')\right) - \left(S^n_{\theta+\theta''}(\omega \otimes_\theta \omega'') - S^n_\theta(\omega)\right).
\end{aligned}$$

(iii) Let $\tau \leq T$ be an $\overline{\mathbb{F}}$−stopping time, then $\left(\widehat{\mathbb{P}}_{\bar{\omega}}\right)_{\bar{\omega} \in \overline{\Omega}}$, defined by

$$\widehat{\mathbb{P}}_{\bar{\omega}} := \overline{\mathbb{Q}}^2_{(\omega, \tau(\bar{\omega}))} \mathbf{1}_{\{\tau(\bar{\omega}) < \theta\}} + \mathbb{P}_0^{\tau(\bar{\omega}), \omega} \otimes \delta_{\{\theta\}} \mathbf{1}_{\{\tau(\bar{\omega}) = \theta\}}$$

provides a family of conditional probability measures of $\overline{\mathbb{P}}^*$ w.r.t. $\overline{\mathcal{F}}_\tau$ (see Lemma 3.1). Recall that $\widehat{\mathbb{P}}^*_{\bar{\omega}} := \overline{\mathbb{Q}}^*_{(\omega, \tau(\bar{\omega}))}$ for all $\bar{\omega} \in \{\tau < T\}$ is the shifted probability measures.

By (3.13), there is some set $\overline{\Gamma}^1_\tau$ such that $\overline{\mathbb{P}}^*[\overline{\Gamma}^1_\tau] = 1$ and

$$\mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\left[\eta^n\right] \longrightarrow 0, \text{ as } n \longrightarrow \infty, \text{ for all } \bar{\omega} \in \overline{\Gamma}^1_\tau. \tag{3.14}$$

Further, (3.13) implies that $0 \geq \mathbb{E}^{\overline{\mathbb{P}}^*}[S^n_T] \to 0$ as $n \to \infty$. Then it follows from the strong supermartingale property of S^n that

$$S^n_\tau - \mathbb{E}^{\overline{\mathbb{P}}^*}\left[S^n_T \big| \overline{\mathcal{F}}_\tau\right] \geq 0, \ \overline{\mathbb{P}}^* - \text{a.s. and } \mathbb{E}^{\overline{\mathbb{P}}^*}\left[S^n_\tau - \mathbb{E}^{\overline{\mathbb{P}}^*}\left[S^n_T \big| \overline{\mathcal{F}}_\tau\right]\right] \leq -\mathbb{E}^{\overline{\mathbb{P}}^*}[S^n_T] \to 0,$$

Hence there is some set $\overline{\Gamma}^2_\tau \subset \overline{\Omega}$ such that $\overline{\mathbb{P}}^*[\overline{\Gamma}^2_\tau] = 1$ and for all $\bar{\omega} \in \overline{\Gamma}^2_\tau$,

$$0 \leq S^n_\tau(\bar{\omega}) - \mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}}[S^n_T] \longrightarrow 0, \text{ as } n \to \infty, \tag{3.15}$$

after possibly taking some subsequence. Moreover, by the definition of $\overline{\mathcal{P}}^0$ in (2.1), B is a $\overline{\mathbb{F}}$−Brownian motion and $B_{T \wedge t}$ is uniformy integrable under $\overline{\mathbb{P}}^*$, and the

property holds still under the conditional probability measures. Then there is some measurable set $\overline{\Gamma}_\tau^3 \subset \overline{\Omega}$ such that $\overline{\mathbb{P}}^*[\overline{\Gamma}_\tau^3] = 1$ and for every $\bar{\omega} \in \overline{\Gamma}_\tau^3 \cap \{\tau < T\}$, one has

$$\widehat{\mathbb{P}}_{\bar{\omega}}^* [T > 0] > 0, \quad \widehat{\mathbb{P}}_{\bar{\omega}}^* [\bar{\omega}_{\tau(\bar{\omega}) \wedge \cdot} \otimes \overline{B} \in \overline{\Gamma}_0^*] = 1 \quad \text{and} \quad \widehat{\mathbb{P}}_{\bar{\omega}}^* \in \mathcal{P}^0. \quad (3.16)$$

Set $\overline{\Gamma}_\tau^0 := \overline{\Gamma}_\tau^1 \cap \overline{\Gamma}_\tau^2 \cap \overline{\Gamma}_\tau^3$, in the rest of this proof, we show that

$$\left((\overline{\Gamma}_\tau^0 \cap \{\tau < T\}) \times \overline{\Omega} \right) \cap \mathrm{SG}^* \cap \left(\overline{\Omega} \times \overline{\Gamma}_0^* \right) = \emptyset, \quad (3.17)$$

which justifies (3.11).

(iv) We finally prove (3.17) by contradiction. Let $(\bar{\omega}, \bar{\omega}') \in (\overline{\Gamma}_\tau^0 \times \overline{\Omega}) \cap \mathrm{SG}^* \cap (\overline{\Omega} \times \overline{\Gamma}_0^*)$. Notice that $\bar{\omega}' = (\omega', \theta') \in \overline{\Gamma}_0^* \subset \overline{\Gamma}_1$ and for some constant L_n,

$$|S_{\theta'+T}^n(\omega' \otimes_\theta B)| \leq L_n |1 + \omega'_{\theta'} + B_T|,$$

it follows by the supermartingale property, together with the Fatou lemma, that

$$\mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}^*} \left[S_{\theta'+T}^n(\omega' \otimes_{\theta'} B) \right] \leq S_{\theta'}^n(\omega').$$

Moreover, one has $\mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}^*}[\eta^n(\bar{\omega}' \otimes B)] \geq 0$. Further, using (3.16) then (3.13), (3.14) and (3.15), we obtain that

$$0 < \mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}^*}[\delta] \leq \mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}^*} \left[\eta^n(\bar{\omega} \otimes \overline{B}) \right] + \eta^n(\bar{\omega}') - \mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}^*} \left[S_{\tau(\bar{\omega})+T}^n(\omega \otimes_{\tau(\bar{\omega})} B) \right] - S_{\tau(\bar{\omega})}^n(\omega) \to 0,$$

as $n \longrightarrow \infty$, which is a contradiction, and we hence conclude the proof. $\qquad\square$

Suppose that $\Pi_S \left(\mathrm{SG}^* \cap (\overline{\Omega} \times \overline{\Gamma}_0^*) \right)$ is Borel measurable on $\overline{\Omega}$, then by Lemma A.2 of [18], the set

$$\left\{ (t, \bar{\omega}, \bar{\omega}') \in \mathbb{R}_+ \times \overline{\Gamma}_0^* \times \overline{\Gamma}_0^* : t < \theta, \text{ and } (\omega, t, \bar{\omega}') \in \mathrm{SG}^* \right\}.$$

is an $\overline{\mathbb{F}}$−optional set. Using Proposition 3.2 together with the classical optional cross-section theorem (see e.g. Theorem IV.86 of Dellacherie and Meyer [12]), it follows immediately that there is some measurable set $\overline{\Gamma}_1^* \subset \overline{\Omega}$ such that $\overline{\mathbb{P}}^*(\overline{\Gamma}_1^*) = 1$ and $\Pi_S \left(\mathrm{SG}^* \cap (\overline{\Omega} \times \overline{\Gamma}_0^*) \right) \cap \overline{\Gamma}_1^{*<} = \emptyset$. However, when the set $\mathrm{SG}^* \cap (\overline{\Omega} \times \overline{\Gamma}_0^*)$ is a Borel set in $\overline{\Omega} \times \overline{\Omega}$, the projection set $\Pi_S \left(\mathrm{SG}^* \cap (\overline{\Omega} \times \overline{\Gamma}_0^*) \right)$ is a priori a $\mathcal{B}(\overline{\Omega})$−analytic set (Definition III.7 of [12]) in $\overline{\Omega}$. Therefore, we need to adapt the arguments of the optional cross-section theorem to our context.

Denote by $\overline{\mathcal{O}}$ the optional σ−field w.r.t. the filtration $\overline{\mathbb{F}}$ on $\mathbb{R}_+ \times \overline{\Omega}$. Let E be some auxiliary space, $A \subset \mathbb{R}_+ \times \overline{\Omega} \times E$, we denote

$$\Pi_2(A) := \{\bar{\omega} : \text{there is some } (t, e) \in \mathbb{R}_+ \times E \text{ such that } (t, \bar{\omega}, e) \in A\},$$

and

$$\Pi_{12}(A) := \{(t, \bar{\omega}) : \text{there is some } e \in E \text{ such that } (t, \bar{\omega}, e) \in A\}.$$

Proposition 3.3. *Let $\overline{\mathbb{P}}$ be an arbitrary probability measure on $(\overline{\Omega}, \overline{\mathcal{F}})$, $(E, \mathcal{E})$ be a Lusin measurable space [1]. Suppose that $A \subset \mathbb{R}_+ \times \overline{\Omega} \times E$ is a $\overline{\mathcal{O}} \times \mathcal{E}$−measurable set. Then for every $\varepsilon > 0$, there is some $\overline{\mathbb{F}}$−stopping time τ such that $\overline{\mathbb{P}}[\tau < \infty] \geq \mathbb{P}[\Pi_2(A)] - \varepsilon$ and $(\tau(\bar{\omega}), \bar{\omega}) \in \Pi_{12}(A)$ whenever $\bar{\omega} \in \overline{\Omega}$ satisfies $\tau(\bar{\omega}) < \infty$.*

---

[1] A measurable space $(E, \mathcal{E})$ is said to be Lusin if it is isomorphic to a Borel subset of a compact metrizable space (Definition III.16 of [12]).

Proof. We follow the lines of Theorem IV.84 of [12].

(i) Notice that every Lusin space is isomorphic to a Borel subset of $[0, 1]$ (see e.g. Theorem III.20 of [12]), we can then suppose without loss of generality that $(E, \mathcal{E}) = ([0, 1], \mathcal{B}([0, 1]))$. Then the projection set $\Pi_{12}(A)$ is clearly $\overline{\mathcal{O}}-$analytic in sense of Definition III.7 of [12].

(ii) Using the measurable section theorem (Theorem III.44 of [12]), there is $\overline{\mathcal{F}}-$random variable $R : \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ such that $\overline{\mathbb{P}}[R < \infty] = \overline{\mathbb{P}}[\Pi_2(A)]$ and $R(\bar{\omega}) < \infty \Rightarrow (R(\bar{\omega}), \bar{\omega}) \in \Pi_{12}(A)$. The variable R is in fact a stopping time w.r.t. the completed filtration $\overline{\overline{\mathbb{F}}}^{\overline{\mathbb{P}}}$ (see e.g. Proposition 2.13 of [16]), but not a $\overline{\mathbb{F}}-$stopping time a priori. We then need to modify R following the measure ν defined on $\mathcal{B}(\mathbb{R}_+) \otimes \overline{\mathcal{F}}$ by

$$\nu(G) := \int \mathbf{1}_G(R(\bar{\omega}), \bar{\omega}) \mathbf{1}_{\{R < \infty\}}(\bar{\omega}) \overline{\mathbb{P}}(d\bar{\omega}), \quad \forall G \in \mathcal{B}(\mathbb{R}_+) \otimes \overline{\mathcal{F}}.$$

(iii) We continue by following the lines of item (b) in the proof of Theorem IV.84 of [12]. Denote by ζ_0 the set of all intervals $[[\sigma, \tau[[$, with $\sigma \leq \tau$ and σ, τ are both $\overline{\mathbb{F}}-$stoping times. Denote also by ζ the closure of ζ_0 under finite union operation, then ζ is a Boolean algebra which generates the optional $\sigma-$filed $\overline{\mathcal{O}}$. Moreover, the debut of a set $C \in \zeta_\delta$ (the smallest collection containing ζ and stable under countable intersection) is a.s. equal to an $\overline{\mathbb{F}}-$stopping time. Further, the projection set $\Pi_{12}(A)$ is $\overline{\mathcal{O}}-$analytic and hence $\overline{\mathcal{O}}-$universally measurable. Therefore, there exists a set $C \in \zeta_\delta$ contained in $\Pi_{12}(A)$ such that $\nu(C) \geq \nu(\Pi_{12}(A)) - \varepsilon$. Let τ_0 be the $\overline{\mathbb{F}}-$stopping time, which equals to the debut of C, $\overline{\mathbb{P}}-$a.s., then define $\tau := \tau_0 \mathbf{1}_{\{(\tau_0(\bar{\omega}), \bar{\omega}) \in B\}}$, which is a new $\overline{\mathbb{F}}-$stopping time since $\{\bar{\omega} : (\tau_0(\bar{\omega}), \bar{\omega}) \in C\} \in \overline{\mathcal{F}}_{\tau_0}$ by Theorem IV.64 of [12]. We then conclude the proof by the fact that τ is the required stopping time. $\qquad \square$

3.3 Proof of Theorem 2.3

Let us define

$$A := \left\{ (t, \bar{\omega}, \bar{\omega}') \in \mathbb{R}_+ \times \overline{\Gamma}_0^* \times \overline{\Gamma}_0^* : t < \theta, \text{ and } (\omega, t, \bar{\omega}') \in SG^* \right\}. \qquad (3.18)$$

Since SG^* is a $\overline{\mathcal{F}}_T \otimes \overline{\mathcal{F}}_T-$measurable set in $\overline{\Omega} \times \overline{\Omega}$, it follows (see Lemma A.2. of [18]) that the set A defined by (3.18) satisfies the conditions in Proposition 3.3 with $E = \overline{\Omega}$.

We next prove that $\Pi_2(A)$ is $\overline{\mathbb{P}}^*-$null set. Indeed, if $\overline{\mathbb{P}}^*[\Pi_2(A)] > 0$, by Proposition 3.3, there is some $\overline{\mathbb{F}}-$stopping time τ such that $(\tau(\bar{\omega}), \bar{\omega}) \in \Pi_{12}(A)$ for all $\bar{\omega} \in \{\tau < \infty\} = \{\tau < T\}$, and $\overline{\mathbb{P}}^*[\tau < \infty] = \overline{\mathbb{P}}^*[\tau = T] > 0$. Notice that $(\tau(\bar{\omega}), \bar{\omega}) \in \Pi_{12}(A)$ implies that $(\omega, \tau(\bar{\omega})) \in \Pi_S(SG^*)$. We then have

$$0 < \overline{\mathbb{P}}^*[\tau < T] \leq \overline{\mathbb{P}}^*\left[\tau < T, \overline{B}_{\tau \wedge \cdot} \in \Pi_S\left(SG^* \cap \left(\overline{\Omega} \times \overline{\Gamma}_0^*\right)\right)\right].$$

This is a contradiction to Proposition 3.2.

Since $\Pi_2(A)$ is a $\overline{\mathbb{P}}^*-$null set, we may find a Borel set $\overline{\Gamma}_1^* \subset (\overline{\Omega} \setminus \Pi_2(A))$ such that $\overline{\mathbb{P}}^*[\overline{\Gamma}_1^*] = 1$ and $\Pi_S(SG^*) \cap \overline{\Gamma}_1^{*<} = \emptyset$. Therefore, $\overline{\Gamma}^* := \overline{\Gamma}_0^* \cap \overline{\Gamma}_1^*$ is the required Borel subset of $\overline{\Omega}$. $\qquad \square$

Remark 3.4. (i) *Proposition 3.2 can be compared to Proposition 6.6 of [2], while the proofs are different. Our proof of Proposition 3.2 is in the same spirit of the*

classical proof for the monotonicity principle of optimal transport problem (see e.g. Chapter 5 of Villani [31]), or martingale optimal transport problem (see e.g. Zaev [32, Theorem 3.6]), based on the existence of optimal transport plan and the duality result.

(ii) Proposition 3.3 should be compared to the so-called filtered Kellerer Lemma (Proposition 6.7 of [2]), where a key argument in their proof is Choquet's capacity theory. Our proof of Proposition 3.3 uses crucially an optional section theorem, which is based on a measurable section theorem, and the latter is also proved in [12] using Choquet's capacity theory (see also the review in [16]).

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