Optimal Skorokhod embedding under finitely-many marginal constraints *

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Abstract

The *Skorokhod embedding problem* aims to represent a given probability measure on the real line as the distribution of Brownian motion stopped at a chosen stopping time. In this paper, we consider an extension of the *optimal Skorokhod embedding problem* in Beiglböck, Cox & Huesmann [1] to the case of finitely-many marginal constraints[1]. Using the classical convex duality approach together with the optimal stopping theory, we obtain the duality results which are formulated by means of probability measures on an enlarged space. We also relate these results to the problem of martingale optimal transport under multiple marginal constraints.

Key words. Skorokhod embedding, martingale optimal transport, model-free pricing, robust hedging.

1 Introduction

Let μ be a probability measure on $\mathbb{R}$, with finite first moment and centered, the Skorokhod embedding problem (SEP) consists in finding a stopping time τ on a Brownian motion W such that $W_\tau \sim \mu$ and the stopped process $W_{\tau\wedge\cdot} := \big(W_{\tau\wedge t}\big)_{t\geq 0}$ is uniformly integrable. We refer the readers to the survey paper [36] of Obłój for a comprehensive account of the field.

In this paper, we consider its extension to the case of multiple marginal constraints. Namely, let $\mu := (\mu_1, \cdots, \mu_n)$ be a given family of centered probability measures such that the family is increasing in convex ordering, i.e. for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, one has

$$\int_{\mathbb{R}} \phi(x)\mu_k(dx) \ \leq \ \int_{\mathbb{R}} \phi(x)\mu_{k+1}(dx) \quad \text{for all } k = 1, \cdots, n-1.$$

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[1]While producing the final version of this paper, we knew from Mathias Beiglböck about the new development in [2] that is also in the final producing version and extends the previous work [1] to the case of finitely-many marginal contraints. We emphasize that our approach is of completely different nature.

The extended SEP is to find an increasing family of stopping times $\tau := (\tau_1, \cdots, \tau_n)$ such that $W_{\tau_k} \sim \mu_k$ for all $k = 1, \cdots, n$ and the stopped process $W_{\tau_n \wedge \cdot}$ is uniformly integrable. We study an associated optimization problem, which consists in maximizing the expected value of some reward function among all such embeddings.

One of the motivations to study this problem is its application in finance to compute the arbitrage-free model-independent price bounds of contingent claims consistent with the market prices of vanilla options. Mathematically, the underlying asset is required to be a martingale according to the no-arbitrage condition and the market calibration allows to recover the marginal laws of the underlying at certain maturities (see e.g. Breeden & Litzenberger [7]). Then by considering all martingales fitting the given marginal distributions, one can obtain the arbitrage-free price bounds. Based on the fact that every continuous martingale can be considered as a time-changed Brownian motion by Dambis-Dubins-Schwarz theorem, Hobson studied the model-free hedging of lookback options in his seminal paper [25] by means of the SEP. The main idea of his poineering work is to exploit some solution of the SEP satisfying some optimality criteria, which yields the model-free hedging strategy and allows to solve together the model-free pricing and hedging problems. Since then, the optimal SEP has received substantial attention from the mathematical finance community and various extensions were acheived in the literature, such as Cox & Hobson [9], Hobson & Klimmek [27], Cox, Hobson & Obłój [10], Cox & Obłój [11] and Davis, Obłój & Raval[12], Obłój & Spoida [37], etc. A thorough literature is provided in Hobson's survey paper [26].

Beiglböck, Cox and Huesmann generalized this heuristic idea and formulated the optimal SEP in [1], which recovered the previous known results by a unifying formulation. Namely, their main results are twofold. First, they establish the expected identity between the optimal SEP and the corresponding model-free superhedging problem. Second, they derive the chacterization of the optimal embedding by a geometric pathwise property which allows to recover all previous known embeddings in the literature.

The problem of model-free hedging has also been approached by means of the martingale optimal transport, as initiated by Beiglböck, Henry-Labordère & Penkner [3] in discrete-time case and Galichon, Henry-Labordère & Touzi [20] in continuous-time case. Further developpement enriches this literature, such as Beiglböck & Juillet [4], , Henry-Labordère & Touzi [24], Henry-Labordère, Tan & Touzi [23]. An remarkable contribution for the continuous-time martingale optimal transport is due to Dolinsky and Soner [14, 15]. We also refer to Tan and Touzi [42] for the optimal transport problem under more general controlled stochastic dynamics.

Our objective of this paper is to revisit the duality result of [1] and to extend the duality to the case of multiple marginal constraints. Our approach uses tools from a completely different nature. First, by following the convex duality approach, we convert the optimal SEP into an infimum of classical optimal stopping problems. Next, we use the standard dynamic programming approach to relate such optimal stopping problems to model-free superhedging problems. Finally, we show that our result induces the duality for a class of martingale optimal transport problems in the space of continuous paths.

The paper is organized as follows. In Section 2, we formulate our optimal SEP under finitely-many marginal constraints and provide two duality results. In Section 3 the duality of optimal SEP together with time-change arguments gives the duality for the martingale optimal transport problem under multi-marginal constraints. We finally provide the related proofs in Section 4.

Notations. (i) Let $\Omega := C(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous paths ω on $\mathbb{R}_+$ such that $\omega_0 = 0$, B be the canonical process, $\mathbb{P}_0$ be the Wiener measure, $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be the canonical filtration generated by B, and $\mathbb{F}^a := (\mathcal{F}^a_t)_{t\geq 0}$ be the augmented filtration under $\mathbb{P}_0$.

(ii) Define for some fixed integer $n \geq 1$ the enlarged canonical space by $\overline{\Omega} := \Omega \times \Theta$ (see El Karoui and Tan [18, 19]), where $\Theta := \big\{(\theta_1, \cdots, \theta_n) \in \mathbb{R}^n_+ : \theta_1 \leq \cdots \leq \theta_n\big\}$. All the elements of $\overline{\Omega}$ are denoted by $\bar{\omega} := (\omega, \theta)$ with $\theta := (\theta_1, \cdots, \theta_n)$. Denote further by (B, T) (with $T := (T_1, \cdots, T_n)$) the canonical element on $\overline{\Omega}$, i.e. $B_t(\bar{\omega}) := \omega_t$ and $T(\bar{\omega}) := \theta$ for every $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$. The enlarged canonical filtration is denoted by $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{t\geq 0}$, where $\overline{\mathcal{F}}_t$ is generated by $(B_s)_{0\leq s\leq t}$ and all the sets $\{T_k \leq s\}$ for all $s \in [0, t]$ and $k = 1, \cdots, n$. In particular, all random variables $T_1, \cdots, T_n$ are $\overline{\mathbb{F}}-$stopping times.

(iii) We endow Ω with the compact convergence topology, and Θ with the classical Euclidean topology, then Ω and $\overline{\Omega}$ are both Polish spaces (separable, complete metrizable space). In particular, $\overline{\mathcal{F}}_\infty := \bigvee_{t\geq 0} \overline{\mathcal{F}}_t$ is the Borel $\sigma-$field of the Polish space $\overline{\Omega}$ (see Lemma A.1).

(iv) Denote by $\mathcal{C}_1 := \mathcal{C}_1(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ with linear growth.

(v) Throughout the paper UI, a.s. and q.s. are respectively the abbreviations of uniformly integrable, almost surely and quasi-surely. Moreover, given a set of probability measures $\mathcal{N}$ (e.g. $\mathcal{N} = \overline{\mathcal{P}}$ and $\mathcal{N} = \mathcal{M}$ in the following) on some measurable space, we write $\mathcal{N}-$q.s. to represent that some property holds under every probability of $\mathcal{N}$.

2 An optimal Skorokhod embedding problem and the dualities

In this section, we formulate an optimal Skorokhod embedding problem (SEP) under finitely-many marginal constraints, as well as its dual problems. We then provide two duality results.

2.1 An optimal Skorokhod embedding problem

Throughtout the paper, $\mu := (\mu_1, \cdots, \mu_n)$ is a vector of n probability measures on $\mathbb{R}$ and we denote, for any integrable function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\mu_k(\phi) \;\; := \;\; \int_{\mathbb{R}} \phi(x)\mu_k(dx) \;\; \text{for all } k = 1, \cdots, n.$$

The vector μ is said to be a peacock if each probability μ_k has finite first moment, i.e. $\mu_k(|x|) < +\infty$, and μ is increasing in convex ordering, i.e, $k \mapsto \mu_k(\phi)$ is non-decreasing for every convex function ϕ. A peacock μ is called centered if $\mu_k(x) = 0$ for all $k = 1, \cdots, n$. Denote by $\mathbf{P}^{\preceq}$ the collection of all centered peacocks.

Optimal SEP As in Beiglböck, Cox & Huesmann [1], we shall consider the problem in a weak setting, i.e. the stopping times may be identified by probability measures on the enlarged space $\overline{\Omega}$. Recall that the elements of $\overline{\Omega}$ are denoted by $\bar{\omega} := \big(\omega, \theta = (\theta_1, \cdots, \theta_n)\big)$ and the canonical element is denoted by $\big(B, T = (T_1, \cdots, T_n)\big)$, and in particular $T_1, \cdots, T_n$ are all $\overline{\mathbb{F}}-$stopping times. Let $\overline{\mathcal{P}}(\overline{\Omega})$ be the space of all probability measures on $\overline{\Omega}$, and define

$$\overline{\mathcal{P}} \;:=\; \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\overline{\Omega}) : B \text{ is an } \overline{\mathbb{F}} - \text{Brownian motion and } B_{T_n \wedge \cdot} \text{ is UI under } \overline{\mathbb{P}}\Big\}. \tag{2.1}$$

Set for any given family of probability measures $\mu = (\mu_1, \cdots, \mu_n)$

$$\overline{\mathcal{P}}(\mu) \;:=\; \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}} : B_{T_k} \overset{\overline{\mathbb{P}}}{\sim} \mu_k \text{ for all } k = 1, \cdots, n\Big\}. \tag{2.2}$$

As a consequence of Kellerer's theorem in [33], $\overline{\mathcal{P}}(\mu)$ is nonempty if and only if $\mu \in \mathbf{P}^{\preceq}$.

Let $\Phi : \overline{\Omega} \to \mathbb{R}$ be a measurable function, then Φ is called non-anticipative if $\Phi(\omega, \theta) = \Phi\big(\omega_{\theta_n \wedge \cdot}, \theta\big)$ for every $(\omega, \theta) \in \overline{\Omega}$. Define the optimal SEP for a non-anticipative function Φ by

$$P(\mu) \;:=\; \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu)} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi(B, T)\big], \tag{2.3}$$

where the expectation of a random variable ξ is defined by $\mathbb{E}^{\overline{\mathbb{P}}}[\xi] = \mathbb{E}^{\overline{\mathbb{P}}}[\xi^+] - \mathbb{E}^{\overline{\mathbb{P}}}[\xi^-]$ with the convention $\infty - \infty = -\infty$. The problem is well-posed if there exists at least a $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu)$ such that $\mathbb{E}^{\overline{\mathbb{P}}}[|\Phi(B, T)|] < +\infty$. We emphasize that Φ is non-anticipative throughout the paper.

Remark 2.1. *A $\mu-$embedding is a collection*

$$\alpha \;=\; \big(\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{P}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}_t^\alpha)_{t \ge 0}, W^\alpha, \tau^\alpha = (\tau_1^\alpha, \cdots, \tau_n^\alpha)\big),$$

where W^α is an $\mathbb{F}^\alpha-$Brownian motion, $\tau_1^\alpha, \cdots, \tau_n^\alpha$ are increasing $\mathbb{F}^\alpha-$stopping times such that $W^\alpha_{\tau_n^\alpha \wedge \cdot}$ is uniformly integrable, and $W^\alpha_{\tau_k^\alpha} \sim \mu_k$ for all $k = 1, \cdots n$. We observe that for every centered peacock μ, there exists a one-to-one correspondence between $\overline{\mathcal{P}}(\mu)$ and the set of $\mu-$embeddings. Indeed, it is clear that every $\mu-$embedding α induces a probability measure $\overline{\mathbb{P}} := \overline{\mathbb{P}}^\alpha \circ (W^\alpha, \tau^\alpha)^{-1} \in \overline{\mathcal{P}}(\mu)$. Conversely, every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu)$ induces a $\mu-$embedding.

2.2 The duality results

We introduce two dual problems. Recall that $\mathbb{P}_0$ is the Wiener measure on $\Omega = C(\mathbb{R}_+, \mathbb{R})$ under which the canonical process B is a standard Brownian motion, $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$ is the canonical filtration and $\mathbb{F}^a = (\mathcal{F}_t^a)_{t \ge 0}$ is the augmented filtration by $\mathbb{P}_0$. Denote by $\mathcal{T}^a$ the collection of all increasing families of $\mathbb{F}^a-$stopping times $\tau = (\tau_1, \cdots, \tau_n)$ such that the process $B_{\tau_n \wedge \cdot}$ is uniformly integrable. Define also the class of functions

$$\Lambda \;:=\; \mathcal{C}_1^n \;=\; \Big\{\lambda := (\lambda_1, \cdots, \lambda_n) \;: \lambda_k \in \mathcal{C}_1 \text{ for all } k = 1, \cdots n\Big\}. \tag{2.4}$$

For $\mu = (\mu_1, \cdots, \mu_n)$, $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $\big(\omega, \theta = (\theta_1, \cdots, \theta_n)\big) \in \overline{\Omega}$, we denote

$$\mu(\lambda) \;:=\; \sum_{k=1}^n \mu_k(\lambda_k) \quad \text{and} \quad \lambda(\omega_\theta) \;:=\; \sum_{k=1}^n \lambda_k(\omega_{\theta_i}) \text{ with } \omega_\theta \;:=\; (\omega_{\theta_1}, \cdots, \omega_{\theta_n}).$$

Then the first dual problem for the optimal SEP (2.3) is given by

$$D_0(\mu) \;:=\; \inf_{\lambda\in\Lambda}\Big\{\sup_{\tau\in\mathcal{T}^a}\mathbb{E}^{\mathbb{P}_0}\big[\Phi(B,\tau)-\lambda(B_\tau)\big]+\mu(\lambda)\Big\}. \tag{2.5}$$

As for the second dual problem, we return to the enlarged space $\overline{\Omega}$. Given $\overline{\mathbb{P}}\in\overline{\mathcal{P}}$, an $\overline{\mathbb{F}}-$adapted process $M=(M_t)_{t\geq 0}$ is called a strong $\overline{\mathbb{P}}-$supermartingale if

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[M_{\tau_2}\big|\overline{\mathcal{F}}_{\tau_1}\big] \;\leq\; M_{\tau_1},\ \overline{\mathbb{P}}-\text{a.s.}$$

for all $\overline{\mathbb{F}}-$stopping times $\tau_1\leq\tau_2$. Let $\mathbb{L}^2_{loc}$ be the space of all $\overline{\mathbb{F}}-$progressively measurable processes $\overline{H}=(\overline{H}_t)_{t\geq 0}$ such that

$$\int_0^t \overline{H}_s^2 ds<+\infty \text{ for every } t\geq 0,\ \overline{\mathcal{P}}-\text{q.s.}.$$

For $\overline{H}\in\mathbb{L}^2_{loc}$, the stochastic integral $(\overline{H}\cdot B):=\int_0^{\cdot}\overline{H}_s dB_s$ is well defined $\overline{\mathbb{P}}-$a.s. for all $\overline{\mathbb{P}}\in\overline{\mathcal{P}}$. We introduce a subset of processes:

$$\overline{\mathcal{H}} \;:=\; \Big\{\overline{H}\in\mathbb{L}^2_{loc}\ :(\overline{H}\cdot B)\text{ is a }\overline{\mathbb{P}}-\text{strong supermartingale for all }\overline{\mathbb{P}}\in\overline{\mathcal{P}}\Big\}.$$

Denote further

$$\mathcal{D} \;:=\; \Big\{(\lambda,\overline{H})\in\Lambda\times\overline{\mathcal{H}}:\lambda(B_T)+(\overline{H}\cdot B)_{T_n}\geq\Phi(B,T),\ \overline{\mathcal{P}}-\text{q.s.}\Big\},$$

and the second dual problem is given by

$$D(\mu) \;:=\; \inf_{(\lambda,\overline{H})\in\mathcal{D}}\mu(\lambda). \tag{2.6}$$

Loosely speaking, the two dual problems dualize respectively different constraints of the primal problem (2.3). By penalizing the marginal constraints, we obtain the first dual problem $D_0(\mu)$ of (2.5), where a multi-period optimal stopping problem appears for every fixed $\lambda\in\Lambda$. Then the second dual problem $D(\mu)$ of (2.6) follows by the resolution of the optimal stopping problem via the Snell envelop approach and the Doob-Meyer decomposition.

Our main result of this section is the following duality results.

Assumption 2.2. *The non-anticipative function Φ is bounded from above, and $\theta\mapsto\Phi(\omega,\theta)$ is upper-semicontinuous for each $\omega\in\Omega$.*

Assumption 2.3. *One of the following conditions holds true.*
(i) *The reward function Φ admits the representation $\Phi(\bar\omega)=\sum_{k=1}^n\Phi_k(\omega,\theta_1,\cdots,\theta_k)$, where for each $k=1,\cdots,n$, $\Phi_k:\Omega\times\mathbb{R}_+^k$ satisfies that $\Phi_k(\omega,\theta_1,\cdots,\theta_k)=\Phi_k(\omega_{\theta_k\wedge\cdot},\theta_1,\cdots,\theta_k)$, the map $\theta_k\mapsto\Phi_k(\bar\omega)$ is càdlàg and*

$$(\theta_1,\cdots,\theta_{k-1})\mapsto\Phi_k(\omega,\theta_1,\cdots,\theta_k)$$

is uniformly continuous for $0\leq\theta_1\leq\cdots\leq\theta_{k-1}\leq\theta_k$, uniformly in θ_k.
(ii) *The map $\bar\omega\mapsto\Phi(\bar\omega)$ is upper-semicontinuous.*

Theorem 2.4. (i) *Under Assumption 2.2, we have $P(\mu)=D_0(\mu)$.*
(ii) *Suppose in addition that Assumption 2.3 holds true, then*

$$P(\mu) \;=\; D_0(\mu) \;=\; D(\mu).$$

Remark 2.5. (i) *Based on the first dual problem* $D_0(\mu)$ *in* (2.5)*, a numerical algorithm has been obtained in Bonnans and Tan [6] for the above optimal SEP.*

(ii) *The second dual problem* $D(\mu)$ *in* (2.6) *is similar to that in Beiglböck, Cox and Huesmann [1], where they use a continuous martingale instead of the stochastic integral* $(H \cdot B)$ *in our formulation.*

(iii) *The duality result in [1, 2] is established under Assumptions 2.2 and 2.3* (ii) *. Our Assumption 2.2* (i) *does not involve any regularity in* ω*, and allows to handle examples as* $\Phi(\bar{\omega}) = \sum_{k=1}^{n} \varphi(\omega, \theta_k)$ *for some non-anticipative càdlàg process* φ *:* $\Omega \times \mathbb{R}_+ \to \mathbb{R}$.

3 Application to a class of martingale transport problems

In this section, we use the previous duality results of the optimal SEP to study a continuous time martingale transport problem under multiple marginal constraints. As an application in finance to study the robust superhedging problem, the multi-marginal case is very natural. Namely, when the vanilla options are available for trading for several maturities, which allows to recover the marginal distributions of the underlying asset at several times, we can formulate the robust superhedging problem as a martingale transport problem under multiple marginal constraints.

3.1 Robust superhedging and martingale transport

Define the canonical process $X := (X_t)_{0 \le t \le 1}$ by $X_t = B_{1 \wedge t}$ for all $t \in [0, 1]$ and its natural filtration $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{0 \le t \le 1}$. Denote further by $\mathcal{M}$ the collection of all martingale measures $\tilde{\mathbb{P}}$, i.e. the probability measures under which X is a martingale. Let $I := (0 < t_1 < \cdots < t_n = 1)$ be a set of time instants and define the set of martingale transport plans for $\mu \in \mathbf{P}^{\preceq}$

$$\mathcal{M}(\mu) \quad := \quad \Big\{ \tilde{\mathbb{P}} \in \mathcal{M} \ : X_{t_k} \overset{\tilde{\mathbb{P}}}{\sim} \mu_k \text{ for all } k = 1, \cdots, n \Big\}.$$

By Karandikar [31], there is a non-decreasing $\tilde{\mathbb{F}}-$progressive process $\langle X \rangle$ which coincides with the quadratic variation of X, $\tilde{\mathbb{P}}-$a.s. for every martingale measure $\tilde{\mathbb{P}} \in \mathcal{M}$. Denote

$$\langle X \rangle_t^{-1} \ := \ \inf \big\{ s \ge 0 \ : \langle X \rangle_s > t \big\} \text{ and } W_t \ := \ X_{\langle X \rangle_t^{-1}} \text{ for every } t \ge 0.$$

For a measurable function $\xi : \Omega \to \mathbb{R}$, the martingale transport problem under multiple marginal constraints is defined by

$$\tilde{P}(\mu) \quad := \sup_{\tilde{\mathbb{P}} \in \mathcal{M}(\mu)} \mathbb{E}^{\tilde{\mathbb{P}}} \big[\xi(X) \big]. \tag{3.7}$$

Denote by $\tilde{\mathcal{H}}$ the collection of all $\tilde{\mathbb{F}}-$progressive processes $\tilde{H} := (\tilde{H}_t)_{0 \le t \le 1}$ such that

$$\int_0^1 \tilde{H}_s^2 d\langle X \rangle_s < +\infty, \ \mathcal{M} - \text{q.s. and } (\tilde{H} \cdot X) \text{ is } \tilde{\mathbb{P}} - \text{supermartingale for all } \tilde{\mathbb{P}} \in \mathcal{M}.$$

Then the two dual problems are given by

$$\tilde{D}_0(\mu) \;:=\; \inf_{\lambda\in\Lambda}\Big\{\sup_{\tilde{\mathbb{P}}\in\mathcal{M}}\mathbb{E}^{\tilde{\mathbb{P}}}\big[\xi\big(X\big)-\lambda(X_I)\big]+\mu(\lambda)\Big\} \quad\text{and}\quad \tilde{D}(\mu) \;:=\; \inf_{(\lambda,\tilde{H})\in\tilde{\mathcal{D}}}\mu(\lambda),$$

where

$$\lambda(X_I) \;:=\; \sum_{i=1}^{n}\lambda_i(X_{t_i}) \text{ with } X_I \;:=\; (X_{t_1},\cdots,X_{t_n})$$

and

$$\tilde{\mathcal{D}} \;:=\; \Big\{(\lambda,\tilde{H})\in\Lambda\times\tilde{\mathcal{H}}:\lambda(X_I)+(\tilde{H}\cdot X)_1\geq\xi\big(X\big),\ \mathcal{M}-\text{q.s.}\Big\}.$$

It is easy to check that the weak dualities hold:

$$\tilde{P}(\mu) \;\leq\; \tilde{D}_0(\mu) \;\leq\; \tilde{D}(\mu). \tag{3.8}$$

Remark 3.1. *The problem $\tilde{D}(\mu)$ has a natural interpretation in financial mathematics as the minimal robust superhedging cost of the exotic option, where λ and $\tilde{H}$ are called the static and dynamic strategy, respectively. Here robustness refers to the fact that the underlying probability measure is not fixed a priori, so that the superhedging requirement is imposed under all possible models.*

3.2 Duality results

Using the duality results of the optimal SEP in Theorem 2.4, we can establish the duality for the above martingale transport problem.

Theorem 3.2. *Assume that the reward function ξ admits the representation*

$$\xi(X) \;=\; \Phi\big(W,\langle X\rangle_{t_1},\cdots,\langle X\rangle_{t_n}\big),$$

for some $\Phi:\overline{\Omega}\to\mathbb{R}$ satisfying Assumptions 2.2 and 2.3. Then

$$\tilde{P}(\mu) \;=\; \tilde{D}_0(\mu) \;=\; \tilde{D}(\mu).$$

Example 3.3. *Let $\phi:\big(\mathbb{R}_+\times\mathbb{R}^3\big)^n\to\mathbb{R}$ be upper-semicontinuous and bounded from above. Then the ξ defined below satisfies the condition in Theorem 3.2.*

$$\xi(X) \;=\; \phi\big(\langle X\rangle_{t_i},X_{t_i},\overline{X}_{t_i},\underline{X}_{t_i},i=1,\cdots n\big), \tag{3.9}$$

where $\overline{X}_t:=\sup_{0\leq s\leq t}X_s$ and $\underline{X}_t:=\inf_{0\leq s\leq t}X_s$.

Remark 3.4. *In Dolinsky & Soner [14], the duality is established (in a stronger sense) for the case $n=1$, for a general payoff function ξ which is Lipschtiz with respect to the uniform metric. In our Theorem 3.2, the reward function ξ is more specific, but it may include the dependence on the quadratic variation of the underlying process, which is related to the variance option in finance. Moreover, our results is given for the multiple marginal case, such an extension for their technique seems not obvious, see also the work of Hou & Obłój [28] and Biagini, Bouchard, Kardaras & Nutz [5]. More recently, an analogous duality is proved in the Skorokhod space under suitable conditions in Dolinsky & Soner [15] , where the underlying asset is assumed to take values in some subspace of càdlàg functions.*

Proof of Theorem 3.2. Combining the dualities $P(\mu) = D_0(\mu) = D(\mu)$ in Theorem 2.4 and the weak dualities $\tilde{P}(\mu) \le \tilde{D}_0(\mu) \le \tilde{D}(\mu)$, it is enough to prove

$$P(\mu) \quad \le \quad \tilde{P}(\mu) \quad \text{and} \quad D(\mu) \quad \ge \quad \tilde{D}(\mu),$$

where $P(\mu)$ and $D(\mu)$ are defined respectively in (2.3) and (2.6) with reward function Φ.

(i) Define the process $M := (M_t)_{0\le t\le 1}$ by

$$M_t \ := \ B_{\left(T_k + \frac{t-t_k}{t_{k+1}-t}\right)\wedge T_{k+1}} \quad \text{for all } t \in [t_k, t_{k+1}) \text{ and } 0 \le k \le n-1,$$

with $T_0 = t_0 = 0$ and $M_1 = B_{T_n}$. It is clear that M is a continuous martingale under every probability $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$ and $M_{t_k} = B_{T_k}$ for all $k = 1, \cdots, n$, which implies in particular $M_{t_k} \overset{\overline{\mathbb{P}}}{\sim} \mu_k$ for every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu)$. Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu)$ be arbitrary, then $\tilde{\mathbb{P}} := \overline{\mathbb{P}} \circ M^{-1} \in \mathcal{M}(\mu)$. Moreover, one finds $\overline{\mathbb{P}}-$a.s., $\langle M\rangle_{t_k} = T_k$ for all $k = 1, \cdots, n$ and $B_t = M_{\langle M\rangle_t^{-1}}$, which yields

$$\xi(M) \ = \ \Phi\big(B, \langle M\rangle_{t_1}, ..., \langle M\rangle_{t_n}\big) \ = \ \Phi(B, T), \ \ \overline{\mathbb{P}} - \text{a.s.}$$

Thus

$$\tilde{P}(\mu) \quad \ge \quad \mathbb{E}^{\overline{\mathbb{P}}}\big[\xi(M)\big] \quad = \quad \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi\big(B, T\big)\big]. \tag{3.10}$$

It follows that

$$P(\mu) \quad \le \quad \tilde{P}(\mu).$$

(ii) Let us now prove $\tilde{D}(\mu) \le D(\mu)$. Let $(\lambda, \overline{H}) \in \mathcal{D}$, i.e. $(\lambda, \overline{H}) \in \Lambda \times \overline{\mathcal{H}}$ be such that

$$\lambda(B_T) + (\overline{H}\cdot B)_{T_n} \quad \ge \quad \Phi\big(B, T\big), \ \ \overline{\mathcal{P}} - \text{q.s.}.$$

For every $\tilde{\mathbb{P}} \in \mathcal{M}$, it follows by Dambis-Dubins-Schwarz theorem that the time-changed process $W_t = X_{\langle X\rangle_t^{-1}}$ with $\langle X\rangle_t^{-1} = \inf\{s \ge 0 : \langle X\rangle_s > t\}$ is a Brownian motion with respect to the time-changed filtration $\big(\tilde{\mathcal{F}}_{\langle X\rangle_t^{-1}}\big)_{t\ge 0}$ under $\tilde{\mathbb{P}}$ and

$$X_t \ = \ W_{\langle X\rangle_t} \ \text{for every } t \in [0,1], \ \tilde{\mathbb{P}} - \text{a.s.}$$

Moreover, $\langle X\rangle_I := (\langle X\rangle_{t_k})_{1\le k\le n}$ are stopping times w.r.t. the time-changed filtration $\big(\tilde{\mathcal{F}}_{\langle X\rangle_t^{-1}}\big)_{t\ge 0}$. Let us define $\overline{\mathbb{P}} := \tilde{\mathbb{P}} \circ \big(W, \langle X\rangle_{t_1}, ..., \langle X\rangle_{t_n}\big)^{-1}$, then $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$ and thus we have $\tilde{\mathbb{P}}-$a.s.

$$\lambda\big(W_{\langle X\rangle_I}\big) + \big(\overline{H}_s \cdot W\big)_{\langle X\rangle_1} \quad \ge \quad \Phi\big(W, \langle X\rangle_{t_1}, ..., \langle X\rangle_{t_n}\big).$$

Define

$$\tilde{H}_s(X) \quad := \quad \overline{H}_{\langle X\rangle_s}\Big(W, \langle X\rangle_{t_1}, ..., \langle X\rangle_{t_n}\Big),$$

then it follows by Propositions V.1.4 and V.1.5 of Revuz and Yor [39] that $\overline{H}$ is $\tilde{\mathbb{F}}-$progressively measurable such that

$$\int_0^1 \tilde{H}_s^2 d\langle X\rangle_s \quad = \quad \int_0^{\langle X\rangle_1} \overline{H}_s^2 ds \quad < \quad +\infty, \ \tilde{\mathbb{P}} - \text{a.s.},$$

and

$$\left(\overline{H}\cdot W\right)_{\langle X\rangle_t} \;=\; (\tilde{H}\cdot X)_t \text{ for every } 0\le t\le 1,\ \tilde{\mathbb{P}}-\text{a.s.}$$

Hence

$$\lambda(X_I)+(\tilde{H}\cdot X)_1 \quad\ge\quad \Phi\big(W,\langle X\rangle_{t_1},...,\langle X\rangle_{t_n}\big) \quad=\quad \xi\big(X\big),\ \tilde{\mathbb{P}}-\text{a.s.} \tag{3.11}$$

Notice that $\overline{H}\in\overline{\mathcal{H}}$, and hence $(\overline{H}\cdot W)$ is a strong supermartingale under $\tilde{\mathbb{P}}$, which implies by the time-change argument that the stochastic integral $\left(\overline{H}\cdot W\right)_{\langle X\rangle_\cdot}$ is a supermartingale under $\tilde{\mathbb{P}}$ (with respect to its natural filtration) and so it is with $(\tilde{H}\cdot X)$. Hence $\tilde{H}\in\tilde{\mathcal{H}}$ and further $(\lambda,\tilde{H})\in\tilde{\mathcal{D}}$. It follows that $\tilde{D}(\mu)\le D(\mu)$, which concludes the proof. □

4 Proof of Theorem 2.4

To prove our main results in Theorem 2.4, we begin by a lemma which is repeatedly used in the following and then cite some technical lemmas from the classical optimal stopping theory.

4.1 Technical lemmas

Recall that $\mathbf{P}^{\preceq}$ denotes the collection of all centered peacocks, which is a collection of vectors of probability measures on $\mathbb{R}$. We first introduce a notion of convergence $\mathcal{W}_1$ on $\mathbf{P}^{\preceq}$ which is stronger than the weak convergence. A sequence of centered peacocks $\big(\mu^m=(\mu_1^m,\cdots,\mu_n^m)\big)_{m\ge1}\subset\mathbf{P}^{\preceq}$ is said to converge under $\mathcal{W}_1$ to $\mu^0=(\mu_1^0,\cdots,\mu_n^0)\in\mathbf{P}^{\preceq}$ if μ_k^m converges to μ_k^0 under the Wasserstein metric for all $k=1,\cdots,m$ (see e.g. Definition 6.1 in Villani [43]). This convergence is denoted by $\xrightarrow{\mathcal{W}_1}$. It follows by Theorem 6.9 of [43] that the convergence $\xrightarrow{\mathcal{W}_1}$ holds if and only if for any $\phi\in\mathcal{C}_1$,

$$\lim_{m\to\infty}\mu_k^m(\phi) \;=\; \mu_k^0(\phi) \text{ for all } k=1,\cdots,n. \tag{4.12}$$

In particular, the convergence $\xrightarrow{\mathcal{W}_1}$ implies the (usual) weak convergence.

Further, in order to apply the Fenchel-Moreau theorem, we shall consider a linear topological space containing all centered peacocks. Let $\mathbf{M}$ denote the space of all finite signed measures ν on $\mathbb{R}$ such that $\int_{\mathbb{R}}\big(1+|x|\big)\,|\nu|(dx)<+\infty$. We endow $\mathbf{M}$ with a topology of Wasserstein kind, denoted still by $\mathcal{W}_1$ without any confusion. Let $(\nu^m)_{m\ge1}\subset\mathbf{M}$ and $\nu^0\in\mathbf{M}$, we say ν^m converges to ν^0 under $\mathcal{W}_1$ if

$$\lim_{m\to\infty}\int_{\mathbb{R}}\phi(x)\nu^m(dx) \;=\; \int_{\mathbb{R}}\phi(x)\nu^0(dx) \text{ for all } \phi\in\mathcal{C}_1. \tag{4.13}$$

Let $\mathbf{M}^n:=\mathbf{M}\times...\times\mathbf{M}$ be the $n-$product of $\mathbf{M}$, endowed with the product topology. It is clear that under $\mathcal{W}_1$, $\mathbf{P}^{\preceq}$ is a closed convex subspace of $\mathbf{M}^n$ and the restriction of this convergence on $\mathbf{P}^{\preceq}$ is as same as the Wasserstein convergence.

It is well known that the space of all finite signed measures equipped with the weak convergence topology is a locally convex topological vector space, and its dual space is the space of all bounded continuous functions (see e.g. Section 3.2 of Deuschel and Stroock [14]). By exactly the same arguments, we have the following similar result.

Lemma 4.1. *There exists a topology $\mathcal{O}_n$ for $\mathbf{M}^n$ which is compatible with the convergence $\xrightarrow{\mathcal{W}_1}$ such that $(\mathbf{M}^n, \mathcal{O}_n)$ is a Hausdorff locally convex space. Moreover, its dual space can be identified by $(\mathbf{M}^n)^* = \Lambda$.*

We next turn to the space $\overline{\mathcal{P}}(\overline{\Omega})$ of all Borel probability measures on the Polish space $\overline{\Omega}$. Denote by $C_b(\overline{\Omega})$ the collection of all bounded continuous functions on $\overline{\Omega}$, and $B_{mc}(\overline{\Omega})$ the collection of all bounded measurable function ϕ, such that $\theta \mapsto \phi(\omega, \theta)$ is continuous for all $\omega \in \Omega$. Notice that the weak convergence topology on $\overline{\mathcal{P}}(\overline{\Omega})$ is defined as the coarsest topology under which $\overline{\mathbb{P}} \mapsto \mathbb{E}^{\overline{\mathbb{P}}}[\xi]$ is continuous for all $\xi \in C_b(\overline{\Omega})$. Following Jacod and Mémin [29], we introduce the stable convergence topology on $\overline{\mathcal{P}}(\overline{\Omega})$ as the coarsest topology under which $\overline{\mathbb{P}} \mapsto \mathbb{E}^{\overline{\mathbb{P}}}[\xi]$ is continuous for all $\xi \in B_{mc}(\overline{\Omega})$. Recall that every probability measure in $\overline{\mathcal{P}}$ (defined by (2.1)) has the same marginal law on Ω. Then as an immediate consequence of Proposition 2.4 of [29], we the following result.

Lemma 4.2. *The weak convergence topology and the stable convergence topology coincide on the space $\overline{\mathcal{P}}$.*

Lemma 4.3. *Let $(\mu^m)_{m\geq 1}$ be a sequence of centered peacocks such that $\mu^m \xrightarrow{\mathcal{W}_1} \mu^0$, $(\overline{\mathbb{P}}_m)_{m\geq 1}$ be a sequence of probability measures with $\overline{\mathbb{P}}_m \in \overline{\mathcal{P}}(\mu^m)$ for all $m \geq 1$. Then $(\overline{\mathbb{P}}_m)_{m\geq 1}$ is relatively compact under the weak convergence topology. Moreover, any accumulation point of $(\overline{\mathbb{P}}_m)_{m\geq 1}$ belongs to $\overline{\mathcal{P}}(\mu^0)$.*

Proof. (i) For any $\varepsilon > 0$, there exists a compact set $D \subset \Omega$ such that $\overline{\mathbb{P}}_m(D \times \Theta) = \mathbb{P}_0(D) \geq 1 - \varepsilon$ for every $m \geq 1$. In addition, by Proposition 7 of Monroe [35], one has for any constant $C > 0$,

$$\overline{\mathbb{P}}_m\big[T_n \geq C\big] \;\leq\; C^{-1/3}\Big(1 + \big(\mu_n^m(|x|)\big)^2\Big) \;\leq\; C^{-1/3}\Big(1 + \big(\sup_{m\geq 1} \mu_n^m(|x|)\big)^2\Big).$$

Choose the cube $[0, C]^n$ large enough such that $\overline{\mathbb{P}}_m\big[T \in [0, C]^n\big] \geq 1 - \varepsilon$ for all $m \geq 1$. The tightness of $(\overline{\mathbb{P}}_m)_{m\geq 1}$ under weak convergence topology follows by

$$\overline{\mathbb{P}}_m\big[D \times [0, C]^n\big] \;\geq\; \overline{\mathbb{P}}_m\big[D \times \Theta\big] + \overline{\mathbb{P}}_m\big[\Omega \times [0, C]^n\big] - 1 \;\geq\; 1 - 2\varepsilon \text{ for all } m \geq 1.$$

Let $\overline{\mathbb{P}}_0$ be any limit point. By possibly subtracting a subsequence, we assume that $\overline{\mathbb{P}}_m \to \overline{\mathbb{P}}_0$ weakly.

(ii) Notice that B is $\overline{\mathbb{F}}-$Brownian motion under each $\overline{\mathbb{P}}_m$ and thus the process $\varphi(B_t) - \int_0^t \frac{1}{2}\varphi''(B_s)ds$ is a $\overline{\mathbb{F}}-$martingale under $\overline{\mathbb{P}}_m$ whenever φ is bounded, smooth and of bounded derivatives. Notice that the maps $(\omega, \theta) \mapsto \varphi(\omega_t) - \int_0^t \varphi''(\omega_s)ds$ is also bounded continuous, then

$$\mathbb{E}^{\overline{\mathbb{P}}_0}\Big[\Big(\varphi(B_t) - \varphi(B_s) - \int_s^t \frac{1}{2}\varphi''(B_u)du\Big)\ \psi\Big] \;=\; 0 \tag{4.14}$$

for every $r < s < t$ and bounded continuous and $\overline{\mathcal{F}}_r-$measurable random variable ψ. Since $\overline{\mathcal{F}}_r$ can be generated by bounded continuous random variables (see Lemma A.1), it follows that (4.14) is still true for every bounded and $\overline{\mathcal{F}}_r-$measurable ψ. Letting $s \to r$, by the dominated convergence theorem, it follows that (4.14) holds for every $r = s < t$ and bounded $\overline{\mathcal{F}}_s-$measurable random variable ψ. This implies that B is a $\overline{\mathbb{F}}-$Brownian motion under $\overline{\mathbb{P}}_0$.

(iii) We next assume that $\overline{\mathbb{P}}_m \in \overline{\mathcal{P}}(\mu^m)$ and prove

$$B_{T_n\wedge\cdot} \text{ is uniformly integrable under } \overline{\mathbb{P}}_0. \tag{4.15}$$

The convergence of $(\mu^m)_{m\geq 1}$ to μ^0 implies in particular

$$\mathbb{E}^{\overline{\mathbb{P}}_m}\big[\big(\big|B_{T_n}\big| - R\big)^+\big] \;=\; \mu_n^m\big((|x|-R)^+\big) \;\longrightarrow\; \mu_n^0\big((|x|-R)^+\big) \;<\; +\infty.$$

Therefore, for every $\varepsilon > 0$, there is $R_\varepsilon > 0$ large enough such that $\mu_n^m\big((|x|-R_\varepsilon)^+\big) < \varepsilon$ for every $m \geq 1$. It follows by Jensen's inequality and $|x|\mathbf{1}_{\{|x|>2R\}} \leq 2(|x|-R)^+$ that

$$\mathbb{E}^{\overline{\mathbb{P}}_m}\big[\big|B_{T_n\wedge t}\big|\mathbf{1}_{\{|B_{T_n\wedge t}|>R_\varepsilon\}}\big] \;\leq\; 2\mathbb{E}^{\overline{\mathbb{P}}_m}\big[\big(\big|B_{T_n}\big| - R_\varepsilon\big)^+\big] \;\leq\; 2\varepsilon \;\text{ for all } t \geq 0.$$

Notice also that the function $|x|\mathbf{1}_{\{|x|>2R_\varepsilon\}}$ is lower semicontinuous and we obtain by Fatou's lemma

$$\mathbb{E}^{\overline{\mathbb{P}}_0}\big[\big|B_{T_n\wedge t}\big|\mathbf{1}_{\{|B_{T_n\wedge t}|>R_\varepsilon\}}\big] \;\leq\; \liminf_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}_m}\big[\big|B_{T_n\wedge t}\big|\mathbf{1}_{\{|B_{T_n\wedge t}|>R_\varepsilon\}}\big] \;\leq\; 2\varepsilon,$$

which justifies the claim (4.15). Moreover, since the map $(\omega, \theta) \mapsto \omega_{\theta_k}$ is continuous, it follows that $B_{T_k} \overset{\overline{\mathbb{P}}_0}{\sim} \mu_k^0$ for all $k = 1, \cdots, n$. Therefore, $\overline{\mathbb{P}}_0 \in \overline{\mathcal{P}}(\mu^0)$, which concludes the proof. □

We next recall some useful results from the classical optimal stopping theory (see e.g. El Karoui [16], Peskir and Shiryaev [38], Karatzas and Shreve [32] etc.) Let $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ be an abstract complete probability space, which is equipped with two filtrations $\mathbb{F}^* = (\mathcal{F}_t^*)_{t\geq 0}$ and $\mathbb{G}^* = (\mathcal{G}_t^*)_{t\geq 0}$, where $\mathcal{F}_t^* \subseteq \mathcal{G}_t^*$ for every $t \geq 0$ and both filtrations satisfy the usual conditions. Denote $\mathcal{F}_\infty^* := \vee_{t\geq 0}\mathcal{F}_t^*$ and $\mathcal{G}_\infty^* := \vee_{t\geq 0}\mathcal{G}_t^*$. We denote further by $\mathcal{T}_{\mathbb{F}^*}$ the class of all $\mathbb{F}^*-$stopping times, and by $\mathcal{T}_{\mathbb{G}^*}$ the collection of all $\mathbb{G}^*-$stopping times. Let Y be a càdlàg $\mathbb{F}^*-$optional process defined on Ω^* of class (D).

The first result is about the equivalence of the optimal stopping problem for different filtrations under the following so-called Assumption (K) in the optimal stopping theory.

Assumption 4.4 (K). *For every $t \geq 0$, every $\mathcal{G}_t-$measurable bounded random variable X satisfies*

$$\mathbb{E}\big[X|\mathcal{F}_t^*\big] \;=\; \mathbb{E}\big[X|\mathcal{F}_\infty^*\big],\; \mathbb{P}^* - a.s.$$

Lemma 4.5. *Under Assumption 4.4 we have*

$$\sup_{\tau\in\mathcal{T}_{\mathbb{F}^*}} \mathbb{E}[Y_\tau] \;=\; \sup_{\tau\in\mathcal{T}_{\mathbb{G}^*}} \mathbb{E}[Y_\tau].$$

Proof. The result follows by Theorem 5 of Szpirglas and Mazziotto [41]. We notice that they assume that Y is positive, which induces immediately the same result when Y is of class (D) since it can be dominated from below by some martingale. □

The next result recalls the so-called Snell envelope characterization of optimal stopping problems.

Lemma 4.6. *There is a càdlàg $\mathbb{F}^*-$adapted process Z, which is the smallest supermartingale dominating Y, and satisfies $\mathbb{E}[Z_0] = \sup_{\tau\in\mathcal{T}_{\mathbb{F}^*}}\mathbb{E}\big[Y_\tau\big]$.*

4.2 Proof of the first duality

We now provide the proof for the first duality result in Theorem 2.4. The main idea is to show that $\mu \mapsto P(\mu)$ is concave and upper-semicontinuous and then to use the Fenchel-Moreau theorem.

Lemma 4.7. *Under Assumption 2.2, the map $\mu \in \mathbf{P}^{\preceq} \mapsto P(\mu) \in \mathbb{R}$ is concave and upper-semicontinuous w.r.t. $\mathcal{W}_1$.*

Proof. (i) Let $\mu^1, \mu^2 \in \mathbf{P}^{\preceq}$, $\overline{\mathbb{P}}_1 \in \overline{\mathcal{P}}(\mu^1)$ and $\overline{\mathbb{P}}_2 \in \overline{\mathcal{P}}(\mu^2)$ and $\alpha \in (0,1)$, it follows by definition that $\alpha\overline{\mathbb{P}}_1 + (1-\alpha)\overline{\mathbb{P}}_2 \in \overline{\mathcal{P}}(\alpha\mu^1 + (1-\alpha)\mu^2)$. Therefore, it is clear that the map $\mu \mapsto P(\mu)$ is concave.
(ii) We now prove that $\mu \mapsto P(\mu)$ is upper-semicontinuous w.r.t. $\mathcal{W}_1$. Let $(\mu^m)_{m\geq 1} \subset \mathbf{P}^{\preceq}$ and $\mu^m \to \mu^0 \in \mathbf{P}^{\preceq}$ in $\mathcal{W}_1$. After possibly passing to a subsequence, we can have a family $(\overline{\mathbb{P}}_m)_{m\geq 1}$ such that

$$\overline{\mathbb{P}}_m \;\in\; \overline{\mathcal{P}}(\mu^m) \text{ and } \limsup_{m\to\infty} P(\mu^m) \;=\; \lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{P}}_m}\Big[\Phi\big(B,T\big)\Big].$$

B Lemma 4.3, we may find a subsequence still named by $(\overline{\mathbb{P}}_n)_{n\geq 1}$, which converges weakly to some $\overline{\mathbb{P}}_0 \in \overline{\mathcal{P}}(\mu^0)$. By Lemma 4.2, the map $\mathbb{P} \mapsto \mathbb{E}^{\mathbb{P}}\big[\Phi(B,T)\big]$ is upper-semicontinuous on $\overline{\mathcal{P}}$ w.r.t. the weak convergence topology for all Φ satisfying Assumption 2.2. We then obtain by Fatou's lemma that

$$\limsup_{m\to\infty} P(\mu^m) \;=\; \lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{P}}_m}\Big[\Phi\big(B,T\big)\Big] \;\leq\; \mathbb{E}^{\overline{\mathbb{P}}_0}\Big[\Phi\big(B,T\big)\Big] \;\leq\; P(\mu^0).$$

□

The results in Lemma 4.7 together with the Fenchel-Moreau theorem implies the first duality in Theorem 2.4. Before providing the proof, we consider the optimal stopping problem arising in the dual formulation (2.5). Denote for every $\lambda \in \Lambda$,

$$\Phi^{\lambda}(\omega,\theta) \;:=\; \Phi(\omega,\theta) - \lambda(\omega_\theta) \text{ for all } (\omega,\theta) \in \overline{\Omega}. \tag{4.16}$$

Recall that $\mathcal{T}^a$ denotes the collection of all increasing families of $\mathbb{F}^a-$stopping times $\tau = (\tau_1, \cdots, \tau_n)$ such that $B_{\tau_n \wedge \cdot}$ is uniformly integrable. Let $N > 0$, denote also by $\mathcal{T}^a_N \subset \mathcal{T}^a$ the subset of families $\tau = (\tau_1, \cdots, \tau_n)$ such that $\tau_n \leq N$, $\mathbb{P}_0-$a.s. Denote further by $\overline{\mathcal{P}}_N \subset \overline{\mathcal{P}}$ the collection of $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$ such that $T_n \leq N$, $\overline{\mathbb{P}}-$a.s.

Lemma 4.8. *Let Φ be bounded and continuous, then for every $\lambda \in \Lambda$*

$$\begin{aligned} \sup_{\tau\in\mathcal{T}^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^{\lambda}(B,\tau)] \;&=\; \lim_{N\to\infty} \sup_{\tau\in\mathcal{T}^a_N} \mathbb{E}^{\mathbb{P}_0}[\Phi^{\lambda}(B,\tau)] \\ &=\; \lim_{N\to\infty} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_N} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^{\lambda}(B,T)\big] \;=\; \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^{\lambda}(B,T)\big]. \end{aligned} \tag{4.17}$$

In particular, let $\phi \in \mathcal{C}_1$ and denote by ϕ^{conc} its concave envelope, then

$$\sup_{\tau\in\mathcal{T}^a} \mathbb{E}^{\mathbb{P}_0}[\phi(B_{\tau_n})] \;=\; \phi^{conc}(0). \tag{4.18}$$

Proof. (i) Given $\lambda \in \Lambda$ there is some constant $C > 0$ such that

$$\big|\Phi^{\lambda}(B,\tau)\big| \;\leq\; C\Big(1 + \sum_{k=1}^{n} \big|B_{\tau_k}\big|\Big). \tag{4.19}$$

Let $\tau \in \mathcal{T}^a$, define $\tau^N := (\tau_1^N, \cdots, \tau_n^N)$ with $\tau_k^N := \tau_k \wedge N$, then it is clear that $\lim_{N\to\infty} \Phi^\lambda(B, \tau^N) = \Phi^\lambda(B, \tau)$, $\mathbb{P}_0-$a.s. By the domination in (4.19) and the fact that $B_{\tau_n \wedge \cdot}$ is uniformly integrable, we have $\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_0}\big[\Phi^\lambda(B, \tau^N)\big] = \mathbb{E}^{\mathbb{P}_0}\big[\Phi^\lambda(B, \tau)\big]$. It follows by the arbitrariness of $\tau \in \mathcal{T}^a$ and the fact $\mathcal{T}_N^a \subset \mathcal{T}^a$ that

$$\sup_{\tau \in \mathcal{T}^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B, \tau)] \quad = \quad \lim_{N\to\infty} \sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B, \tau)].$$

By the same arguments, it is clear that we also have

$$\sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^\lambda(B, T)\big] \quad = \quad \lim_{N\to\infty} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_N} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^\lambda(B, T)\big].$$

(ii) We now apply Lemma 4.5 to prove that for every fixed constant $N > 0$,

$$\sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B, \tau)] \quad = \quad \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_N} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^\lambda(B, T)\big]. \tag{4.20}$$

First, let us suppose that $n = 1$. Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_N$, denote $Y_t := \Phi^\lambda(B, t \wedge N)$, it is clear that $\mathbb{E}^{\overline{\mathbb{P}}}\big[\sup_{t\geq 0} Y_t\big] < \infty$. Denote by $\overline{\mathbb{F}}^{\overline{\mathbb{P}}} = (\overline{\mathcal{F}}_t^{\overline{\mathbb{P}}})_{t\geq 0}$ the augmented filtration of $\overline{\mathbb{F}}$ under $\overline{\mathbb{P}}$ and by $\overline{\mathbb{F}}^{B,\overline{\mathbb{P}}}$ the filtration generated by B on $\overline{\Omega}$ and by $\overline{\mathbb{F}}^{B,\overline{\mathbb{P}}} = (\overline{\mathcal{F}}_t^{B,\overline{\mathbb{P}}})_{t\geq 0}$ its $\overline{\mathbb{P}}-$augmented filtration. It is clear that $\overline{\mathcal{F}}_t^{B,\overline{\mathbb{P}}} \subset \overline{\mathcal{F}}_t^{\overline{\mathbb{P}}}$. More importantly, by the fact that B is a $\overline{\mathbb{F}}^{\overline{\mathbb{P}}}-$Brownian motion under $\overline{\mathbb{P}}$, it is easy to check that the probability space $(\overline{\Omega}, \overline{\mathcal{F}}^{\overline{\mathbb{P}}}, \overline{\mathbb{P}})$ together with the filtration $\overline{\mathbb{F}}^{\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}^{B,\overline{\mathbb{P}}}$ satisfies Hypothesis (K) (Assumption 4.4). Then by Lemma 4.5, $\mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^\lambda(B, T)\big] \leq \sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B, \tau)]$ and hence $\sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_N} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^\lambda(B, T)\big] \leq \sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B, \tau)]$. We then have equality (4.20) since the inverse inequality is clear. Finally, when $n > 1$, it is enough to use the arguments together with induction to prove (4.20).
(iii) To prove (4.18) it suffices to set $\Phi \equiv 0$ and $n = 1$. Then by (4.17), it follows that

$$\sup_{\tau \in \mathcal{T}^a} \mathbb{E}^{\mathbb{P}_0}\big[\phi(B_\tau)\big] \quad = \quad \lim_{N\to\infty} \sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}\big[\phi(B_\tau)\big] \quad \leq \quad \phi^{conc}(0).$$

The other inequality is obvious by considering the exiting time of the Brownian motion from an open interval. We then conclude the proof. □

Proof of Theorem 2.4 (i)**.** For the first duality result, we shall use the Fenchel-Moreau theorem. Let us first extend the map $\mu \mapsto P(\mu)$ from $\mathbf{P}^{\preceq}$ to $\mathbf{M}^n$ by setting that $P(\mu) = -\infty$, for every $\mu \in \mathbf{M}^n \setminus \mathbf{P}^{\preceq}$. It is easy to check, using Lemma 4.7, that the extended map $\mu \mapsto P(\mu)$ from the topological vector space $\mathbf{M}^n$ to $\mathbb{R}$ is still concave and upper-semicontinuous. Then by the Fenchel-Moreau theorem together with Lemma 4.1 and, it follows that

$$\begin{aligned} P(\mu) &= P^{**}(\mu) \\ &= \inf_{\lambda \in \Lambda} \Big\{ \sup_{\nu \in \mathbf{P}^{\preceq}} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\nu)} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi^\lambda(B, T)\big] + \mu(\lambda) \Big\} \\ &= \inf_{\lambda \in \Lambda} \Big\{ \sup_{\tau \in \mathcal{T}^a} \mathbb{E}^{\mathbb{P}_0}\big[\Phi^\lambda(B, \tau)\big] + \mu(\lambda) \Big\}, \end{aligned}$$

where the last equality follows by (4.17). Hence we have $P(\mu) = D_0(\mu)$. □

Remark 4.9. *When Φ is bounded (which is the relevant case by the reduction of Section 4.3.1), we can prove further that*

$$D_0(\mu) \quad = \quad \inf_{\lambda\in\Lambda^+}\Big\{\sup_{\tau\in\mathcal{T}^a}\mathbb{E}^{\mathbb{P}_0}\big[\Phi^\lambda(B,\tau)\big]+\mu(\lambda)\Big\}, \tag{4.21}$$

where

$$\Lambda^+ \quad := \quad \big\{\lambda=(\lambda_1,\cdots,\lambda_n)\in\Lambda\ :\lambda_k\geq 0 \text{ for all } k=1,\cdots,n\big\}.$$

Indeed, using (4.18)*, it is easy to see that in the definition of $D_0(\mu)$, it is enough to take the infimum over the class of all functions $\lambda\in\Lambda^+$ such that the convex envelop $\lambda_k^{conv}(0)>-\infty$ for all $k=1,\cdots,m$, since by* (4.18) *and the boundedness of Φ, $\sup_{\tau\in\mathcal{T}^a}\mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda]=+\infty$ whenver $(-\lambda_k)^{conc}(0)=\infty$ for some k. Hence the infimum is taken among all $\lambda\in\Lambda$ such that $\lambda_k^{conv}(0)>-\infty$ for all $k=1,\cdots,m$, and consequently λ_k is dominated from below by some affine function. Since $\mathbb{E}^{\mathbb{P}_0}[B_{\tau_k}]=0$ for every $\tau\in\mathcal{T}^a$, we see that by possibly subtracting from λ_k the last affine function, it is enough to take infimum over the class Λ^+.*

4.3 Proof of the second duality

For the second duality, we shall use the Snell envelop characterization of the optimal stopping problem, together with the Doob-Meyer decomposition. We will provide the proof progressively.

Recall that $\mathcal{T}^a$ denotes the collection of all increasing families of $\mathbb{F}^a-$stopping times $\tau=(\tau_1,\cdots,\tau_n)$ such that $B_{\tau_n\wedge\cdot}$ is uniformly integrable, and $\mathcal{T}^a_N\subset\mathcal{T}^a$ the subset of families $\tau=(\tau_1,\cdots,\tau_n)$ such that $\tau_n\leq N$, $\mathbb{P}_0-$a.s. Denote also by $\mathcal{T}^0$ the collection of all $\mathbb{F}^a-$stopping times τ_0 such that $B_{\tau_0\wedge\cdot}$ is uniformly integrable, and by $\mathcal{T}^0_N$ the collection of all stopping times $\tau_0\in\mathcal{T}^0$ such that $\tau_0\leq N$, $\mathbb{P}_0-$a.s.

4.3.1 Reduction to bounded reward functions

Proposition 4.10. *To prove Theorem 2.4* (ii)*, it is enough to prove the duality $P(\mu)=D(\mu)$ under additional condition that Φ is bounded.*

Proof. Assume that the duality $P(\mu)=D(\mu)$ holds true whenever Φ is bounded and satisfies Assumptions 2.2 and 2.3.

We now consider the case without boundedness of Φ. Let $\Phi_m:=\Phi\vee(-m)$ (or $\Phi_m:=\sum_{k=1}^n(-m)\vee\Phi_k$ in case of Assumption 2.3 (i)), then Φ_m is bounded and satisfies Assumptions 2.2 and 2.3.

Denote by $P^m(\mu)$ and $D^m(\mu)$ the corresponding primal and dual values associated to the reward function Φ_m, so that we have the duality

$$P^m(\mu) \quad = \quad D^m(\mu).$$

Further, notice that $\Phi_m\geq\Phi$, one has $P^m(\mu)=D^m(\mu)\geq D(\mu)\geq P(\mu)$. Then it is enough to show that

$$\limsup_{m\to\infty}P^m(\mu) \quad \leq \quad P(\mu).$$

Let $\overline{\mathbb{P}}_m\in\overline{\mathcal{P}}(\mu)$ such that $\limsup_{m\to\infty}P^m(\mu)=\limsup_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}_m}[\Phi_m]$. Then after possibly passing to a subsequence we may assume that $\limsup_{m\to\infty}P^m(\mu)=$

$\lim_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}_m}[\Phi_m]$. By Lemma 4.3, we know that $(\overline{\mathbb{P}}_m)_{m\geq 1}$ is tight and every limit point belongs to $\overline{\mathcal{P}}(\mu)$. Let $\overline{\mathbb{P}}_0$ be a limit point of $(\overline{\mathbb{P}}_m)_{m\geq 1}$, and label again the convergent subsequence by m, i.e. $\overline{\mathbb{P}}_m \to \overline{\mathbb{P}}_0$. Then by the monotone convergence theorem

$$\begin{aligned} P(\mu) \quad &\geq \quad \mathbb{E}^{\overline{\mathbb{P}}_0}[\Phi] \quad = \quad \lim_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}_0}[\Phi_m] \quad = \quad \lim_{m\to\infty}\Big(\lim_{l\to\infty}\mathbb{E}^{\overline{\mathbb{P}}_l}[\Phi_m]\Big) \\ &\geq \quad \lim_{m\to\infty}\Big(\lim_{l\to\infty}\mathbb{E}^{\overline{\mathbb{P}}_l}[\Phi_l]\Big) \quad = \quad \limsup_{l\to\infty} P^l(\mu), \end{aligned}$$

which is the required result. □

4.3.2 The second duality for a separable reward function

We first provide the duality result in a simplified context, so that the analysis of the multiple stopping problem in (2.5) is much easier. We observe that the following result is not needed for the proof of our main duality result, and is only provided here for the convenience of the reader.

Let us first introduce a stronger version of the second dual problem. Denote by $\mathcal{H}$ the collection of all $\mathbb{F}^0-$predictable process $H^0:\mathbb{R}^+\times\Omega\to\mathbb{R}$ such that the stochastic integral $(H^0\cdot B)_t := \int_0^t H^0_s dB_s$ is a martingale under $\mathbb{P}_0$, and $(H^0\cdot B)_t \geq -C(1+|B_t|)$ for some constant $C>0$. Define then

$$\begin{aligned} \mathcal{D}' \quad := \quad \Big\{ (\lambda, H^1,\cdots,H^n)\in\Lambda\times(\mathcal{H})^n \ : \sum_{k=1}^n \big(\lambda_k(\omega_{\theta_k}) + \int_{\theta_{k-1}}^{\theta_k} H^k_s dB_s\big) \geq \Phi\big(\omega,\theta\big) \\ \text{for all } 0\leq\theta_1\leq\cdots\leq\theta_n, \text{ and } \mathbb{P}_0-\text{a.e. } \omega\in\Omega \Big\}. \end{aligned}$$

Proposition 4.11. *Suppose that Assumption 2.2 and Assumption 2.3* (i) *hold true with $\Phi_k : \Omega\times(\mathbb{R})^k$. Suppose in addition that $\Phi_k(\omega,\theta_1,\cdots,\theta_k)$ depends only on (ω,θ_k). Then*

$$P(\mu) \quad = \quad D'(\mu) \quad := \quad \inf_{(\lambda,H)\in\mathcal{D}'} \mu(\lambda).$$

Proof. Given the first duality $P(\mu)=D_0(\mu)$, it suffices to study the optimal stopping problem

$$\sup_{\tau\in\mathcal{T}^a}\Big[\Phi^\lambda\big(B,\tau\big)\Big] \quad = \quad \lim_{N\to\infty}\sup_{\tau\in\mathcal{T}^a_N}\mathbb{E}[\Phi^\lambda(B,\tau)], \tag{4.22}$$

for a given $\lambda\in\Lambda^+$ (Remark 4.9) and under the additional condition that each Φ_k is bounded (Proposition 4.10). Notice that in this case, there is some C such that

$$-C\Big(1+\sum_{k=1}^n|\omega_{\theta_k}|\Big) \ \leq \ \Phi^\lambda(\bar\omega) \ \leq \ C.$$

Suppose that $n=1$, then by Lemma 4.6, there is a supermartingale $(Z^{1,N}_t)$ for every $N\in\mathbb{N}$ which is the Snell envelop for the optimal stopping problem $\sup_{\tau\in\mathcal{T}^a_N}\mathbb{E}[\Phi^{\boldsymbol{\lambda}}(B,\tau)]$. Clearly, $Z^{1,N}$ increases in N. Moreover, one has $-C(1+|B_t|)\leq Z^{1,N}_t\leq C$ for some constant C independent of N. Then by the dominated convergence theorem together with Lemma 4.8, $Z^1:=\sup_{N\in\mathbb{N}}Z^{1,N}$ is still a supermartingale, of class (DL), such that

$$Z^1_0 = \sup_{\tau\in\mathcal{T}^a}\mathbb{E}[\Phi^\lambda(B,\tau)], \text{ and } Z^1_t\geq\Phi^\lambda(B,t), \text{ for all } t\geq 0, \ \mathbb{P}_0-\text{a.s.}$$

We claim that the supermartingale Z^1 is right-continuous in expectation and hence admits a right-continuous modification. Indeed, let τ be a bounded stopping time, and $(\tau_n)_{n\geq 1}$ be a sequence of bounded stopping time such that $\tau_n \searrow \tau$. Then by the supermartingale property, we have $\mathbb{E}[Z^1_{\tau_n}] \leq \mathbb{E}[Z^1_\tau]$. Further, for every $\varepsilon > 0$, by the definition of Z^1, there is some bounded stopping time $\sigma_\varepsilon \geq \tau$ such that $\mathbb{E}[Z^1_\tau] \leq \mathbb{E}\big[\Phi^\lambda(B,\sigma_\varepsilon)]+\varepsilon$. It follows that $\mathbb{E}[Z^1_{\tau_n}] \geq \mathbb{E}[\Phi^\lambda(B,\sigma_\varepsilon \wedge \tau_n)] \to \mathbb{E}[\Phi^\lambda(B,\sigma_\varepsilon)]$. Thus $\lim_{n\to\infty} \mathbb{E}[Z^1_{\tau_n}] = \mathbb{E}[Z^1_\tau]$ by arbitrariness of $\varepsilon > 0$.

Now, by the Doob-Meyer decomposition for right-continuous supermartingales together with the martingale representation, there is an $\mathbb{F}^a-$predictable process H^1 such that

$$\lambda_1(B_t) \;+\; (H^1 \cdot B)_t \;\geq\; \Phi(B,t), \text{ for all } t \geq 0, \quad \mathbb{P}_0 - \text{a.s.}$$

Further, we can also choose H^1 to be $\mathbb{F}-$predictable (see e.g. Theorem IV.78 and Remark IV.74 of Dellacherie and Meyer [13]). This proves in particular that

$$D'(\mu) \;\leq\; D_0(\mu) \;=\; P(\mu).$$

Hence by weak duality, we get

$$P(\mu) \;=\; D_0(\mu) \;=\; D'(\mu).$$

Suppose now $n = 2$, we first consider the optimal stopping problem

$$\sup_{\tau \in \mathcal{T}^0_N} \big[\Phi_2(B,\tau) - \lambda_2(B_\tau)\big],$$

whose Snell envelop is given by $Z^{2,N}$ by Lemma 4.6, where in particular $-C(1+|B_t|) \leq Z^{2,N}_t \leq C$ for some constant C independent of N, and

$$Z^{2,N}_{\theta_2} \geq \Phi_2(B,\theta_2) - \lambda_2(B_{\theta_2}), \quad \text{for all } \theta_2 \leq N, \quad \mathbb{P}_0 - \text{a.s.}$$

We then reduce the multiple optimal stopping problem (4.22) to the $n = 1$ case, i.e.

$$\sup_{\tau \in \mathcal{T}^a_N} \mathbb{E}\big[\Phi^\lambda(B,\tau)\big] \;=\; \sup_{\tau_1 \in \mathcal{T}^0_N} \mathbb{E}\big[Z^{2,N}_{\tau_1} + \Phi_1(B,\tau_1) - \lambda_1(B_{\tau_1})\big].$$

Using again the procedure for the case $n = 1$, we obtain a new Snell envelop, denoted by $Z^{1,N}$, such that $Z^{1,N}_t \geq -C(1+|B_t|)$.

Thus, $Z^{1,N}, Z^{2,N}$ are both supermartingales of class (D), bounded from above by C, and dominated from below by $-C(1+|B_t|)$ for some constant $C > 0$ independent of N. More importantly, we have $Z^{1,N}_0 = \sup_{\tau\in\mathcal{T}^a_N} \mathbb{E}\big[\Phi^\lambda(B,\tau)\big]$, and

$$Z^{1,N}_{\theta_1} \;+\; \big(Z^{2,N}_{\theta_2} - Z^{2,N}_{\theta_1}\big) \;\geq\; \Phi^\lambda(B,\theta_1,\theta_2), \text{ for all } \theta_1 \leq \theta_2 \leq N, \;\; \mathbb{P}_0 - \text{a.s.}$$

Since $Z^{1,N}$ and $Z^{2,N}$ both increase in N, define $Z^1 := \sup_N Z^{1,N}$ and $Z^2 := \sup_N Z^{2,N}$, it follows by the dominated convergence theorem that Z^1 and Z^2 are both supermartingales of class (DL). By similar arguments as in the case $n = 1$, we may consider Z^1 and Z^2 in their right-continuous modifications. Moreover, it follows from Lemma 4.8 that $Z^1_0 = \sup_{\tau\in\mathcal{T}^a} \mathbb{E}\big[\Phi^\lambda(B,\tau)\big]$ and

$$Z^1_{\theta_1} \;+\; \big(Z^2_{\theta_2} - Z^2_{\theta_1}\big) \;\geq\; \Phi^\lambda(B,\theta_1,\theta_2), \text{ for all } \theta_1 \leq \theta_2, \;\; \mathbb{P}_0 - \text{a.s.}$$

Now, using the Doob-Meyer decomposition, together with the martingale representation on Z^1 and Z^2, we obtain the process $H = (H^1, H^2)$ as we need in the dual formulation $\mathcal{D}'$.

Finally, the case $n > 2$ can be handled by exactly the same recursive argument as for the case $n = 2$. $\square$

4.3.3 The second duality under Assumption 2.3 (i)

Let $N > 0$, we first study the multiple optimal stopping problem

$$\sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}\big[\Phi^\lambda(B,\tau)\big] \;=\; \sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}\Big[\sum_{k=1}^n \Big(\Phi_k(B,\tau_1,\cdots,\tau_k) - \lambda_k(B_{\tau_k})\Big)\Big], \tag{4.23}$$

where $\lambda \in \Lambda^+$ and Φ_k is bounded, so that

$$-C\Big(1+\sum_{k=1}^n |\omega_{\theta_k}|\Big) \;\le\; \Phi^\lambda(\bar\omega) \;\le\; C,$$

for some constant C. Denote $v_{n+1}^N(\omega,\theta_1,\cdots,\theta_n,\theta_n) := \Phi^\lambda(\omega,\theta_1,\cdots,\theta_n)$.

Lemma 4.12. *There are functionals $(v_k^N)_{k=1,\cdots,n}$, where $v_k^N : \Omega\times(\mathbb{R}_+)^k \to \mathbb{R}$, such that*

$$v_1^N(\omega,0) \;=\; \sup_{\tau\in\mathcal{T}_N^a} \mathbb{E}\big[\Phi^\lambda(B,\tau)\big],$$

and under $\mathbb{P}_0$, for each $k=1,\cdots,n$, and $\theta_1\le\cdots\le\theta_{k-1}$, the process

$$\theta \mapsto v_k^N(B,\theta_1,\cdots,\theta_{k-1},\theta) \textit{ is a right-continuous } \mathbb{F}-\textit{supermartingale},$$

$$v_k^N(B,\theta_1,\theta_{k-1},\theta) \;\ge\; v_{k+1}^N(B,\theta_1,\cdots,\theta_{k-1},\theta,\theta), \;\; a.s.$$

Moreover, v_k^N increases in N and satisfies $-C(1+|B_t|)\le v_k^N(B,\cdots,t)\le C$ for some constant C independent of N.

Proof of Theorem 2.4 (ii) By Remark 4.9 and Proposition 4.10, we can assume without loss of generality that each Φ_k is bounded and choose $\lambda\in\Lambda^+$ in the dual formulation $D_0(\mu)$.

(i) Let v_k^N be given by Lemma 4.12, we define further

$$v_k(\cdot) \;:=\; \sup_N v_k^N(\cdot), \;\; \text{so that } v_0(\omega,0) = \sup_{\tau\in\mathcal{T}^a} \mathbb{E}\big[\Phi^\lambda(B,\tau)\big].$$

It follows from the dominated convergence theorem that, for all $k=1,\cdots,n$ and $0=:\theta_0\le\theta_1\le\cdots\le\theta_{k-1}$, the process $\big(v_k(B,\theta_1,\cdots,\theta_{k-1},t)\big)_{t\ge\theta_{k-1}}$ is a $\mathbb{F}-$supermartingale and

$$v_k(B,\theta_1,\cdots,\theta_{k-1},t) \;\ge\; v_{k+1}(B,\theta_1,\cdots,\theta_{k-1},t,t), \text{for all } t\ge\theta_{k-1}, \;\mathbb{P}_0-\text{a.s.}$$

By the same arguments as in Proposition 4.11, it can be modified to be a right-continuous supermartingale since it is right-continuous in expectation.

(ii) By the Doob-Meyer decomposition and the martingale representation theorem, it follows that for each $k=1,\cdots,n$, there is some $\mathbb{F}^a-$predictable process $H_t^k(\omega) := H_t^k(\omega,\theta_1,...,\theta_{k-1})$ such that $\mathbb{P}_0-$a.s.

$$\begin{aligned} v_k(\omega,\theta_1,...,\theta_{k-1},\theta_{k-1}) + \int_{\theta_{k-1}}^{\theta_k} H_u^k dB_u \;&\ge\; v_k(\omega,\theta_1,...\theta_{k-1},\theta_k) \\ &\ge\; v_{k+1}(\omega,\theta_1,...\theta_{k-1},\theta_k,\theta_k). \end{aligned} \tag{4.24}$$

Moreover, since the quadratic co-variation $\langle v_k(B, \theta_1, \cdots, \theta_{k-1}, \cdot), B.\rangle_t$ can be defined pathwisely (see e.g. Karandikar [31]), H^k can also be defined pathwisely and to be $\mathbb{F}-$progressively measurable by

$$H_t^k(\theta_1, \cdots, \theta_{k-1}) \;:=\; \limsup_{\varepsilon \to 0} \frac{\langle v_k(\theta_1, \cdots, \theta_{k-1}, \cdot), B\rangle_t - \langle v_k(\theta_1, \cdots, \theta_{k-1}, \cdot), B\rangle_{t-\varepsilon}}{\varepsilon}.$$

In particular, the map $(\omega, \theta_1, \cdots, \theta_k) \mapsto H^k_{\theta_k}(\omega, \theta_1, \cdots, \theta_{k-1})$ is Borel measurable, and

$$\int_{\theta_{k-1}}^t \big(H_s^k(\cdot, \theta_1, \cdots, \theta_{k-1})\big)^2 ds \;< +\infty \text{ for all } t \geq \theta_{k-1},\ \mathbb{P}_0 - \text{a.s.} \tag{4.25}$$

(iii) Next, we define a process $\overline{H} : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R}$ by

$$\overline{H}_u(\bar\omega) \;:=\; \sum_{k=1}^n \mathbf{1}_{(\theta_{k-1}, \theta_k]}(u) H_u^k(\omega, \theta_1, ..., \theta_{k-1}) \text{ for all } \bar\omega = (\omega, \theta) \in \overline{\Omega},$$

where by convention $\theta_0 = 0$. Moreover, since

$$(\omega, \theta_1, \cdots, \theta_k) \;\mapsto\; H^k_{\theta_k}(\omega, \theta_1, \cdots, \theta_{k-1}) \text{ is Borel measurable,}$$

the process $\overline{H}$ is $\overline{\mathbb{F}}-$progressively measurable by Lemma A.2 in Appendix.

(iv) Now, let us take an arbitrary $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$ and consider a family of r.c.p.d. (regular conditional probability distributions) $(\overline{\mathbb{P}}_{\bar\omega})_{\bar\omega \in \overline{\Omega}}$ of $\overline{\mathbb{P}}$ with respect to $\overline{\mathcal{F}}_{T_k}$ for $0 \leq k \leq n-1$ (see Lemma A.2 for the existence of r.c.p.d.). Then for $\overline{\mathbb{P}}-$almost every $\bar\omega \in \overline{\Omega}$, under the conditional probability $\overline{\mathbb{P}}_{\bar\omega}$, the process $t \mapsto B_t$ for $t \geq T_k$ is still a Brownian motion. Moreover, we have $\overline{\mathbb{P}}_{\bar\omega}(T_k = \theta_k, B_{T_k \wedge \cdot} = \omega_{\theta_k \wedge \cdot}) = 1$. Then it follows by (4.24) that

$$\begin{aligned} v_{k+1}(B, T_1, ..., T_k, T_k) \;&\leq\; v_k(B, T_1, ..., T_k) \\ &\leq\; v_k(B, T_1, ..., T_{k-1}, T_{k-1}) + \int_{T_{k-1}}^{T_k} H_s^k dB_s,\ \overline{\mathbb{P}}_{\bar\omega} - \text{a.s.} \end{aligned}$$

This means that the set $A_k := \big\{v_{k+1} \leq v_k + \int_{T_{k-1}}^{T_k} H_s^k dB_s\big\}$ is of full measure under $\overline{\mathbb{P}}_{\bar\omega}$ for $\overline{\mathbb{P}}-$almost every $\bar\omega \in \overline{\Omega}$, and hence by the tower property $\overline{\mathbb{P}}(A_k) = 1$ for all $k = 0, \cdots, n$. This yields that

$$\Phi^\lambda(B, T) \;=\; v_{n+1}(B, T_1, ..., T_n, T_n) \;\leq\; v_1(B, 0) + (\overline{H} \cdot B)_{T_n},\ \overline{\mathbb{P}} - \text{a.s.} \tag{4.26}$$

(v) To conclude the proof, it suffices to check that $\overline{H} \in \overline{\mathcal{H}}$. First, for any probability measure $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$, by taking r.c.p.d and using (4.25), it is clear that

$$\int_0^t \overline{H}_s^2 ds < +\infty \text{ for every } t \geq 0,\ \overline{\mathbb{P}} - \text{a.s.}$$

Further, by Remark 4.9, Φ is bounded and $\lambda \in \Lambda_+$. Notice also that (4.26) holds true for every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$, and by the tower property, it follows that for any $\overline{\mathbb{F}}-$stopping time τ, we have for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$,

$$(\overline{H} \cdot B)_{T_n \wedge \tau} \;\;\geq\;\; -C\Big(1 + \sup_{1 \leq k \leq n} |B_{T_k \wedge \tau}|\Big),\ \overline{\mathbb{P}} - \text{a.s.},$$

where the r.h.s. is uniformly integrable under $\overline{\mathbb{P}}$. Using Fatou's Lemma, it follows that $(\overline{H} \cdot B)_{T_n \wedge \cdot}$ is a strong supermartingale under every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$. $\Box$

Proof of Lemma 4.12. We provide here a proof for the case $n = 2$ for ease of presentation. The general case can be treated by exactly the same backward iterative procedure. We will use the classical aggregation procedure in the optimal stopping theory (see e.g. El Karoui [16], Peskir and Shiryaev [38], Karatzas and Shreve [32] etc.) A good resume can also be found in Kobylanski, Quenez and Rouy-Mironescu [34].

1. For every $\tau_1 \in \mathcal{T}_N^0$, we first consider the optimal stopping problem

$$\sup_{\tau_2 \in \mathcal{T}_N^0,\ \tau_2 \geq \tau_1} \mathbb{E}\Big[\Phi_2\big(B, \tau_1, \tau_2\big) - \lambda_2\big(B_{\tau_2}\big)\Big],$$

whose Snell envelop is denoted by $(Z^{2,N}_{\tau_1,t})_{\tau_1 \leq t \leq N}$. We shall prove in Step 2 below that the above process can be aggregated into a function $u^{2,N}(\omega, \theta_1, \theta_2)$ which is Borel measurable as a map from $\Omega \times (\mathbb{R}_+)^2 \to \mathbb{R}$, and

$$\begin{array}{c} \text{uniformly continuous in } \theta_1, \text{ right continuous in } \theta_2,\ \mathbb{P}_0\text{−a.s.} \\ u^{2,N}(\cdot, \tau_1, \tau_2) \ = \ Z^{2,N}_{\tau_1,\tau_2}, \ \ \mathbb{P}_0 - \text{a.s. for all } \tau_1 \leq \tau_2 \leq N. \end{array} \tag{4.27}$$

In particular, it is clear that $t \mapsto u^{2,N}(\omega, t, t)$ is right-continuous $\mathbb{P}_0$−a.s. Let

$$v^{2,N}(\omega, \theta_1, \theta_2) \ := \ u^{2,N}(\omega, \theta_1, \theta_2) \ + \ \Phi_1(\omega, \theta_1) \ - \ \lambda_1(\omega_{\theta_1}),$$

and consider the optimal stopping problem

$$\sup_{\tau_1 \in \mathcal{T}_N^0} \mathbb{E}\big[v^{2,N}(\cdot, \tau_1, \tau_1)\big] \qquad \Big(= \sup_{\tau \in \mathcal{T}_N^a} \mathbb{E}\big[\Phi^{\lambda}(B, \tau)\big].\Big).$$

Denoted by $(Z^{1,N}_t)_{0 \leq t \leq N}$ the corresponding Snell envelop. Notice that $Z^{1,N}_t$ is $\mathbb{F}^a$−adapted and right-continuous, and hence it is $\mathbb{F}^a$−optional (or equivalently $\mathbb{F}^a$−predictable, since $\mathbb{F}^a$ is the augmented Brownian filtration), then $Z^{1,N}_t$ can be chosen to be $\mathbb{F}$−predictable (see e.g. Theorem IV.78 and Remark IV.74 of Dellacherie and Meyer [13]). Define $v^{1,N}(\omega, \theta_1) := Z^{1,N}(\omega, \theta_1)$. Then $v^{1,N}(\cdot), v^{2,N}(\cdot)$ are the required functionals.

2. We now construct the measurable map $u^{2,N}$ satisfying (4.27). Let $\tau_1 \leq \tau_2 \in \mathcal{T}_N^0$, define a random variable

$$\overline{Z}^{2,N}_{\tau_1,\tau_2} \ := \ \operatorname*{ess\,sup}_{\tau_3 \in \mathcal{T}_N^0,\ \tau_3 \geq \tau_2} \mathbb{E}\Big[\Phi_2\big(B, \tau_1, \tau_3\big) - \lambda_2\big(B_{\tau_3}\big) \Big| \mathcal{F}^a_{\tau_2}\Big]. \tag{4.28}$$

It is clear that for two stopping times τ_1^1 and τ_1^2 smaller than τ_2, we have

$$\overline{Z}^{2,N}_{\tau_1^1,\tau_2} \ = \ \overline{Z}^{2,N}_{\tau_1^2,\tau_2}, \ \mathbb{P}_0 - \text{a.s. on } A = \{\tau_1^1 = \tau_1^2\}. \tag{4.29}$$

Notice that the process $t \mapsto \Phi_2(B, \tau_1, t) - \lambda_2(B_t)$ is right-continuous and hence right-continuous in expectation, then for fixed τ_1, the family of random variables $(\overline{Z}^{2,N}_{\tau_1,\tau_2})_{\tau_2}$ is right-continuous in expectation (see e.g. Proposition 1.5 of [34]). Then, for every fixed τ_1, it can be aggregated into a right-continuous supermartingale, denoted by $Z^{2,N}_{\tau_1,t}$ (see e.g. Proposition 4.1 of [34]), such that $\overline{Z}^{2,N}_{\tau_1,\tau_2} = Z^{2,N}_{\tau_1,\tau_2}$, $\mathbb{P}_0$−a.s.

Notice that $Z^{2,N}_{\tau_1,t}$ is $\mathbb{F}^a-$optional and equivalently $\mathbb{F}^a-$predictable, we can choose $Z^{2,N}_{\tau_1,t}$ to be $\mathbb{F}-$predictable.

Further, since $\Phi_2(\omega, \theta_1, \theta_2)$ is uniformly continuous in θ_1, denote by ρ the continuity modulus. Then it follows by its definition in (4.28) that the family of random variables $\overline{Z}^{2,N}_{\tau_1,\tau_2}$ is uniformly continuous w.r.t. τ_1, in sense that

$$\big|\overline{Z}^{2,N}_{\tau_1^1,\tau_2} - \overline{Z}^{2,N}_{\tau_1^2,\tau_2}\big| \;\le\; \rho(|\tau_1^1 - \tau_1^2|), \; \mathbb{P}_0 - \text{a.s. for stopping times } \tau_1^i \le \tau_2.$$

We now define $u^{2,N}$ by

$$u^{2,N}(\omega, \theta_1, \theta_2) \;\;:=\;\; Z^{2,N}_{\theta_1,\theta_2}(\omega), \;\text{for all } \theta_1 \in \mathbb{Q},$$

and

$$u^{2,N}(\omega, \theta_1, \theta_2) \;\;:=\;\; \lim_{\mathbb{Q}\ni\theta_1'\to\theta_1} u^{2,N}(\omega, \theta_1, \theta_2), \;\text{for all } \theta_1 \notin \mathbb{Q}.$$

It is clear that $u^{2,N}$ is Borel measurable w.r.t. each variable since $Z^{2,N}_{\theta_1,\theta_2}(\omega)$ is chosen to be $\mathbb{F}-$predictable. Furthermore, by (4.29), we have $u^{2,N}(\omega, \tau_1, \theta_2) = Z^{2,N}(\omega, \tau_1, \theta_2)$ for all $\theta \ge \tau_1$, $\mathbb{P}_0-$a.s., for every stopping times τ_1 taking values in $\mathbb{Q}$. Since we can approximate any stopping time by stopping times taking values in $\mathbb{Q}$, then by the uniform continuity of $\overline{Z}^{2,N}_{\tau_1,\tau_2}$ w.r.t. τ_1, we obtain that

$$\overline{Z}^{2,N}_{\tau_1,\tau_2} = Z^{2,N}_{\tau_1,\tau_2} = u^{2,N}(\cdot, \tau_1, \tau_2)\; \mathbb{P}_0 - \text{a.s. for all stopping times } \tau_1 \le \tau_2 \in \mathcal{T}^1_N.$$

In particular, $u^{2,N}(\omega, \theta_1, \theta_2)$ is uniformly continuous in θ_1 and right-continuous in θ_2, $\mathbb{P}_0-$a.s., which is the required functional in claim (4.27). □

Remark 4.13. *We notice that a general multiple optimal stopping problem has been studied in Kobylanski, Quenez & Rouy-Mironescu [34], where they proved the existence of optimal multiple stopping times by a constructive method. Here we are in a specified context with Brownian motion and we are interested in finding a process $\overline{H}$ whose stochastic integral dominates the value process.*

4.3.4 The second duality under Assumption 2.3 (ii)

Let Φ satisfy Assumption 2.2 and Assumption 2.3 (ii), i.e. $\bar{\omega} \mapsto \Phi(\bar{\omega})$ is upper-semicontinuous and bounded from above. Define a metric d of Polish space $\overline{\Omega}$ by

$$d(\bar{\omega}, \bar{\omega}') \;:=\; \sum_{k=1}^{n} \big(|\theta_k - \theta_k'| \;+\; \|\omega_{\theta_k\wedge\cdot} - \omega'_{\theta_k'\wedge\cdot}\|\big),$$

and then define $\Phi_m : \overline{\Omega} \to \mathbb{R}$ by

$$\Phi_m(\bar{\omega}) := \sup_{\bar{\omega}'\in\overline{\Omega}} \big\{\Phi(\bar{\omega}') - m d(\bar{\omega}, \bar{\omega}')\big\}. \tag{4.30}$$

Then Φ_m is a $d-$Lipschitz reward function, and satisfies in particular Assumption 2.2 and Assumption 2.3 (i). Moreover, $\Phi_m(\bar{\omega})$ decreases to $\Phi(\bar{\omega})$ as m goes to infinity for all $\bar{\omega} \in \overline{\Omega}$.

Denote by $P^m(\mu)$ and $D^m(\mu)$ the corresponding primal and dual values associated to the reward function Φ_m. Since Φ_m satisfies Assumption 2.2 and Assumption 2.3 (i), we have proved in Section 4.3.3 the duality

$$P^m(\mu) \;\;=\;\; D^m(\mu).$$

Then by following the same line of argument as in Proposition 4.10, we deduce that $P(\mu) = D(\mu)$. □

A Appendix

We finally provide some properties of the canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$ of canonical space $\overline{\Omega}$. Recall that the canonical element of $\overline{\Omega}$ is denoted by $\big(B, T = (T_1, \cdots, T_n)\big)$, the $\sigma-$field $\overline{\mathcal{F}}_t$ is generated by the processes $B_{t\wedge\cdot}$ and $(T_k^t, k = 1, \cdots, n)$, where $T_k^t(\bar{\omega}) := \theta_k \mathbf{1}_{\theta_k \leq t} - \infty \mathbf{1}_{\theta_k > t}$ for all $\bar{\omega} = \big(\omega, \theta = (\theta_1, \cdots, \theta_n)\big) \in \overline{\Omega}$. Equivalently, $\overline{\mathcal{F}}_t$ is generated by random variables B_s and the sets $\{T_k \leq s\}$ for all $k = 1, \cdots, n$ and $s \in [0, t]$. More importantly, $(T_k, k = 1, \cdots, n)$ are all $\overline{\mathbb{F}}-$stopping times.

Lemma A.1. *The $\sigma-$field $\overline{\mathcal{F}}_\infty$ is the Borel $\sigma-$filed of $\overline{\Omega}$. Moreover, the class of all bounded continuous, $\overline{\mathcal{F}}_t-$measurable functions on $\overline{\Omega}$ generates the $\sigma-$field $\overline{\mathcal{F}}_{t-} := \bigvee_{s<t} \overline{\mathcal{F}}_s$.*

Proof. (i) Since T_k and B are all $\mathcal{B}(\overline{\Omega})-$measurable, one has $\overline{\mathcal{F}}_\infty \subseteq \mathcal{B}(\overline{\Omega})$. On the other hand, the process $(B_t, t \geq 0)$ generates the Borel $\sigma-$field $\mathcal{B}(\Omega)$ and the collection of all sets $\{T_k \leq s\}$ generates the Borel $\sigma-$filed $\mathcal{B}(\Theta)$, it follows that $\mathcal{B}(\overline{\Omega}) = \mathcal{B}(\Omega) \otimes \mathcal{B}(\Theta) \subseteq \overline{\mathcal{F}}_\infty$.
(ii) Let $t \geq 0$, denote $\mathcal{F}_t^B := \sigma(B_s, 0 \leq s \leq t)$, $\mathcal{F}_t^{T_k} := \sigma\big(\{T_k \leq s\}, s \in [0, t]\big)$ and by $\mathcal{G}_t^{T_k}$ the $\sigma-$field generated by all bounded continuous and $\mathcal{F}_t^{T_k}-$measurable functions. First, for every $s < t$, it is clear that $\mathcal{F}_s^{T_k} \subset \mathcal{G}_t^{T_k}$, thus $\mathcal{F}_{t-}^{T_k} \subset \mathcal{G}_t^{T_k}$. Further, let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded continuous function such that $\phi(T_k)$ is $\mathcal{F}_t^{T_k}-$measurable, then we have $\phi(t_1) = \phi(t_2)$ for every $t_1 \geq t_2 \geq t$. It follows that $\Phi(T_k)$ is $\mathcal{F}_{t-}^{T_k}-$measurable. Therefore, we have $\mathcal{F}_{t-}^{T_k} = \mathcal{G}_t^{T_k}$. Besides, it is well known that $\mathcal{F}_{t-}^B = \mathcal{F}_t^B$ is the $\sigma-$field generated by all bounded, continuous and $\mathcal{F}_t^B-$measurable functions. It follows that $\overline{\mathcal{F}}_{t-} = \cup_{k=1}^n \mathcal{F}_{t-}^{T_k} \cup \mathcal{F}_{t-}^B$ is in fact the $\sigma-$field generated by all bounded, continuous and $\overline{\mathcal{F}}_t-$measurable functions. $\square$

We now consider the filtration $\overline{\mathbb{F}}$. Let $t \geq 0$ and $\bar{\omega} = (\omega, \theta_1, \cdots, \theta_n) \in \overline{\Omega}$, we introduce $[\bar{\omega}]_t = (\omega_{t\wedge\cdot}, [\theta_1]_t, \cdots, [\theta_n]_t)$ by $[\theta_k]_t := \theta_k \mathbf{1}_{\theta_k \leq t} + (t+1)\mathbf{1}_{\theta > t}$.

Lemma A.2. (i) *$Y : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R}$ is $\overline{\mathbb{F}}-$optional if and only if it is $\mathcal{B}(\mathbb{R}_+ \times \overline{\Omega})-$measurable and satisfies*

$$Y_s(\bar{\omega}) \quad = \quad Y_s([\bar{\omega}]_s) \text{ for all } s \geq 0 \text{ and } \bar{\omega} \in \overline{\Omega}. \tag{A.31}$$

(ii) *Consequently, $\overline{\mathcal{F}}_{T_k}$ is countable generated and every probability measure $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}_\infty)$ admits a r.c.p.d. $(\overline{\mathbb{P}}_{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ with respect to $\overline{\mathcal{F}}_{T_k}$ which satisfies that*

a) $(\overline{\mathbb{P}}_{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ is a family of conditional probabilities of $\overline{\mathbb{P}}$ with respect to $\overline{\mathcal{F}}_{T_k}$,

b) $\overline{\mathbb{P}}_{\bar{\omega}}(T_k = \theta_k, B_{T_k\wedge\cdot} = \omega_{T_k\wedge\cdot}) = 1$ for all $\bar{\omega} = (\omega, \theta_1, \cdots, \theta_n) \in \overline{\Omega}$.

Proof. (i) First, if Y is $\overline{\mathbb{F}}-$optional, then Y is measurable and $\overline{\mathbb{F}}-$adapted, i.e. Y_s is $\overline{\mathcal{F}}_s-$measurable. Since $\overline{\mathcal{F}}_s$ is generated by $\bar{\omega} \mapsto (\omega_{s\wedge\cdot}, [\theta]_s)$, it follows that (A.31) holds true. On the other hand, the process $(s, \bar{\omega}) \mapsto (\omega_{s\wedge\cdot}, [\theta]_s)$ is adapted and right-continuous, and hence $\overline{\mathbb{F}}-$optional. Therefore, for every measurable process Y, the process Y defined by (A.31) is $\overline{\mathbb{F}}-$optional.

(ii) Notice that $\mathcal{B}(\overline{\Omega})$ is countably generated. And by the representation (A.31), the $\overline{\mathbb{F}}-$optional $\sigma-$field is generated by the map $(s, \bar{\omega}) \in \mathbb{R}_+ \times \overline{\Omega} \mapsto [\bar{\omega}]_s \in \overline{\Omega}$, and hence is also countably generated. Moreover, by Theorem IV-64 of Dellacherie and Meyer [13, page 122], we have

$$\overline{\mathcal{F}}_{T_k} \quad = \quad \sigma\{B_{T_k\wedge\cdot},\ T_k\},$$

and hence $\overline{\mathcal{F}}_{T_k}$ is countably generated. Therefore, it follows by Theorem 1.1.6 in Stroock and Varadhan [40] that every probability measure $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}_\infty)$ admits a r.c.p.d. with respect to the $\sigma-$field $\overline{\mathcal{F}}_{T_k}$ satisfying the condition in item (ii) of the lemma. $\square$

References

[1] Beiglbock, M., Cox, A., Huesmann, M. : *Optimal Transport and Skorokhod Embedding*, preprint, 2013.

[2] Beiglbock, M., Cox, A., Huesmann, M., Perkowski, N., Prömel, J. : *Robust super-replication for time-invariant derivatives*, preprint, 2015.

[3] Beiglböck, M., Henry-Labordère, P., Penkner, F. : *Model-independent Bounds for Option Prices: A Mass-Transport Approach*, Finance and Stochastics, to appear.

[4] Beiglböck, M. , Juillet, N. : *On a problem of optimal transport under marginal martingale constraints*, preprint, 2012, arXiv:1208.1509.

[5] Biagini, S., Bouchard, B., Kardaras, C. Nutz, M. : *Robust fundamental theorem for continuous processes*, preprint, 2014.

[6] Bonnans J.F. and Tan X. : *A model-free no-arbitrage price bound for variance options*, Applied Mathematics & Optimization, Vol. 68, Issue 1, 43-73, 2013.

[7] Breeden, D.T., Litzenberger, R.H. : *Prices of state-contingent claims implicit in options prices*, J. Business, 51, 1978.

[8] Bouchard, B. and Touzi, N. : *Weak Dynamic Programming Principle for Viscosity Solutions*, SIAM Journal on Control and Optimization, 49(3):948-962, 2011.

[9] Cox, A.M.G and Hobson, D.G. : *Skorokhod embeddings, minimality and non-centred target distributions*, Probability Theory and Related Fields, 135(3):395-414, 2006.

[10] Cox, A. M. G., Hobson, D. and Obłój, J. (2008) : *Pathwise inequalities for local time: Applications to Skorokhod embeddings and optimal stopping*, Ann. Appl. Probab.

[11] Cox, A.M.G.and Obłój, J. (2011) : *Robust pricing and hedging of double no-touch options*, Finance Stoch.

[12] Davis, M., Obłój, J. and Raval, V. (2013) : *Arbitrage bounds for weighted variance swap prices*, Math. Finance.

[13] Dellacherie, D. and Meyer, P. A.: *Probabilities and potential A*, volume 29 of *North-Holland Mathematics Studies*, North-Holland Publishing Co., Amsterdam, 1978.

[14] Dolinsky, Y., Soner, H.M. : *Robust hedging and martingale optimal transport in continuous time*, Probability Theory and Related Fields, to appear.

[15] Dolinsky, Y., Soner, H.M. : *Martingale optimal transport in the Skorokhod space*, preprint, 2015.

[16] El Karoui, N. *Les aspects probabilistes du controle stochastique*, Ecole d'Ete de Probabilites de Saint Flour IX, Lecture Notes in Mathematics 876, Springer Verlag, 1979.

[17] El Karoui, N., Lepeltier, J.P. and Millet A., *A probabilistic approach to the reduite in optimal stopping*, Probab. Math. Statist. 13(1):97-121, 1992.

[18] El Karoui, N. and Tan, X. *Capacities, measurable selection and dynamic programming Part I: abstract framework*, preprint, 2013.

[19] El Karoui, N. and Tan, X. *Capacities, measurable selection and dynamic programming Part II: application in stochastic control problems*, preprint, 2013.

[20] Galichon A., Henry-Labordère P., Touzi N. : *A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options* Annals of Applied Probability, to appear.

[21] Guo, G., Tan, X. and Touzi, N.: *On the monotonicity principle of optimal SEP*, in preparation.

[22] Henry-Labordère, P., Obłój, J., Spoida, P. and Touzi, N. : *The maximum maximum of a martingale with given n marginals*, Annals of Applied Probability, to appear.

[23] Henry-Labordère, P., Tan, X. and Touzi, N. *An Explicit Martingale Version of the One-dimensional Brenier's Theorem with Full Marginals Constraint*, preprint, 2014.

[24] Henry-Labordère, P., and Touzi, N. : *An Explicit Martingale Version of Brenier's Theorem.* Preprint 2013.

[25] Hobson, D. : *Robust hedging of the lookback option*, Finance and Stochastics, 2:329-347, 1998.

[26] Hobson, D. : *The Skorokhod embedding problem and model-independent bounds for option prices*, In Paris-Princeton Lectures on Mathematical Finance 2010, volume 2003 of Lecture Notes in Math., pages 267318. Springer, Berlin, 2011.

[27] Hobson, D., Klimmek, M. : *Robust price bounds for the forward starting straddle*, preprint, 2013.

[28] Hou, Z., Obłój, J. : *On robust pricing-hedging duality in continuous time*, preprint, 2015.

[29] Jacod, J. and Mémin, J. : *Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité*, Séminaire de Probabilité XV 1979/80, Lecture Notes in Mathematics, Vol 850, 529-546, 1981.

[30] Källblad, S., Tan, X., Touzi, N. : *Optimal Skorokhod embedding given full marginals and Azéma-Yor peacocks*, preprint, 2015.

[31] Karandikar, R.L. : *On pathwise stochastic integration*, Stoch. Proc. and their App. 57(1):11-18, 1995.

[32] Karatzas, I. and Shreve, S.E, : *Methods of Mathematical Finance*, Springer Verlag, 1998.

[33] Kellerer, H.G. : *Markov-Komposition und eine anwendung auf martingale*, Math. Ann. 198:99-122, 1972.

[34] Kobylanski, M., Quenez, M.C., and Rouy-Mironescu, E. : *Optimal multiple stopping time problem*, Ann. Appl. Prob. 21(4):1365-1399, 2011.

[35] Monroe, I. : *On embedding right continuous martingales in Brownian motion*, The Annals of Mathematical Statistics, 43(4):1293-1311, 1972.

[36] Obłój, J. : *The Skorokhod embedding problem and its offspring*, Probab. Surv., 1:321390, 2004.

[37] Obłój, J., Spoida, P. : *An Iterated Azema-Yor Type Embedding for Finitely Many Marginals*, preprint, 2013

[38] Peskir, G. and Shiryaev, A.: *Optimal Stopping and Free-Boundary Problems*, Birkhauser, Basel, 2006.

[39] Revuz, D. and Yor, M.: *Continuous martingales and Brownian motion*, Springer, 2005.

[40] Stroock, D.W. and Varadhan, S.R.S. *Multidimensional Diffusion Processes*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 233. Springer, Berlin, 1979.

[41] J. Szpirglas and G. Mazziotto, *Théorème de séparation dans le problème d'arrêt optimal*, Séminaires de Probabilités, XIII: 32, 378-384, LNM 721, 1979.

[42] Tan, X. and Touzi, N. : *Optimal Transportation under Controlled Stochastic Dynamics*, Annals of Probability, Vol. 41, No. 5, 3201-3240, 2013.

[43] C. Villani, *Optimal Transport: Old and New*, Grundlehren der mathematischen Wissenschaften, Vol 338, Springer, 2008.