Martingale Inequalities for the Maximum via Pathwise Arguments *

Jan Obłój† Peter Spoida‡ Nizar Touzi§

March 24, 2015

Abstract

We study a class of martingale inequalities involving the running maximum process. They are derived from pathwise inequalities introduced by Henry-Labordère et al. [15] and provide an upper bound on the expectation of a function of the running maximum in terms of marginal distributions at n intermediate time points. The class of inequalities is rich and we show that in general no inequality is *uniformly sharp* – for any two inequalities we specify martingales such that one or the other inequality is sharper. We then use our inequalities to recover Doob's L^p inequalities. For $p \leqslant 1$ we obtain new, or refined, inequalities.

1 Introduction

In this article we study certain martingale inequalities for the terminal maximum of a stochastic process. We thus contribute to a research area with a long and rich history. In seminal contributions, Blackwell and Dubins [7], Dubins and Gilat [14] and Azéma and Yor [4; 3] showed that the distribution of the maximum $\bar{X}_T := \sup_{t \leqslant T} X_t$ of a martingale (X_t) is bounded above, in stochastic order, by the so called Hardy-Littlewood transform of the distribution of X_T, and the bound is attained. This led to series of studies on the possible distributions of $(X_T, \bar{X}_T)$, see Carraro, El Karoui and Obłój [10] for a discussion and further references. More recently, such problems appeared very naturally within the field of mathematical finance. The original result was extended to the case of a non trivial starting law in Hobson [16] and to the case of a fixed intermediate law in Brown, Hobson and Rogers [9].

*Jan Obłój thankfully acknowledges support from the ERC Starting Grant RobustFinMath 335421, the Oxford-Man Institute of Quantitative Finance and St John's College in Oxford. Peter Spoida gratefully acknowledges scholarships from the Oxford-Man Institute of Quantitative Finance and the DAAD. Nizar Touzi gratefully acknowledges financial support from the ERC Advanced Grant 321111 ROFIRM, the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, and the Chair *Finance and Sustainable Development* sponsored by EDF and CA-CIB.

†University of Oxford, Mathematical Institute, the Oxford-Man Institute of Quantitative Finance and St John's College, Jan.Obloj@maths.ox.ac.uk

‡University of Oxford, Mathematical Institute and the Oxford-Man Institute of Quantitative Finance, Peter.Spoida@maths.ox.ac.uk

§Ecole Polytechnique Paris, Centre de Mathématiques Appliquées, nizar.touzi@polytechnique.edu

The novelty of our study here, as compared with the works mentioned above, is that we look at inequalities which use the information about the process at n intermediate time points. One of our goals is to understand how the bound induced by these more elaborate inequalities compares to simpler inequalities which do not use information about the process at intermediate time points. We show that in our context these bounds can be both, better or worse. We also note that knowledge of intermediate moments does not induce a necessarily tighter bound in Doob's L^p-inequalities. Our main result is split into two Theorems. First, in Theorem 2.1, we present our class of inequalities, indexed with an n-tuple of functions ζ, and show that they are sharp: for a given ζ we find a martingale which attains equality. Second, in Theorem 3.1, we show that no inequality is universally better than another: for $\zeta \neq \tilde{\zeta}$ we find two processes X and $\tilde{X}$ which show that either of the inequalities can be strictly better than the other.

Throughout, we emphasise the simplicity of our arguments, which are all elementary. This is illustrated in Sections 2.2–2.4 where we obtain amongst others the sharp versions of Doob's L^p-inequalities for all $p > 0$. While the case $p \geqslant 1$ is already known in the literature, our Doob's L^p-inequality in the case $p \in (0, 1)$ appears new.

The idea of deriving martingale inequalities from pathwise inequalities is already present in work on robust pricing and hedging by Hobson [16]. Other authors have used pathwise arguments to derive martingale inequalities, e.g. Doob's inequalities are considered by Acciaio et al. [1] and Obłój and Yor [19]. The Burkholder-Davis-Gundy inequality is rediscovered with pathwise arguments by Beiglböck and Siorpaes [6]. In this context we also refer to Cox and Wang [13] and Cox and Peskir [12] whose pathwise inequalities relate a process and time. In a similar spirit, bounds for local time are obtained by Cox et al. [11]. Beiglböck and Nutz [5] look at general martingale inequalities and explain how they can be obtained from deterministic inequalities. This approach builds on the so-called Burkholder's method, a classical tool in probability used to construct sharp martingale inequalities, see Osękowski [20, Chp. 2] for a detailed discussion.

In a discrete time and quasi-sure setup, the results of Bouchard and Nutz [8] can be seen as general theoretical underpinning of many ideas we present here in the special case of martingale inequalities involving the running maximum.

Organization of the article We first recall a remarkable pathwise inequality obtain by Henry-Labordère et al. [15] and some related results. The body of the paper is then split into two sections. In Section 2 we derive our class of submartingale inequalities and demonstrate how they can be used to derive, amongst others, Doob's inequalities. Then, in Section 3, we study if a given inequality can be universally better than another one for all submartingales.

1.1 Preliminaries

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is fixed which supports a standard real-valued Brownian motion B with some initial value $X_0 \in \mathbb{R}$. We will typically use $X = (X_t)$ to denote a (sub/super) martingale and, unless otherwise specified, we always mean this with respect to X's natural filtration. Throughout, we fix arbitrary times $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n =: T$.

Before we proceed to the main result, we recall a remarkable pathwise inequality from Henry-Labordère et al. [15]. The version we give below appears in the proof of Proposition 3.1 in [15] and is best suited to our present context.

Proposition 1.1 (Proposition 3.1 of Henry-Labordère et al. [15])**.** *Let ω be a càdlàg path and denote $\bar{\omega}_t := \sup_{0\leqslant s\leqslant t}\omega_s$. Then, for $m > \omega_0$ and $\zeta_1 \leqslant \cdots \leqslant \zeta_n < m$:*

$$\mathbb{1}_{\{\bar{\omega}_{t_n}\geqslant m\}} \leqslant \Upsilon_n(\omega, m, \boldsymbol{\zeta}) := \sum_{i=1}^{n}\left(\frac{(\omega_{t_i}-\zeta_i)^+}{m-\zeta_i} + \mathbb{1}_{\{\bar{\omega}_{t_{i-1}}<m\leqslant\bar{\omega}_{t_i}\}}\frac{m-\omega_{t_i}}{m-\zeta_i}\right) \tag{1.1}$$
$$- \sum_{i=1}^{n-1}\left(\frac{(\omega_{t_i}-\zeta_{i+1})^+}{m-\zeta_{i+1}} + \mathbb{1}_{\{m\leqslant\bar{\omega}_{t_i},\zeta_{i+1}\leqslant\omega_{t_i}\}}\frac{\omega_{t_{i+1}}-\omega_{t_i}}{m-\zeta_{i+1}}\right).$$

Next, we recall a process with some special structure in view of (1.1). This process has been analysed in more detail by Obłój and Spoida [18].

Definition 1.2 (Iterated Azéma-Yor Type Embedding)**.** *Let $\xi_1, \ldots, \xi_n$ be non-decreasing functions on (X_0, ∞) and denote $\bar{B}_t := \sup_{u\leqslant t} B_u$. Set $\tau_0 \equiv 0$ and for $i = 1, \ldots, n$ define*

$$\tau_i \quad := \quad \inf\left\{t \geqslant \tau_{i-1} :\ B_t \leqslant \xi_i(\bar{B}_t)\right\}. \tag{1.2}$$

A continuous martingale X is called an iterated Azéma-Yor type embedding based on $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ if

$$(X_{t_i}, \bar{X}_{t_i}) = (B_{\tau_i}, \bar{B}_{\tau_i}) \text{ a.s.} \qquad \text{for } i = 0, \ldots, n. \tag{1.3}$$

Note from the non-decrease of the ξ_i's that $\tau_0 \leqslant \inf\{t \geqslant H_1 : B_t \leqslant \xi_1(1)\}$ for $H_1 = \inf\{t \geqslant 0 : B_t \geqslant 1\}$ and then $\tau_i \leqslant \inf\{t \geqslant \tau_{i-1} : B_t \leqslant \xi_i(\bar{B}_{\tau_{i-1}})\}$, $i = 2, \ldots, n$. It follows that $\tau_i < \infty$ a.s. for all $i = 1, \ldots, n$. Further, X being a martingale implies that B_{τ_i} are integrable and all have mean X_0. In particular, $\tau_n < \infty$ a.s. More importantly, it follows from the characterisation of uniform integrable martingales in Azéma et al. [2] that $(B_{t\wedge\tau_n}, t \geqslant 0)$ is uniformly integrable. Indeed, we have, with $H_x = \inf\{t \geqslant 0 : B_t = x\}$,

$$\lim_{x\to\infty} x\mathbb{P}\left[\sup_{t\geqslant 0}|B_{t\wedge\tau_n}| > x\right] \leqslant \lim_{x\to\infty} x\mathbb{P}\left[H_x < H_{\max_i \xi_i^{-1}(-x)}\right] + x\mathbb{P}\left[\bar{B}_{t\wedge\tau_n} > x\right]$$
$$= \lim_{x\to\infty}\left(\frac{x(\max_i \xi_i^{-1}(-x) - X_0)}{\max_i \xi_i^{-1}(-x) + x} + x\mathbb{P}\left[\bar{X}_{t_n} > x\right]\right) = 0,$$

since $(X_t : t \leqslant t_n)$ is uniformly integrable and $\max_i \xi_i^{-1}(-x) - X_0 \searrow 0$. Conversely, if $(B_{t\wedge\tau_n} : t \geqslant 0)$ is uniformly integrable then an example of an iterated Azéma-Yor type embedding is obtained by taking

$$X_t := B_{\tau_i\wedge\left(\tau_{i-1}\vee\frac{t-t_{i-1}}{t_i-t}\right)}, \quad \text{for} \quad t_{i-1} < t \leqslant t_i,\ i = 1, \ldots, n. \tag{1.4}$$

Finally, we recall a version of Lemma 4.1 from Henry-Labordère et al. [15].

Proposition 1.3 (Pathwise Equality)**.** *Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be non-decreasing right-continuous functions and let X be an iterated Azéma-Yor embedding based on $\boldsymbol{\xi}$. Then, for any $m > X_0$ with $\xi_n(m) < m$, X achieves equality in* (1.1)*, i.e.*

$$\mathbb{1}_{\{\bar{X}_{t_n} \geqslant m\}} = \Upsilon_n\big(X, m, \boldsymbol{\zeta}(m)\big) \quad a.s., \tag{1.5}$$

where

$$\zeta_i(m) = \min_{j \geqslant i} \xi_j(m), \qquad i = 1, \ldots, n. \tag{1.6}$$

We note that if we work on the canonical space of continuous functions then (1.5) holds pathwise and not only a.s. We also note that the assumption that X is an iterated Azéma-Yor type embedding, or that $(B_{\tau_n \wedge t})$ is a uniformly integrable martingale, may be relaxed as long as X satisfies (1.3).

2 (Sub)martinagle inequality and its applications

We present now an inequality on the expected value of a function of the running maximum of a submartingale which is obtained by taking expectations in the pathwise inequality of Proposition 1.1. We then demonstrate how this inequality can be used to derive and improve Doob's inequalities. Related work on pathwise interpretations of Doob's inequalities can be found in Acciaio et al. [1] and Obłój and Yor [19]. Peskir [21, Section 4] derives Doob's inequalities and shows that the constants he obtains are optimal. We give below an alternative proof of these statements and provide new sharp inequalities for the case $p < 1$.

2.1 Submartingale inequality

We first deduce a general martingale inequality for $\mathbb{E}\left[\phi(\bar{X}_T)\right]$, similarly as in Proposition 3.2 in [15], and prove that it is attained under some conditions. Define

$$\begin{aligned} \mathscr{Z} := \Big\{ \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \ : \zeta_i : (X_0, \infty) \to \mathbb{R} \ \text{ is right-continuous,} \\ \zeta_1(m) \leqslant \cdots \leqslant \zeta_n(m) < m, \quad n \in \mathbb{N} \Big\}. \end{aligned} \tag{2.1}$$

In order to ensure that the expectations we consider are finite we will occasionally need the technical condition that

$$\zeta_1^\infty := \liminf_{m \to \infty} \frac{\zeta_1(m)}{m} > 0 \ \text{ and } \ \limsup_{m \to \infty} \frac{\phi(m)}{m^\gamma} = 0 \text{ for some } \gamma < \frac{1}{1 - \zeta_1^\infty}. \tag{2.2}$$

Theorem 2.1. *Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathscr{Z}$. Then,*
(i) *for any càdlàg submartingale X: for any $m > X_0$ we have*

$$\mathbb{P}\left[\bar{X}_T \geqslant m\right] \leqslant \mathbb{E}\left[\sum_{i=1}^n \frac{(X_{t_i} - \zeta_i(m))^+}{m - \zeta_i(m)} - \sum_{i=1}^{n-1} \frac{(X_{t_i} - \zeta_{i+1}(m))^+}{m - \zeta_{i+1}(m)}\right] \tag{2.3}$$

and, more generally, for a right-continuous non-decreasing function ϕ,

$$\mathbb{E}\left[\phi(\bar{X}_T)\right] \leqslant \mathrm{UB}\,(X, \phi, \boldsymbol{\zeta}) := \phi(X_0) + \int_{(X_0,\infty)} \sum_{i=1}^n \mathbb{E}\left[\lambda_i^{\boldsymbol{\zeta},m}(X_{t_i})\right] \mathrm{d}\phi(m) \tag{2.4}$$

where

$$\lambda_i^{\boldsymbol{\zeta},m}(x) := \frac{(x-\zeta_i(m))^+}{m-\zeta_i(m)} - \frac{(x-\zeta_{i+1}(m))^+}{m-\zeta_{i+1}(m)} \mathbb{1}_{\{i<n\}}, \tag{2.5}$$

(ii) *if ζ_1 is non-decreasing and satisfies, together with ϕ, the condition* (2.2), *there exists a continuous martingale which achieves equality in* (2.4).

Remark 2.2 (Optimization over $\boldsymbol{\zeta}$). If X and $t_1, \ldots, t_n$ are fixed we can optimize (2.4) over $\boldsymbol{\zeta} \in \mathscr{Z}$ to obtain a minimizer $\boldsymbol{\zeta}^\star$. Clearly, more intermediate points t_i in (2.4) can only improve the bound for this particular process X. However, only for very special processes (e.g. the iterated Azéma-Yor type embedding) there is hope that (2.4) will hold with equality. This is, loosely speaking, because a finite number of intermediate marginal law constraints does not, in general, determine uniquely the law of the maximum at terminal time t_n.

Proof of Theorem 2.1. Equation (2.3) follows from (1.1) by taking expectations. Then, (2.4) follows from (2.3) by integration and Fubini's theorem:

$$\mathbb{E}\left[\phi(\bar{X}_T)\right] = \mathbb{E}\left[\phi(X_0) + \int_{(X_0,\infty)} \mathbf{1}_{\{\bar{X}_T \geqslant m\}} \mathrm{d}\phi(m)\right].$$

Note that for a fixed m, $\mathbb{E}\left[|\lambda_i^{\boldsymbol{\zeta},m}(X_{t_i})|\right] < \infty$ for $i = 1, \ldots, n$, since $\mathbb{E}\left[X_{t_i}^+\right] < \infty$ by the submartingale property.

If ζ_1 is non-decreasing and $\zeta_1(m) \geqslant \alpha m$ for m large, $\alpha > 0$, we define X by

$$X_t = \begin{cases} B_{\frac{t}{t_1-t} \wedge \tau_{\zeta_1}} & \text{if } t < t_1, \\ B_{\tau_{\zeta_1}} & \text{if } t \geqslant t_1. \end{cases}$$

where B is a Brownian motion, $B_0 = X_0$, and $\tau_{\zeta_1} := \inf\left\{u > 0 \ : \ B_u \leqslant \zeta_1(\bar{B}_u)\right\}$. Excursion theoretical considerations, cf. e.g. Rogers [22], combined with asymptotic bounds on ζ_1 in (2.2), allow us to compute

$$\begin{aligned} \mathbb{P}\left[\bar{X}_{t_n} \geqslant y\right] &= \exp\left(-\int_{(X_0,y]} \frac{1}{z-\zeta_1(z)} \mathrm{d}z\right) \leqslant \text{const} \cdot \exp\left(-\int_{(1,y]} \frac{1}{z-\alpha z} \mathrm{d}z\right) \\ &= \text{const} \cdot y^{-\frac{1}{1-\alpha}} \end{aligned}$$

for large y. We may take α such that $\gamma < 1/(1-\alpha)$ in (2.2) which then ensures that $\mathbb{E}\left[\phi(\bar{X}_{t_n})\right] < \infty$. Further, note that for large y, $\inf_{t\geqslant 0} X_t \leqslant -y$ implies $\bar{X}_\infty = \bar{X}_{t_n} \geqslant y/\alpha$ and hence it follows that

$$\lim_{y\to\infty} y\mathbb{P}\left[\sup_{t\geqslant 0} |X_t| \geqslant y\right] \leqslant \text{const} \cdot \lim_{y\to\infty} y^{1-\frac{1}{1-\alpha}} = 0$$

which in turn implies that $(X_t : t \geqslant 0)$ is a uniformly integrable martingale, see Azéma et al. [2]. Finally, one readily verifies together with Proposition 1.3 that

$$\Upsilon_n(X, m, \boldsymbol{\zeta}) = \Upsilon_1(X, m, \boldsymbol{\zeta}) = \mathbb{1}_{\left\{\bar{X}_{t_1} \geqslant m\right\}} = \mathbb{1}_{\left\{\bar{X}_{t_n} \geqslant m\right\}}.$$

and then the claim follows from

$$\begin{aligned}\mathbb{E}\left[\phi(\bar{X}_{t_n})\right] &= \phi(X_0) + \int_{(X_0,\infty)} \mathbb{E}\left[\mathbb{1}_{\{\bar{X}_{t_n}\geqslant m\}}\right] \mathrm{d}\phi(m)\\ &= \phi(X_0) + \int_{(X_0,\infty)} \mathrm{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right) \mathrm{d}\phi(m)\\ &= \mathrm{UB}\left(X, \phi, \boldsymbol{\zeta}\right)\end{aligned}$$

where we applied Fubini's theorem. □

2.2 Doob's L^p-Inequalities, $p > 1$

Using a special case of Theorem 2.1 we obtain an improvement to Doob's inequalities. Denote $\mathrm{pow}^p(m) = m^p, \;\; \zeta_\alpha(m) := \alpha m$.

Proposition 2.3 (Doob's L^p-Inequalities, $p > 1$)**.** *Let $(X_t)_{t\leqslant T}$ be a non-negative càdlàg submartingale.*

(i) *Then,*

$$\mathbb{E}\left[\bar{X}_T^p\right] \;\leqslant\; \mathrm{UB}\left(X, \mathrm{pow}^p, \zeta_{\frac{p-1}{p}}\right) \tag{2.6a}$$

$$\leqslant\; \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1} X_0^p. \tag{2.6b}$$

(ii) *For every $\epsilon > 0$, there exists a martingale X such that*

$$0 \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1} X_0^p - \mathbb{E}\left[\bar{X}_T^p\right] < \epsilon. \tag{2.7}$$

(iii) *The inequality in* (2.6b) *is strict if and only if either holds:*

$$\mathbb{E}\left[\bar{X}_T^p\right] < \infty \text{ and } X_T < \frac{p-1}{p} X_0 \text{ with positive probability.} \tag{2.8a}$$

$$\mathbb{E}\left[\bar{X}_T^p\right] < \infty \text{ and } X \text{ is a strict submartingale.} \tag{2.8b}$$

Proof. Let us first prove (2.6a) and (2.6b). If $\mathbb{E}\left[X_T^p\right] = \infty$ there is nothing to show. In the other case, equation (2.6a) follows from Theorem 2.1 applied with $n = 1,\; \phi(y) = \mathrm{pow}^p(y) = y^p$ and $\zeta_1 = \zeta_{\frac{p-1}{p}}$. To justify this choice of ζ_1 and to simplify further the upper bound we start with a more general $\zeta_1 = \zeta_\alpha,\; \alpha < 1$ and compute

$$\begin{aligned}&\mathbb{E}\left[\bar{X}_T^p\right] - X_0^p \;\leqslant\; \mathrm{UB}\left(X, \mathrm{pow}^p, \zeta_\alpha\right) - X_0^p \;=\; \mathbb{E}\left[\int_{X_0}^\infty p y^{p-1} \frac{(X_T - \alpha y)^+}{y - \alpha y} \mathrm{d}y\right]\\ =\;& \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha}\vee X_0} p y^{p-1} \frac{X_T - \alpha y}{y - \alpha y} \mathrm{d}y\right] \;\leqslant\; \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha}} p y^{p-1} \frac{X_T - \alpha y}{y - \alpha y} \mathrm{d}y\right]\\ =\;& \frac{p}{p-1}\frac{1}{1-\alpha}\mathbb{E}\left[\left\{\left(\frac{X_T}{\alpha}\right)^{p-1} - X_0^{p-1}\right\} X_T\right] - \frac{\alpha}{1-\alpha}\mathbb{E}\left[\left(\frac{X_T}{\alpha}\right)^p - X_0^p\right]\\ \leqslant\;& \frac{1}{p-1}\frac{1}{(1-\alpha)\alpha^{p-1}}\mathbb{E}\left[X_T^p\right] - \frac{p - \alpha(p-1)}{(p-1)(1-\alpha)} X_0^p,\end{aligned} \tag{2.9}$$

where we used Fubini in the first equality and the submartingale property of X in the last inequality. We note that the function $\alpha \mapsto \frac{1}{(1-\alpha)\alpha^{p-1}}$ attains its minimum at $\alpha^\star = \frac{p-1}{p}$. Plugging $\alpha = \alpha^\star$ into the above yields (2.6b).

We turn to the proof that Doob's L^p-inequality is attained asymptotically in the sense of (2.7), a fact which was also proven by Peskir [21, Section 4]. Let $X_0 > 0$, otherwise the claim is trivial. Set $\alpha^\star = \frac{p-1}{p}$ and take $\alpha^\star < \alpha := \frac{p+\epsilon-1}{p+\epsilon} < 1$. Let $X_T = B_{\tau_\alpha}$ where B is a Brownian motion started at X_0 and $\tau_\alpha := \inf\{u > 0 : B_u \leqslant \alpha \bar{B}_u\}$. Then by using excursion theoretical results, cf. e.g. Rogers [22],

$$\mathbb{P}\left[\bar{X}_T \geqslant y\right] = \exp\left(-\int_{X_0}^{y} \frac{1}{z-\alpha z}\mathrm{d}z\right) = \left(\frac{y}{X_0}\right)^{-\frac{1}{1-\alpha}}$$

and then direct computation shows

$$\mathbb{E}\left[\bar{X}_T^p\right] = \frac{p+\epsilon}{\epsilon}X_0^p.$$

By Doob's L^p-inequality,

$$\mathbb{E}\left[\bar{X}_T^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1}X_0^p = \left(\frac{\alpha}{\alpha^\star}\right)^p \mathbb{E}\left[\bar{X}_T^p\right] - \frac{p}{p-1}X_0^p$$

and one verifies

$$\left\{\left(\frac{p}{p-1}\right)^p \cdot \left[\frac{p+\epsilon-1}{p+\epsilon}\right]^p - 1\right\} \cdot \frac{p+\epsilon}{\epsilon}X_0^p \xrightarrow[\epsilon\downarrow 0]{} \frac{p}{p-1}X_0^p.$$

This establishes the claim in (2.7).

Finally, we note that in the calculations (2.9) which led to (2.6b) there are three inequalities: the first one comes from Theorem 2.1 and does not concern the claim regarding (2.8a)–(2.8b). The second one is clearly strict if and only if (2.8a) holds. The third one is clearly strict if and only if (2.8b) holds. □

Remark 2.4 (Asymptotic Attainability). For the martingales in (ii) of Proposition 2.3 we have

$$\mathrm{UB}\left(X, \mathrm{pow}^p, \zeta_{\frac{p-1}{p}}\right) = \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1}X_0^p$$

and $\mathbb{E}\left[X_T^p\right] \to \infty$ as $\epsilon \to 0$.

2.3 Doob's L^1-Inequality

Using a special case of Theorem 2.1 we focus on Doob's $L\log L$ type inequalities. We recover here the classical constant $e/(e-1)$, see (2.11b) , with a refined structure on the inequality. A further improvement to the constant will be obtained in subsequent section in Corollary 2.7. Denote $\mathrm{id}(m) = m$, and

$$\underline{\zeta}_\alpha(m) := \begin{cases} -\infty & \text{if } m < 1, \\ \alpha m & \text{if } m \geqslant 1. \end{cases} \tag{2.10}$$

Proposition 2.5 (Doob's L^1-Inequality)**.** *Let $(X_t)_{t\leqslant T}$ be a non-negative càdlàg submartingale. Then:*

(i) *with* $0\log(0) := 0$ *and* $V(x) := x - x\log(x)$,

$$\begin{aligned}\mathbb{E}\left[\bar{X}_T\right] &\leqslant \mathrm{UB}\left(X, \mathrm{id}, \underline{\zeta}_{\frac{1}{e}}\right) & (2.11a)\\ &\leqslant \frac{e}{e-1}\Big(\mathbb{E}\left[X_T\log\left(X_T\right)\right] + V(1\vee X_0)\Big). & (2.11b)\end{aligned}$$

(ii) *in the case $X_0 \geqslant 1$ there exists a martingale which achieves equality in both,* (2.11a) *and* (2.11b) *and in the case $X_0 < 1$ there exists a submartingale which achieves equality in both,* (2.11a) *and* (2.11b)*.*

(iii) *the inequality in* (2.11b) *is strict if and only if either holds:*

$$\mathbb{E}\left[\bar{X}_T\right] < \infty \text{ and } \bar{X}_T \geqslant 1, \quad X_T < \frac{1}{e}X_0 \text{ with positive probability,} \quad (2.12a)$$

$$\mathbb{E}\left[\bar{X}_T\right] < \infty \text{ and } \bar{X}_T \geqslant 1, \quad \mathbb{E}\left[X_T\right] > X_0 \vee 1. \quad (2.12b)$$

$$\mathbb{E}\left[\bar{X}_T\right] < \infty \text{ and } \bar{X}_T < 1 \text{ with positive probability.} \quad (2.12c)$$

Proof. Let us first prove (2.11a) and (2.11b). If $\mathbb{E}\left[\bar{X}_T\right] = \infty$ there is nothing to show. In the other case, equation (2.11a) follows from Theorem 2.1 applied with $n = 1$, $\phi(y) = \mathrm{id}(y) = y$ and $\zeta_1 = \underline{\zeta}_{\frac{1}{e}}$.

In the case $X_0 \geqslant 1$ we further compute using $\zeta_1 = \underline{\zeta}_\alpha$, $\alpha < 1$,

$$\begin{aligned}\mathbb{E}\left[\bar{X}_T\right] - X_0 &\leqslant \mathrm{UB}\left(X, \mathrm{id}, \underline{\zeta}_\alpha\right) - X_0\\ = \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha}\vee X_0} \frac{X_T - \alpha y}{y - \alpha y}\mathrm{d}y\right] &\leqslant \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha}} \frac{X_T - \alpha y}{(1-\alpha)y}\mathrm{d}y\right]\\ = \frac{\alpha}{1-\alpha}\mathbb{E}\left[\frac{X_T}{\alpha}\left\{\log\left(\frac{X_T}{\alpha}\right) - \log(X_0)\right\}\right] &- \frac{\alpha}{1-\alpha}\mathbb{E}\left[\frac{X_T}{\alpha} - X_0\right]\end{aligned}$$

Choosing $\alpha = e^{-1}$ gives a convenient cancellation. Together with the submartingale property of X, this provides

$$\begin{aligned}\mathbb{E}\left[\bar{X}_T\right] - X_0 &\leqslant \frac{e}{e-1}\mathbb{E}\left[X_T\log\left(X_T\right)\right] - \frac{e}{e-1}\mathbb{E}\left[X_T\right]\log(X_0) + \frac{1}{e-1}X_0\\ &\leqslant \frac{e}{e-1}\mathbb{E}\left[(X_T)\log\left(X_T\right)\right] - \frac{eX_0\log(X_0)}{e-1} + \frac{X_0}{e-1}. \qquad (2.13)\end{aligned}$$

This is (2.11b) in the case $X_0 \geqslant 1$.

For the case $0 < X_0 < 1$ we obtain from Proposition 1.1 for $n = 1$,

$$\mathbb{P}\left[\bar{X}_T \geqslant y\right] \leqslant \inf_{\zeta < y}\frac{\mathbb{E}\left[(X_T - \zeta)^+\right]}{y - \zeta} \leqslant \frac{\mathbb{E}\left[(X_T - \alpha y)^+\right]}{y - \alpha y}$$

for $\alpha < 1$ and therefore

$$\begin{aligned}\mathbb{E}\left[\bar{X}_T\right] - X_0 &= \int_{X_0}^{\infty}\mathbb{P}\left[\bar{X}_T \geqslant y\right]\mathrm{d}y\\ &\leqslant (1 - X_0) + \int_1^{\infty}\mathbb{P}\left[\bar{X}_T \geqslant y\right]\mathrm{d}y\\ &\leqslant (1 - X_0) + \frac{e}{e-1}\mathbb{E}\left[(X_T)\log\left(X_T\right)\right] + \frac{1}{e-1} \qquad (2.14)\end{aligned}$$

by (2.13). This is (2.11b) in the case $X_0 < 1$.

Now we prove that Doob's L^1-inequality is attained. This was also proven by Peskir [21, Section 4]. Firstly, let $X_0 \geqslant 1$. Then the martingale

$$X = \left(B_{\frac{t}{T-t} \wedge \tau_{\frac{1}{e}}} \right)_{t \leqslant T}, \quad \text{where } \tau_{\frac{1}{e}} = \inf\{t : eB_t \leqslant \overline{B}_t\}, \tag{2.15}$$

and B is a Brownian motion with $B_0 = X_0$, achieves equality in both (2.11a) and (2.11b). Secondly, let $X_0 < 1$. Then the submartingale X defined by

$$\begin{cases} X_0 & \text{if } t < T/2, \\ B_{\frac{t-T/2}{T/2-(t-T/2)} \wedge \tau_{\frac{1}{e}}} & \text{if } t \geqslant T/2, \end{cases} \tag{2.16}$$

where B is a Brownian motion, $B_0 = 1$, achieves equality in both, (2.11a) and (2.11b).

Finally, we note that in the calculations (2.13) which led to (2.6b) there are three inequalities: the first one comes from Theorem 2.1 and does not concern the claim regarding (2.12a)–(2.12c). The second one is clearly strict if and only if (2.12a) holds. The third one is clearly strict if and only if (2.12b) holds. In addition, in the case $X_0 < 1$ there is an additional error coming from (2.14). Note that, in the case when $\mathbb{E}\left[\bar{X}_T\right] < \infty$,

$$\left.\frac{\mathbb{E}\left[(X_T - \zeta)^+\right]}{y - \zeta}\right|_{\zeta=\infty} := \lim_{\zeta \to -\infty} \frac{\mathbb{E}\left[(X_T - \zeta)^+\right]}{y - \zeta} = 1.$$

Hence, the first inequality in (2.14) is strict if and only if (2.12c) holds. The second inequality in (2.14) is strict if and only if (2.12a) or (2.12b) holds. □

2.4 Doob Type Inequalities, $0 < p < 1$

It is well known that if X is a positive continuous local martingale converging a.s. to zero, then

$$\bar{X}_\infty \sim \frac{X_0}{U} \tag{2.17}$$

where U is a uniform random variable on $[0, 1]$. More generally, for any non-negative supermartingale X, with a deterministic X_0, we have $\mathbb{P}\left[\bar{X}_\infty \geqslant x\right] \leqslant X_0/x$, for all $x \geqslant X_0$. Hence, for any non-negative supermartingale X and $p > 1$

$$\mathbb{E}\left[\bar{X}_T^p\right] \leqslant \mathbb{E}\left[\left(\frac{X_0}{U}\right)^p\right] = \int_0^1 \left(\frac{X_0}{u}\right)^p \mathrm{d}u = \frac{X_0^p}{1-p} \tag{2.18}$$

and (2.18) is attained. We now generalize (2.18) to a non-negative submartingale.

Proposition 2.6 (Doob Type Inequalities, $0 < p < 1$)**.** *Let X be a non-negative càdlàg submartingale, $X_0 > 0$, and $p \in (0, 1)$. Denote $m_r := X_0^{-r}\mathbb{E}\left[X_T^r\right]$ for $r \leqslant 1$. Then:*

(i) *there is a unique* $\hat\alpha \in (0,1]$ *which solves*

$$m_p \hat\alpha^{-p} = \frac{1-p+pm_1}{1-p+p\hat\alpha} \tag{2.19}$$

and for which we have

$$\mathbb{E}\left[\bar X_T^p\right] \leqslant X_0^p m_p \hat\alpha^{-p} = \frac{X_0^p}{1-p+p\hat\alpha} + X_0^{p-1}\frac{p}{1-p+p\hat\alpha}\Big(\mathbb{E}\left[X_T\right] - X_0\Big) \tag{2.20a}$$

$$< \frac{X_0^p}{1-p} + X_0^{p-1}\frac{p}{1-p}\Big(\mathbb{E}\left[X_T\right] - X_0\Big). \tag{2.20b}$$

(ii) *there exists a martingale which attains equality in* (2.20a)*. Further, for every* $\epsilon > 0$ *there exists a martingale such that*

$$0 \leqslant \frac{X_0^p}{1-p} + X_0^{p-1}\frac{p}{1-p}\Big(\mathbb{E}\left[X_T\right] - X_0\Big) - \mathbb{E}\left[\bar X_T^p\right] < \epsilon. \tag{2.21}$$

Proof. Following the calculations in (2.9), we see that

$$\mathbb{E}\left[\bar X_T^p\right] \leqslant \frac{1}{1-\alpha}X_0^p + \frac{1}{(1-\alpha)(1-p)}\mathbb{E}\Big[-\alpha^{1-p}X_T^p + pX_0^{p-1}X_T\Big] = X_0^p f(\alpha),$$

where, with the notation m_r introduced in the statement of the Proposition,

$$f(\alpha) := \frac{1}{1-\alpha} + \frac{-\alpha^{1-p}m_p + pm_1}{(1-\alpha)(1-p)}, \qquad \alpha \in [0,1].$$

Next we prove the existence of a unique $\hat\alpha \in (0,1]$ such that $f(\hat\alpha) = \min_{\alpha\in[0,1]} f(\alpha)$. To do this, we first compute that

$$f'(\alpha) = \frac{h(\alpha)}{(1-p)(1-\alpha)^2}, \quad \text{where} \quad h(\alpha) := 1-p+pm_1 - (1-p+p\alpha)m_p\alpha^{-p}.$$

By direct calculation, we see that h is continuous and strictly increasing on $(0,1]$, with $h(0+) = -\infty$ and $h(1) = 1-p+pm_1-m_p$. Moreover, it follows from the Jensen inequality and the submartingale property of X that $m_p \leqslant m_1^p$ and $m_1 \geqslant 1$. This implies that $h(1) \geqslant 0$ since $1-p+px-x^p \geqslant 0$ for $x \geqslant 1$. In consequence, there exists $\hat\alpha \in (0,1]$ such that $h \leqslant 0$ on $(0,\hat\alpha]$ and $h \geqslant 0$ on $[\hat\alpha,1]$. This implies that f is decreasing on $[0,\hat\alpha]$ and increasing on $[\hat\alpha,1]$, proving that $\hat\alpha$ is the unique minimizer of f.

Now the first inequality (2.20a) follows by plugging the equation $h(\hat\alpha) = 0$ into the expression for f. The bound in (2.20b) is then obtained by adding strictly positive terms. It also corresponds to taking $\alpha = 0$ in the expression for f. This completes the proof of the claim in (i).

As for (ii), the claim regarding a martingale attaining equality in (2.20a) follows precisely as in the proof of Proposition 2.3. Let $\alpha \in (0,1)$ and recall that $\tau_\alpha = \inf\{t : B_t \leqslant \alpha \bar B_t\}$ for a standard Brownian motion B with $B_0 = X_0 > 0$. Then, similarly to the proof of Proposition 2.3, we compute directly

$$\mathbb{P}(\bar B_{\tau_\alpha} \geqslant y) = \mathbb{P}(B_{\tau_\alpha} \geqslant \alpha y) = \left(\frac{X_0}{y}\right)^{\frac{1}{1-\alpha}}, \quad y \geqslant X_0. \tag{2.22}$$

Computing and simplifying we obtain $\mathbb{E}\left[\bar{B}^p_{\tau_\alpha}\right] = \frac{1}{1-p+p\alpha}X_0^p$, and hence $\mathbb{E}\left[B^p_{\tau_\alpha}\right] = \frac{\alpha^p}{1-p+p\alpha}X_0^p$, while $\mathbb{E}\left[B_{\tau_\alpha}\right] = X_0$. It follows that $\hat{\alpha} = \alpha$ solves (2.19) and equality holds in (2.20a). Taking α arbitrarily small shows (2.21) holds true. $\square$

We close this section with a new type of Doob's $L \ln L$ type of L^1 inequality obtained taking $p \nearrow 1$ in Proposition 2.6. Since $\hat{\alpha}(p)$ defined in (2.19) belongs to $[0,1]$ there is a converging subsequence. So without loss of generality, we may assume $\hat{\alpha}(p) \longrightarrow \hat{\alpha}(1)$ for some $\hat{\alpha}(1) \in [0,1]$. In order to compute $\hat{\alpha}(1)$, we re-write (2.19) into

$$\frac{g(p)-g(1)}{p-1} = m_p \quad \text{where} \quad g(p) := pm_p\hat{\alpha}(p) - (1-p+pm_1)\hat{\alpha}(p)^p. \tag{2.23}$$

We see by a direct differentiation, invoking implicit functions theorem, that

$$g'(1) \quad = \quad \hat{\alpha}(1)\left(1+\mathbb{E}\left[\frac{X_T}{X_0}\ln\frac{X_T}{X_0}\right]\right) - \hat{\alpha}(1)\ln\hat{\alpha}(1)\mathbb{E}\left[\frac{X_T}{X_0}\right].$$

Then, sending $p \to 1$ in (2.23), we get the following equation for $\hat{\alpha}(1)$:

$$\hat{\alpha}(1)\left(1+\mathbb{E}\left[\frac{X_T}{X_0}\ln\frac{X_T}{X_0}\right]\right) \quad = \quad \mathbb{E}\left[\frac{X_T}{X_0}\right](1+\hat{\alpha}(1)\ln\hat{\alpha}(1)). \tag{2.24}$$

We note that this equation does not solve explicitly for $\hat{\alpha}(1)$. Sending $p \to 1$ in the inequality of Proposition 3.4 we obtain the following improvement to the classical Doob's $L\log L$ inequality presented in Proposition 2.5 above.

Corollary 2.7 (Improved Doob's L^1 Inequality)**.** *Let X be a non-negative càdlàg submartingale, $X_0 > 0$. Then:*

$$\mathbb{E}\left[\bar{X}_T\right] \quad \leqslant \quad \frac{\mathbb{E}[X_T]}{\hat{\alpha}} = \frac{\mathbb{E}\left[X_T\ln X_T\right] + X_0 - \mathbb{E}\left[X_T\right]\ln X_0}{1+\hat{\alpha}\ln\hat{\alpha}} \tag{2.25}$$

where $\hat{\alpha} \in (0,1)$ is uniquely defined by (2.25)*.*

Note that the equality in (2.25) is a rewriting of (2.24). To the best of our knowledge the above inequality in (2.25) is new. It bounds $\mathbb{E}\left[\bar{X}_T\right]$ in terms of a function of $\mathbb{E}\left[X_T\right]$ and $\mathbb{E}\left[X_T\ln X_T\right]$, similarly to the classical inequality in (2.11b). However here the function depends on $\hat{\alpha}$ which is only given implicitly and not explicitly. In exchange, the bound refines and improves the classical inequality in (2.11b). This follows from the fact that

$$1+\alpha\ln\alpha \geqslant \frac{e-1}{e}, \quad \alpha \in (0,1).$$

We note also that for $X_t := B_{\frac{t}{T-t}\wedge\tau_\alpha}$, $\alpha \in (0,1)$, we have $\hat{\alpha} = \alpha$ and equality is attained in (2.25). This follows from the proof above or is verified directly using (2.22). The corresponding classical upper bound in (2.11b) is strictly greater expect for $\alpha = 1/e$ when the two bounds coincide.

3 Universally best submartingale inequalities

As mentioned in the introduction, the novelty of our martingale inequality from Theorem 2.1 is that it uses information about the process at intermediate times. In the previous section we saw that careful choice of functions ζ in Theorem 2.1 allowed us to recover and improve the classical Doob's inequalities. In this section we study the finer structure of our class of inequalities and the question whether the information from the intermediate marginals gives us more accurate bounds than e.g. in the case when no information about the process at intermediate times is used. In short, the answer is negative, i.e. we demonstrate that for a large class of $\tilde{\boldsymbol{\zeta}}$'s there is no "universally better" choice of $\boldsymbol{\zeta}$ in the sense that it yields a tighter bound in the class of inequalities for $\mathbb{E}\left[\phi(\bar{X}_T)\right]$ from Theorem 2.1.

3.1 No inequality is universally better than other

To avoid elaborate technicalities, we impose additional conditions on $\boldsymbol{\zeta} \in \mathscr{Z}$ and ϕ below. Many of these conditions could be relaxed to obtain a slightly stronger, albeit more involved, statement in Theorem 3.1. We define

$$\mathscr{Z}^{\mathrm{cts}} := \Big\{\boldsymbol{\zeta} \in \mathscr{Z} \ : \ \boldsymbol{\zeta} \text{ are continuous}\Big\} \tag{3.1}$$

and

$$\begin{aligned}\tilde{\mathscr{Z}} := \Big\{\boldsymbol{\zeta} \in \mathscr{Z}^{\mathrm{cts}} \ : \ &\boldsymbol{\zeta} \text{ are strictly increasing, } \liminf_{m\to\infty} \frac{\zeta_1(m)}{m} > 0, \\ &\text{and } \zeta_1 = \cdots = \zeta_n \text{ on } (X_0, X_0+\epsilon], \text{ for some } \epsilon > 0\Big\}.\end{aligned} \tag{3.2}$$

Before we proceed, we want to argue that the set $\tilde{\mathscr{Z}}$ arises quite naturally. In the setting of Remark 2.2, if X is a martingale such that its marginal laws

$$\mu_1 := \mathcal{L}\left(X_{t_1}\right), \quad \ldots, \quad \mu_n := \mathcal{L}\left(X_{t_n}\right)$$

satisfy Assumption ⊛ of Obłój and Spoida [18], $\int(x-\zeta)^+\mu_i(\mathrm{d}x) < \int(x-\zeta)^+\mu_{i+1}(\mathrm{d}x)$ for all ζ in the interior of the support of μ_{i+1} and their barycenter functions satisfy the mean residual value property of Madan and Yor [17] close to X_0 and have no atoms at the left end of support, then the optimization over $\boldsymbol{\zeta}$ as described in Remark 2.2 yields a unique $\tilde{\boldsymbol{\zeta}}^{\star} \in \tilde{\mathscr{Z}}$. Hence, the set of these $\tilde{\mathscr{Z}}$ seems to be a "good candidate set" for $\boldsymbol{\zeta}$'s to be used in Theorem 2.1.

The statement of the Theorem 3.1 concerns the negative orthant of $\mathscr{Z}^{\mathrm{cts}}$,

$$\begin{aligned}\mathscr{Z}^{\mathrm{cts}}_{-}(\phi, \tilde{\boldsymbol{\zeta}}) := \Big\{\boldsymbol{\zeta} \in \mathscr{Z}^{\mathrm{cts}} \ : \mathrm{UB}\left(X, \phi, \boldsymbol{\zeta}\right) \leqslant \mathrm{UB}\left(X, \phi, \tilde{\boldsymbol{\zeta}}\right) \text{ for all càdlàg} \\ \text{submartingales } X \text{ and} < \text{for at least one } X\Big\},\end{aligned} \tag{3.3}$$

and hence it complements Theorem 2.1. Part (ii) in Theorem 2.1 studied sharpness of (2.4) for a fixed ζ with varying X while Theorem 3.1 studies (2.4) for a fixed X with varying ζ.

Theorem 3.1. *Let ϕ be a right-continuous, strictly increasing function. Then, for $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ such that (2.2) holds we have*

$$\mathscr{Z}^{\mathrm{cts}}_{-}(\phi, \tilde{\boldsymbol{\zeta}}) = \varnothing. \tag{3.4}$$

The above result essentially says that no martingale inequality in (2.4) is universally better than another one. For any choice $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$, the corresponding martingale inequality (2.4) can not be strictly improved by some other choice of $\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}}$, i.e. no other $\boldsymbol{\zeta}$ would lead to a better upper bound for all submartingales and strictly better for some submartingale. The key ingredient to prove this statement is isolated in the following Proposition.

Proposition 3.2 (Positive Error)**.** *Let $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ and $\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}}$ satisfy $\tilde{\boldsymbol{\zeta}} \neq \boldsymbol{\zeta}$. Then there exists a non-empty interval $(m_1, m_2) \subseteq (X_0, \infty)$ such that*

$$\mathrm{UB}\left(X, \mathbb{1}_{[m,\infty)}, \tilde{\boldsymbol{\zeta}}\right) < \mathrm{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right) \quad \textit{for all} \quad m \in (m_1, m_2),$$

where X is an iterated Azéma-Yor type embedding based on some $\tilde{\boldsymbol{\xi}}$.

Proof. To each $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ we can associate non-decreasing and continuous stopping boundaries $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ which satisfies

$$\tilde{\zeta}_i(m) = \min_{j \geqslant i} \tilde{\xi}_j(m) \qquad \forall m > X_0. \tag{3.5}$$

Further, since $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ implies that $\tilde{\zeta}_i$ are all equal on some $(X_0, X_0 + \epsilon]$ we may take $\tilde{\boldsymbol{\xi}}$ such that

$$\tilde{\xi}_n(m) < \cdots < \tilde{\xi}_1(m) < m \qquad \forall m \in (X_0, X_0 + \epsilon), \tag{3.6a}$$
$$\tilde{\boldsymbol{\xi}}(m) = \tilde{\boldsymbol{\zeta}}(m) \qquad \forall m \geqslant X_0 + \epsilon, \tag{3.6b}$$

for some $\epsilon > 0$. A possible choice is given by

$$\tilde{\xi}_i(m) = \tilde{\zeta}_i(m) + (m - \tilde{\zeta}_i(m)) \frac{n-i}{n} \frac{(X_0 + \epsilon - m)^+}{\epsilon}, \quad m > X_0, \ i = 1, \ldots, n,$$

but we may take any $\tilde{\boldsymbol{\xi}}$ satisfying (3.5)–(3.6b). Let X be an iterated Azéma-Yor type embedding based on this $\tilde{\boldsymbol{\xi}}$, e.g. we may take X given by (1.4) since $(B_{t \wedge \tau_n} : t \geqslant 0)$ is uniformly integrable by the same argument as in the proof of Theorem 2.1. Let $j \geqslant 1$. Using the notation of Definition 1.2, it follows by monotonicity of $\tilde{\boldsymbol{\xi}}$, (3.6b) and (3.5) that on the set $\{B_{\tau_j} = \tilde{\xi}_j(\bar{B}_{\tau_j}),\ \bar{B}_{\tau_j} \geqslant X_0 + \epsilon\}$ we have $B_{\tau_j} = \tilde{\xi}_j(\bar{B}_{\tau_j}) \leqslant \tilde{\xi}_{j+1}(\bar{B}_{\tau_j})$. Therefore, the condition of (1.2) in the definition of the iterated Azéma-Yor type embedding is not satisfied and hence $\tau_{j+1} = \tau_j$. Consequently,

$$\begin{aligned} &X_{t_j} = X_{t_{j+1}} = \cdots = X_{t_n} \quad \text{and} \quad \bar{X}_{t_j} = \bar{X}_{t_{j+1}} = \cdots = \bar{X}_{t_n} \\ &\text{on the set } \left\{X_{t_j} = \tilde{\xi}_j(\bar{X}_{t_j}),\ \bar{X}_{t_j} \geqslant X_0 + \epsilon\right\} \end{aligned} \tag{3.7}$$

for all $j \geqslant 1$.

Take $1 \leqslant j \leqslant n$. Denote $\chi := \max\{k \leqslant n : \exists t \leqslant H_{X_0+\epsilon} \text{ s.t. } B_t \leqslant \tilde{\xi}_k(\bar{B}_t)\} \vee 0$, where $H_x := \inf\{u > 0 : B_u = x\}$ and $\mathcal{H} := \{\chi = j-1,\ H_{X_0+\epsilon} < \infty\}$. By (3.6a) we have $\mathbb{P}[\mathcal{H}] > 0$. Further, by using $\tilde{\zeta}_1(m) \leqslant \cdots \leqslant \tilde{\zeta}_n(m) < m$ we conclude by the properties of Brownian motion that $\mathbb{P}\left[\mathcal{H} \cap \{\bar{B}_{\tau_j} \in \mathcal{O}\}\right] > 0$ for $\mathcal{O} \subseteq (X_0 + \epsilon, \infty)$ an open set. Relabelling and using (3.6b) yields

$$\mathbb{P}\left[X_{t_j} = \tilde{\zeta}_j(\bar{X}_{t_j}), \bar{X}_{t_j} \in \mathcal{O}, \bar{X}_{t_{j-1}} < X_0 + \epsilon\right] > 0 \text{ for all open } \mathcal{O} \subseteq (X_0 + \epsilon, \infty). \tag{3.8}$$

By $\tilde{\boldsymbol{\zeta}} \neq \boldsymbol{\zeta}$ either Case A or Case B below holds (possibly by changing ϵ above). In our arguments we refer to the proof of the pathwise inequality of Proposition 1.1 given by Henry-Labordère et al. [15] and argue that certain inequalities in this proof become strict.

Case A: There exist $m_2 > m_1 > X_0 + \epsilon$ and $j \leqslant n$ s.t. $\tilde{\zeta}_j(m_1) > \zeta_j(m_2)$. Set $\mathcal{O} := (m_1, m_2)$, and take $m > m_2$. Then, on $\left\{X_{t_j} = \tilde{\zeta}_j(\bar{X}_{t_j}),\ \bar{X}_{t_j} \in \mathcal{O}\right\}$, it follows from (3.7) and Proposition 1.3 that

$$\Upsilon_n(X, m, \boldsymbol{\zeta}) = \Upsilon_j(X, m, \boldsymbol{\zeta}) > 0 = \mathbb{1}_{\{m \leqslant \bar{X}_{t_j}\}} = \mathbb{1}_{\{m \leqslant \bar{X}_{t_n}\}} = \Upsilon_n(X, m, \tilde{\boldsymbol{\zeta}}), \text{ a.s.}$$

where the strict inequality holds by noting that $(X_{t_j} - \zeta_j(m))^+ > 0$ for all $m \in (m_1, m_2)$ on the above set and then directly verifying that the second inequality of equation (4.3) of Henry-Labordère et al. [15] applied with $\boldsymbol{\zeta}$ and X is strict.

Case B: There exist $m_2 > m_1 > X_0 + \epsilon$ and $j \leqslant n$ s.t. $\tilde{\zeta}_j(m_2) < \zeta_j(m_1)$. Take $m \in \mathcal{O}$. Then, on $\left\{X_{t_j} = \tilde{\zeta}_j(\bar{X}_{t_j}),\ \bar{X}_{t_j} \in \mathcal{O} \cap (m, \infty),\ \bar{X}_{t_{j-1}} < X_0 + \epsilon\right\}$, it follows again from (3.7) and Proposition 1.3 that

$$\Upsilon_n(X, m, \boldsymbol{\zeta}) = \Upsilon_j(X, m, \boldsymbol{\zeta}) > 1 = \mathbb{1}_{\{m \leqslant \bar{X}_{t_j}\}} = \mathbb{1}_{\{m \leqslant \bar{X}_{t_n}\}} = \Upsilon_n(X, m, \tilde{\boldsymbol{\zeta}}), \text{ a.s.}$$

where the strict inequality holds by observing that the last inequality in equation (4.3) of Henry-Labordère et al. [15] applied with $\boldsymbol{\zeta}$ and X is strict because $(X_j - \zeta_j(m))^+ = 0 > X_j - \zeta_j(m)$ for all $m \in \mathcal{O}$ on the above set.

Combining, in both cases A and B the claim (3.5) follows from (3.8). □

Proof of Theorem 3.1. Take $\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}}$ such that strict inequality holds for one submartingale in the definition of $\mathscr{Z}_{-}^{\text{cts}}$, see (3.3). We must have $\boldsymbol{\zeta} \neq \tilde{\boldsymbol{\zeta}}$.

As in the proof of Proposition 3.2 we choose a $\tilde{\boldsymbol{\xi}}$ such that (3.6a)–(3.6b), (3.5) hold and let X be an iterated Azéma-Yor type embedding based on this $\tilde{\boldsymbol{\xi}}$. Propositions 1.1 and 1.3 yield

$$\mathbb{E}\left[\mathbb{1}_{[m,\infty)}(\bar{X}_{t_n})\right] = \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \tilde{\boldsymbol{\zeta}}\right) \leqslant \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right) \qquad \forall m > X_0$$

and by Proposition 3.2

$$\text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \tilde{\boldsymbol{\zeta}}\right) < \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right)$$

for all $m \in \mathcal{O}$ where $\mathcal{O} \subseteq (X_0, \infty)$ is some open set. Now the claim follows as in the proof of Theorem 2.1. □

Remark 3.3. In the setting of Theorem 3.1 let $\tilde{\boldsymbol{\zeta}}^1, \tilde{\boldsymbol{\zeta}}^2 \in \tilde{\mathscr{Z}},\ \tilde{\boldsymbol{\zeta}}^1 \neq \tilde{\boldsymbol{\zeta}}^2$, and assume that (2.2) holds for $(\phi, \tilde{\boldsymbol{\zeta}}^1)$ and $(\phi, \tilde{\boldsymbol{\zeta}}^2)$. Then there exist martingales X^1 and X^2 such that

$$\begin{aligned}\text{UB}\left(X^1, \phi, \tilde{\boldsymbol{\zeta}}^1\right) &< \text{UB}\left(X^1, \phi, \tilde{\boldsymbol{\zeta}}^2\right), \\ \text{UB}\left(X^2, \phi, \tilde{\boldsymbol{\zeta}}^1\right) &> \text{UB}\left(X^2, \phi, \tilde{\boldsymbol{\zeta}}^2\right).\end{aligned}$$

This follows by essentially reversing the roles of $\tilde{\boldsymbol{\zeta}}^1$ and $\tilde{\boldsymbol{\zeta}}^2$ in the proof of Theorem 3.1.

3.2 No Further Improvements with Intermediate Moments

We now use the results of the previous section to show that beyond the improvement stated in Proposition 2.3 no sharper Doob's L^p bounds can be obtained from the inequalities of Theorem 2.1.

Proposition 3.4 (No Improvement of Doob's L^p-Inequality from Theorem 2.1)**.** *Let $p > 1$ and $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ be such that $\tilde{\zeta}_j(m) \neq \zeta_{\frac{p-1}{p}}(m) = \frac{p-1}{p} m$ for some $m > X_0$ and some j. Then, there exists a martingale X such that*

$$\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1} X_0^p < \mathrm{UB}\left(X, \mathrm{pow}^p, \tilde{\boldsymbol{\zeta}}\right). \tag{3.9}$$

Proof. Let $\alpha > \frac{p-1}{p} =: \alpha^\star$ and take X^α satisfying

$$0 = X^\alpha_{t_1} = \cdots = X^\alpha_{t_{j-1}}, \qquad\qquad B_{\tau_\alpha} = X^\alpha_{t_j} = \cdots = X^\alpha_{t_n}$$

where B is a Brownian motion started at X_0 and $\tau_\alpha = \inf\{u > 0 : B_u \leqslant \zeta_\alpha(\bar{B}_u)\}$. It follows easily that for this process X^α,

$$\mathrm{UB}\left(X^\alpha, \mathrm{pow}^p, \tilde{\zeta}_j\right) \leqslant \mathrm{UB}\left(X^\alpha, \mathrm{pow}^p, \tilde{\boldsymbol{\zeta}}\right)$$

and hence it is enough to prove the claim for $n = 1$ and $\tilde{\boldsymbol{\zeta}} = \tilde{\zeta}_j$.

For all $\alpha \in (\alpha^\star, \alpha^\star + \epsilon)$, $\epsilon > 0$, Proposition 3.2 yields existence of a non-empty, open interval $\mathcal{I}_\alpha$ such that

$$\mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \zeta_\alpha\right) < \mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \tilde{\zeta}_j\right) \quad \text{for all} \quad m \in \mathcal{I}_\alpha.$$

In fact, taking $\epsilon > 0$ small enough, $\mathcal{I}_\alpha$ can be chosen such that

$$\bigcap_{\alpha \in (\alpha^\star, \alpha^\star + \epsilon)} \mathcal{I}_\alpha \quad \supseteq \quad (m_1, m_2), \qquad X_0 < m_1 < m_2. \tag{3.10}$$

We can further (recalling the arguments in Case A and Case B in the proof of Proposition 3.2) assume that for all $\alpha \in (\alpha^\star, \alpha^\star + \epsilon)$:

$$\mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \tilde{\zeta}_j\right) - \mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \zeta_\alpha\right) \geqslant \delta > 0 \quad \text{for all} \quad m \in (m_1, m_2).$$

The claim follows by letting $\alpha \downarrow \alpha^\star$ and using the asymptotic optimality of $(X^\alpha)_\alpha$, see (2.7). $\square$

In addition to the result of Proposition 3.4 we prove that there is no "intermediate moment refinement of Doob's L^p-inequalities" in the sense formalized in the next Proposition. Intuitively, this could be explained by the fact that the p^{th} moment of a continuous martingale is continuously non-decreasing and hence does not add relevant information about the p^{th} moment of the maximum. Only the final p^{th} moment matters in this context.

Proposition 3.5 (No Intermediate Moment Refinement of Doob's L^p-Inequality)**.** *Let $p > 1$, $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = T$ and $a_0, \ldots, a_n \in \mathbb{R}$.*

(i) *If* $\mathbb{E}\left[\bar{X}_T^p\right] \leqslant \sum_{i=0}^n a_i \mathbb{E}\left[X_{t_i}^p\right]$ *for every continuous non-negative submartingale* X*, then also*

$$\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1} X_0^p \quad \leqslant \quad \sum_{i=0}^n a_i \mathbb{E}\left[X_{t_i}^p\right].$$

(i) *If* $\left(\mathbb{E}\left[\bar{X}_T^p\right]\right)^{1/p} \leqslant \sum_{i=1}^n a_i \left(\mathbb{E}\left[|X_{t_i} - X_{t_{i-1}}|^p\right]\right)^{1/p}$ *for every continuous non-negative submartingale* X *with* $X_0 = 0$*, then also*

$$\left(\frac{p}{p-1}\right) \left(\mathbb{E}\left[X_T^p\right]\right)^{1/p} \quad \leqslant \quad \sum_{i=1}^n a_i \left(\mathbb{E}\left[|X_{t_i} - X_{t_{i-1}}|^p\right]\right)^{1/p}.$$

Proof. From Peskir [21, Example 4.1] or our Proposition 2.3 we know that Doob's L^p-inequality given in (2.6b) is enforced by a sequence of continuous martingales (Y^ϵ) in the sense of (2.7), i.e.

$$\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|Y_T^\epsilon|^p\right] \leqslant \mathbb{E}\left[\max_{t \leqslant T} |Y_t^\epsilon|^p\right] + \frac{p}{p-1} |Y_0^\epsilon|^p + \epsilon.$$

Recall that $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n$.

We first prove (i). We take $Y_0^\epsilon = X_0$. By scalability of the asymptotically optimal martingales (Y^ϵ) we can assume

$$\mathbb{E}\left[X_{t_n}^p\right] = \mathbb{E}\left[|Y_{t_n}^\epsilon|^p\right].$$

In addition we can find times $u_1 \leqslant \cdots \leqslant u_{n-1}$ such that

$$\mathbb{E}\left[X_{t_i}^p\right] = \mathbb{E}\left[|Y_{u_i}^\epsilon|^p\right], \quad 1 \leqslant i \leqslant n-1.$$

Furthermore, by a simple time-change argument, we may take $u_i = t_i$. Therefore, writing $u_n = t_n = T$ and using asymptotic optimality of (Y^ϵ),

$$\begin{aligned}
\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_{t_n}^p\right] \quad &= \quad \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|Y_{t_n}^\epsilon|^p\right] \\
&\leqslant \quad \mathbb{E}\left[\max_{t \leqslant T} |Y_t^\epsilon|^p\right] + \frac{p}{p-1} |Y_0^\epsilon|^p + \epsilon \\
&\leqslant \quad \sum_{i=0}^n \tilde{a}_i \mathbb{E}\left[|Y_{t_i}^\epsilon|^p\right] + \epsilon \ = \ \sum_{i=}^n \tilde{a}_i \mathbb{E}\left[X_{t_i}^p\right] + \epsilon
\end{aligned}$$

where $\tilde{a}_0 = a_0 + p/(p-1)$, $\tilde{a}_i = a_i$ for $i = 1, \ldots, n$. We obtain the required inequality by sending $\epsilon \searrow 0$ in the above.

We next prove (ii). Taking a martingale which is constant until time t_{i-1} and constant after time t_i and using the fact that Doob's L^p inequality is sharp yields

$$\left(\frac{p}{p-1}\right) \leqslant a_i \quad \text{for all} \quad i = 1, \ldots, n.$$

The required inequality follows using triangular inequality for the L^p norm. □

Remark 3.6. Note that it follows instantly from the previous proof that we may also formulate Proposition 3.5 in terms of L^p norms instead of the expectations of the $p-$th moment. Also, analogous statements as in Proposition 3.5 hold for Doob's L^1 inequality. This can be argued in the same way by using that Doob's L^1 inequality is attained (cf. e.g. Peskir [21, Example 4.2] or our Proposition 2.5) and observing that the function $x \mapsto x\log(x)$ is convex.

References

[1] Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W., and Temme, J. (2013). A trajectorial interpretation of Doob's martingale inequalities. *Ann. Appl. Probab.*, 23(4):1494–1505.

[2] Azéma, J., Gundy, R. F., and Yor, M. (1980). Sur l'intégrabilité uniforme des martingales continues. In *Seminar on Probability, XIV (Paris, 1978/1979) (French)*, volume 784 of *Lecture Notes in Math.*, pages 53–61. Springer, Berlin.

[3] Azéma, J. and Yor, M. (1979a). Le problème de Skorokhod: compléments à "Une solution simple au problème de Skorokhod". In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 625–633. Springer, Berlin.

[4] Azéma, J. and Yor, M. (1979b). Une solution simple au problème de Skorokhod. In Dellacherie, C., Meyer, P.-A., and Weil, M., editors, *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 90–115. Springer, Berlin.

[5] Beiglböck, M. and Nutz, M. (2014). Martingale inequalities and deterministic counterparts. *Electron. J. Probab.*, 19(95):1–15.

[6] Beiglböck, M. and Siorpaes, P. (2013). Pathwise versions of the Burkholder-Davis-Gundy inequality. *arXiv.org*, 1305.6188v1.

[7] Blackwell, D. and Dubins, L. E. (1963). A converse to the dominated convergence theorem. *Illinois J. Math.*, 7:508–514.

[8] Bouchard, B. and Nutz, M. (2015). Arbitrage and duality in nondominated discrete-time models. *Ann. Appl. Prob*, 25(2):823–859.

[9] Brown, H., Hobson, D., and Rogers, L. C. G. (2001). The maximum maximum of a martingale constrained by an intermediate law. *Probab. Theory Rel. Fields*, 119(4):558–578.

[10] Carraro, L., El Karoui, N., and Obłój, J. (2012). On Azéma-Yor processes, their optimal properties and the Bachelier-Drawdown equation. *Ann. Probab.*, 40(1):372–400.

[11] Cox, A. M. G., Hobson, D., and Obłój, J. (2008). Pathwise inequalities for local time: Applications to Skorokhod embeddings and optimal stopping. *Ann. Appl. Prob*, 18(5):1870–1896.

[12] Cox, A. M. G. and Peskir, G. (2015). Embedding Laws in Diffusions by Functions of Time. *Ann. Probab.*, to appear. Available at arXiv:1201.5321v3.

[13] Cox, A. M. G. and Wang, J. (2013). Root's barrier: Construction, optimality and applications to variance options. *Ann. Appl. Prob*, 23(3):859–894.

[14] Dubins, L. E. and Gilat, D. (1978). On the distribution of maxima of martingales. *Proc. Amer. Math. Soc.*, 68(3):337–338.

[15] Henry-Labordère, P., Obłój, J., Spoida, P., and Touzi, N. (2015). The maximum maximum of a martingale with given n marginals. *Ann. Appl. Probab.* To appear. Available at arxiv:1203.6877v3.

[16] Hobson, D. G. (1998). Robust hedging of the lookback option. *Finance Stoch.*, 2(4):329–347.

[17] Madan, D. B. and Yor, M. (2002). Making Markov Martingales Meet Marginals: With Explicit Constructions. *Bernoulli*, 8(4):509–536.

[18] Obłój, J. and Spoida, P. (2013). An Iterated Azéma-Yor Type Embedding for Finitely Many Marginals. *arXiv.org*, 1304.0368v2.

[19] Obłój, J. and Yor, M. (2006). On local martingale and its supremum: harmonic functions and beyond. In *From Stochastic Calculus to Mathematical Finance*, pages 517–534. Springer-Verlag.

[20] Osękowski, A. (2012). *Sharp martingale and semimartingale inequalities.* Birkhäuser.

[21] Peskir, G. (1998). Optimal Stopping of the Maximum Process: The Maximality Principle. *Ann. Probab.*, 26(4):1614–1640.

[22] Rogers, L. C. G. (1989). A guided tour through excursions. *Bull. London Math. Soc.*, 21(4):305–341.