Wellposedness of Second Order Backward SDEs

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Abstract

We provide an existence and uniqueness theory for an extension of backward SDEs to the second order. While standard Backward SDEs are naturally connected to semilinear PDEs, our second order extension is connected to fully nonlinear PDEs, as suggested in [4]. In particular, we provide a fully nonlinear extension of the Feynman-Kac formula. Unlike [4], the alternative formulation of this paper insists that the equation must hold under a non-dominated family of mutually singular probability measures. The key argument is a stochastic representation, suggested by the optimal control interpretation, and analyzed in the accompanying paper [16].

Key words: Backward SDEs, non-dominated family of mutually singular measures, viscosity solutions for second order PDEs.
AMS 2000 subject classifications: 60H10, 60H30.

1 Introduction

Backward stochastic differential equations (BSDEs) appeared in Bismut [1] in the linear case, and received considerable attention since the seminal paper of Pardoux and Peng [11]. The various developments are motivated by applications in probabilistic numerical methods for partial differential equations (PDEs), stochastic control, stochastic differential games, theoretical economics and financial mathematics.

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On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$ generated by a Brownian motion W with values in $\mathbb{R}^d$, a solution to a one-dimensional BSDE consists of a pair of progressively measurable processes (Y, Z) taking values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, such that

$$Y_t = \xi - \int_t^1 f_s(Y_s, Z_s)ds - \int_t^1 Z_s dW_s, \quad t \in [0,1], \quad \mathbb{P}-\text{a.s.}$$

where f is a progressively measurable function from $[0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}$, and ξ is an $\mathcal{F}_1$-measurable random variable.

If the randomness in the parameters f and ξ is induced by the current value of a state process defined by a forward stochastic differential equation (SDE), then the BSDE is referred to as a Markov BSDE and its solution can be written as a deterministic function of time and the current value of the state process. For simplicity, we assume the forward process to be reduced to the Brownian motion, then under suitable regularity assumptions, this function can be shown to be the solution of a parabolic semilinear PDE.

$$-\partial_t v - h^0(t, x, v, Dv, D^2 v) = 0 \quad \text{where} \quad h^0(t, x, y, z, \gamma) := \frac{1}{2}\text{Tr}[\gamma] - f(t, x, y, z).$$

In particular, this connection is the main ingredient for the Pardoux and Peng extension of the Feynman-Kac formula to semilinear PDEs. For a larger review of the theory of BSDEs, we refer to El Karoui, Peng and Quenez [8].

Motivated by applications in financial mathematics and probabilistic numerical methods for PDEs, Cheridito, Soner, Touzi and Victoir [4] introduced the notion of Second Order BSDEs (2BSDEs). The key issue is that, in the Markov case studied by [4], 2BSDEs are connected to the larger class of fully nonlinear PDEs. This is achieved by introducing a further dependence of the generator f on a process γ which essentially identifies to the Hessian of the solution of the corresponding PDE. Then, a uniqueness result is proved in an appropriate set $\mathcal{Z}$ for the process Z. The linear 2BSDE example reported in Section 7.1 below shows clearly that the specification of the class $\mathcal{Z}$ is crucial, and can not recover the natural class of square integrable processes, as in classical BSDEs. However, except for the trivial case where the PDE has a sufficiently smooth solution, the existence problem was left open in [4].

In this paper, we provide a complete theory of existence and uniqueness for 2BSDEs. The key idea is a slightly different definition of 2BSDEs which consists in reinforcing the condition that the 2BSDE must hold $\mathbb{P}-$a.s. for every probability measure $\mathbb{P}$ in a non-dominated class of mutually singular measures introduced in Section 2 below. The precise definition is reported in Section 3. This new point of view is inspired from the quasi-sure analysis of Denis & Martini [6] who established the connection between the so-called hedging problem in uncertain volatility models and the so-called Black-Scholes-Barrenblatt PDE. The latter is fully nonlinear and has a simple piecewise linear dependence on the second order term. We also observe an intimate connection between [6] and the $G-$stochastic integration theory of Peng [12], see Denis, Hu and Peng [7], and our paper [15].

In the present framework, uniqueness follows from a stochastic representation suggested by the optimal control interpretation. Our construction follows the idea of Peng [12]. When the terminal random variable ξ is in the space $\mathrm{UC}_b(\Omega)$ of bounded uniformly continuous maps of ω, the former stochastic representation is shown in our accompanying paper [16] to be the solution of the 2BSDE . Then, we define the closure of $\mathrm{UC}_b(\Omega)$ under an appropriate norm. Our main result then shows that for any terminal random variable in this closure, the solution of the 2BSDE can be obtained as a limit of a sequence of solutions corresponding to bounded uniformly continuous final datum $(\xi_n)_n$. These are the main results of this paper and are reported in Section 4.

Finally, we explore in Sections 5 and 6 the connection with fully nonlinear PDEs. In particular, we prove a fully nonlinear extension of the Feynman-Kac stochastic representation formula. Moreover, under some conditions, we show that the solution of a Markov 2BSDE is a deterministic function of the time and the current state which is a viscosity solution of the corresponding fully nonlinear PDE.

2 Preliminaries

Let $\Omega := \{\omega \in C([0,1],\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\le t\le 1}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le 1}$ the filtration generated by B, and $\mathbb{F}^+ := \{\mathcal{F}_t^+, 0 \le t \le 1\}$ the right limit of $\mathbb{F}$.

2.1 The local martingale measures

We say a probability measure $\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Karandikar [9], there exists an $\mathbb{F}-$progressively measurable process, denoted as $\int_0^t B_s dB_s$, which coincides with the Itô's integral, $\mathbb{P}-$a.s. for all local martingale measure $\mathbb{P}$. In particular, this provides a pathwise definition of

$$\langle B\rangle_t := B_t B_t^{\mathrm{T}} - 2\int_0^t B_s dB_s^{\mathrm{T}} \quad \text{and} \quad \hat{a}_t := \overline{\lim_{\varepsilon\downarrow 0}} \frac{1}{\varepsilon}\Big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\Big),$$

where $^{\mathrm{T}}$ denotes the transposition, and the $\overline{\lim}$ is componentwise. Clearly, $\langle B\rangle$ coincides with the $\mathbb{P}-$quadration variation of B, $\mathbb{P}-$a.s. for all local martingale measure $\mathbb{P}$.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \quad \text{is absolutely continuous in } t \text{ and } \quad \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}, \ \mathbb{P}-\text{a.s.} \qquad (2.1)$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d\times d$ real valued positive definite matrices. We note that, for different $\mathbb{P}_1, \mathbb{P}_2 \in \overline{\mathcal{P}}_W$, in general $\mathbb{P}_1$ and $\mathbb{P}_2$ are mutually singular. For any $\mathbb{P} \in \overline{\mathcal{P}}_W$, it follows from the Lévy characterization that the Itô's stochastic integral under $\mathbb{P}$

$$W_t^{\mathbb{P}} := \int_0^t \hat{a}_s^{-1/2} dB_s, \quad t \in [0,1], \quad \mathbb{P} - \text{a.s.} \tag{2.2}$$

defines a $\mathbb{P}-$Brownian motion.

This paper concentrates on the subclass $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0,1], \mathbb{P}_0 - \text{a.s.} \tag{2.3}$$

for some $\mathbb{F}-$progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ with $\int_0^1 |\alpha_t| dt < \infty$, $\mathbb{P}_0-$a.s. With $\overline{\mathbb{F}}^{\mathbb{P}}$ (resp. $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}$) denoting the $\mathbb{P}-$augmentation of the filtration generated by B (resp. by $W^{\mathbb{P}}$), we recall from [16] that

$$\overline{\mathcal{P}}_S = \left\{\mathbb{P} \in \overline{\mathcal{P}}_W : \overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}\right\}, \tag{2.4}$$

and every $\mathbb{P} \in \overline{\mathcal{P}}_S$ satisfies the Blumenthal zero-one law
and the martingale representation property. (2.5)

2.2 The nonlinear generator

Our nonlinear generator is a map $H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0. We assume throughout that

Assumption 2.1 *For fixed (y, z, γ), H is $\mathbb{F}-$progressively measurable; H is uniformly Lipschitz continuous in (y, z), uniformly continuous in ω under the $\|.\|_\infty-$norm, and lower semi-continuous in γ (i.e. $\underline{\lim}_{n\to\infty} H_t(y, z, \gamma_n) \geq H_t(y, z, \gamma)$ whenever $\gamma_n \to \gamma$).*

Define the corresponding conjugate of H with respect to γ by:

$$\begin{aligned} F_t(\omega, y, z, a) &:= \sup_{\gamma \in D_H} \left\{\frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma)\right\}, \quad a \in \mathbb{S}_d^{>0}; \\ \hat{F}_t(y, z) &:= F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0). \end{aligned} \tag{2.6}$$

Here and in the sequel $a : \gamma := \text{tr}\,(a\gamma)$. We note that F is a $\mathbb{R} \cup \{\infty\}-$valued measurable map, by the lower semi-continuity of H in γ. Since H is uniformly continuous in ω and uniformly Lipschitz continuous in (y, z), the domain of F as a function of a is independent of (ω, y, z), and thus denoted as D_{F_t}, and

$$\begin{aligned} &F(\cdot, a) \text{ is uniformly Lischitz continuous in } (y, z) \text{ and uniformly continuous in } \omega, \\ &\qquad \text{uniformly on } (t, a), \text{ for every } a \in D_{F_t}. \end{aligned} \tag{2.7}$$

For the reason explained in Remark 2.4 below, in this paper we shall fix a constant κ:

$$1 < \kappa \leq 2, \tag{2.8}$$

and restrict the probability measures in the following subset $\mathcal{P}_H^\kappa \subset \overline{\mathcal{P}}_S$:

Definition 2.2 *Let $\mathcal{P}_H^\kappa$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that*

$$\underline{a}_{\mathbb{P}} \le \hat{a} \le \overline{a}_{\mathbb{P}}, \quad dt \times d\mathbb{P} - a.s. \text{ for some } \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0}, \text{ and } \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^1 |\hat{F}_t^0|^\kappa dt\Big)^{\frac{2}{\kappa}}\Big] < \infty. \quad (2.9)$$

It is clear that $\mathcal{P}_H^\kappa$ is decreasing in κ, and $\hat{a}_t \in D_{F_t}$, $dt \times d\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.

Definition 2.3 *We say a property holds $\mathcal{P}_H^\kappa-$quasi-surely ($\mathcal{P}_H^\kappa-$q.s. for short) if it holds $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$.*

2.3 The spaces and norms

We now introduce the spaces and norms which will be needed for the formulation of the second order BSDEs. Notice that all subsequent notations extend to the case $\kappa = 1$.

For $p \ge 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_1-$measurable scalar r.v. ξ with

$$\|\xi\|_{L_H^{p,\kappa}} \;:=\; \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^p] \;<\; \infty;$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+-$progressively measurable $\mathbb{R}^d-$valued processes Z with

$$\|Z\|_{\mathbb{H}_H^{p,\kappa}} \;:=\; \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\Big)^{p/2}\Big] \;<\; \infty;$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+-$progressively measurable $\mathbb{R}-$valued processes Y with

$$\mathcal{P}_H^\kappa - \text{q.s. càdlàg paths}, \quad \text{and} \quad \|Y\|_{\mathbb{D}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1} |Y_t|^p\Big] < \infty.$$

For each $\xi \in L_H^{1,\kappa}$, $\mathbb{P} \in \mathcal{P}_H^\kappa$, and $t \in [0,1]$, denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi] \quad \text{where} \quad \mathcal{P}_H^\kappa(t,\mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\}.$$

Here $\mathbb{E}_t^{\mathbb{P}}[\xi] := \mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t]$. We remark that, by (2.4) and the fact that the augmented Brownian filtration is continuous, $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_t^+]$, $\mathbb{P}-$a.s. Similarly, for $\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})$, we have $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+$. Then we define, for each $p \ge \kappa$,

$$\mathbb{L}_H^{p,\kappa} := \big\{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < \infty\big\} \text{ where } \|\xi\|_{\mathbb{L}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\le t\le 1} \Big(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\Big)^{p/\kappa}\Big]. \quad (2.10)$$

The norm $\|\cdot\|_{\mathbb{L}_H^{p,\kappa}}$ is somewhat less standard. We justify this definition at below.

Remark 2.4 Assume $\mathcal{P}_H := \mathcal{P}_H^\kappa$ and $L_H^p := L_H^{p,\kappa}$ do not depend on κ (e.g. when $\hat{F}^0$ is bounded).

(i) For $1 \le \kappa_1 \le \kappa_2 \le p$, it is clear that

$$\|\xi\|_{L_H^p} \le \|\xi\|_{\mathbb{L}_H^{p,\kappa_1}} \le \|\xi\|_{\mathbb{L}_H^{p,\kappa_2}} \quad \text{and thus} \quad \mathbb{L}_H^{p,\kappa_2} \subset \mathbb{L}_H^{p,\kappa_1} \subset L_H^p.$$

Moreover, as in our paper [15] Lemma 6.2, under certain technical conditions, we have

$$\|\xi\|_{\mathbb{L}_H^{p_1,p_1}} \le C_{p_2/p_1}\|\xi\|_{L_H^{p_2}} \quad \text{and thus} \quad L_H^{p_2} \subset \mathbb{L}_H^{p_1,p_1}, \quad \text{for any} \quad 1 \le p_1 < p_2.$$

(ii) In our paper [15], we used the norm $\|\cdot\|_{\mathbb{L}_H^{p,1}}$. However, this norm does not work in the present paper due to the presence of the nonlinear generator, see Lemma 4.3. So in this paper we shall assume $\kappa > 1$ in order to obtain the norm estimates.
(iii) In the special case where $\mathcal{P}_H$ is reduced to a single measure $\mathcal{P}_H = \{\mathbb{P}\}$, we have $\mathbb{E}_t^{H,P} = \mathbb{E}_t^{\mathbb{P}}$ and the process $\{\mathbb{E}_t^{H,P}[|\xi|^\kappa], t \in [0,1]\}$ is a $\mathbb{P}-$martingale, then it follows immediately from the Doob's maximal inequality that, for all $1 \le \kappa < p$,

$$\|\xi\|_{L^p(\mathbb{P})} = \|\xi\|_{L_H^p} \le \|\xi\|_{\mathbb{L}_H^{p,\kappa}} \le C_{p,\kappa}\|\xi\|_{L_H^p} \quad \text{and thus} \quad \mathbb{L}_H^{p,\kappa} = L_H^p = L^p(\mathbb{P}). \tag{2.11}$$

However, the above equivalence does not hold when $\kappa = p$. □

Remark 2.5 As in [15], in order to estimate $\|Y\|_{\mathbb{D}_H^{p,\kappa}}$ for the solution Y to the 2BSDE with terminal condition ξ, it is natural to consider the supremum over t in the norm of ξ. In fact we can show that the process $M_t := \mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]$ a $\mathbb{P}-$supermartingale. Therefore it admits a càdlàg version and thus the term $\sup_{t\in[0,1]} M_t$ is measurable. □

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \longrightarrow \mathbb{R}$ with respect to the $\|.\|_\infty-$norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}, \text{ for every } 1 \le \kappa \le p. \tag{2.12}$$

Similar to (2.11), we have

Remark 2.6 In the case $\mathcal{P}_H^\kappa = \{\mathbb{P}\}$, we have $\mathcal{L}_H^{p,\kappa} = \mathbb{L}_H^{p,\kappa} = L_H^{p,\kappa} = L^p(\mathbb{P})$ for $1 \le \kappa < p$.

3 The second order BSDEs

We shall consider the following second order BSDE (2BSDE for short):

$$Y_t = \xi - \int_t^1 \hat{F}_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t, \quad 0 \le t \le 1, \quad \mathcal{P}_H^\kappa\text{-q.s.} \tag{3.1}$$

Definition 3.1 *For $\xi \in L_H^{2,\kappa}$, we say $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to 2BSDE (3.1) if*
- *$Y_T = \xi$, $\mathcal{P}_H^\kappa-q.s.$*
- *For each $\mathbb{P} \in \mathcal{P}_H^\kappa$, the process $K^{\mathbb{P}}$ defined below has nondecreasing paths, $\mathbb{P}-a.s.$:*

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \quad 0 \le t \le 1, \quad \mathbb{P} - a.s. \tag{3.2}$$

- *The family* $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ *defined in* (3.2) *satisfies the following minimum condition:*

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})}{}^{\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'}[K_1^{\mathbb{P}'}], \quad \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}_H^\kappa, t \in [0,1]. \tag{3.3}$$

Moreover, if the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ *can be aggregated into a universal process* K*, we call* (Y, Z, K) *a solution of 2BSDE* (3.1)*.*

Clearly, we may rewrite (3.2) as

$$Y_t = \xi - \int_t^1 \hat{F}_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1^{\mathbb{P}} - K_t^{\mathbb{P}}, \quad 0 \le t \le 1, \quad \mathbb{P} - \text{a.s.} \tag{3.4}$$

In particular, if (Y, Z, K) is a solution of 2BSDE (3.1) in the sense of the above definition, then it satisfies (3.1) $\mathcal{P}_H^\kappa$−q.s.

Finally, we note that, if $\mathbb{P}' \in \mathcal{P}_H^\kappa(t, \mathbb{P})$, then $K_s^{\mathbb{P}} = K_s^{\mathbb{P}'}$, $0 \le s \le t$, $\mathbb{P}$−a.s. and $\mathbb{P}'$−a.s.

3.1 Connection with the second order stochastic target problem [16]

Let (Y, Z) be a solution of 2BSDE (3.1). If the conjugate in (2.6) has measurable maximizer, that is, there exists a process Γ such that

$$\frac{1}{2}\hat{a}_t : \Gamma_t - H_t(Y_t, Z_t, \Gamma_t) = \hat{F}_t(Y_t, Z_t), \tag{3.5}$$

then (Y, Z, Γ) satisfies

$$Y_t = \xi - \int_t^1 \big[\frac{1}{2}\hat{a}_s : \Gamma_s - H_s(Y_s, Z_s, \Gamma_s)\big]ds - \int_t^1 Z_s dB_s + K_1 - K_t, 0 \le t \le 1, \mathcal{P}_H^\kappa\text{−q.s.} \tag{3.6}$$

If Z is a semi-martingale under each $\mathbb{P} \in \mathcal{P}$ and $d\langle Z, B\rangle_t = \Gamma_t d\langle B\rangle_t$, $\mathcal{P}_H^\kappa$−q.s. Then

$$Y_t = \xi + \int_t^1 H_s(Y_s, Z_s, \Gamma_s)ds - \int_t^1 Z_s \circ dB_s + K_1 - K_t, \quad 0 \le t \le 1, \quad \mathcal{P}_H^\kappa\text{−q.s.} \tag{3.7}$$

Here $\circ$ denotes the Stratonovich integral. We note that (3.7), (3.6), and (3.1) correspond to the second order target problem which was first introduced in [13] under a slightly different formulation. The present form, together with its first and second relaxations, were introduced in [16]. In particular, in the Markovian case, the process Γ essentially corresponds to the second order derivative of the solution to a fully nonlinear PDE, see Section 5. This justifies the denomination as "Second Order" BSDE of [4]. We choose to define 2BSDE in the form of (3.1), rather than (3.6) or (3.7), because this formulation is most appropriate for establishing the wellposedness result, which is the main result of this paper and will be reported in Section 4 below.

3.2 An alternative formulation of 2BSDEs

In [4], the authors investigate the following so called 2BSDE in Markovian framework:

$$\begin{cases} Y_t = g(B_1) + \int_t^1 h(s, B_s, Y_s, Z_s, \Gamma_s)ds - \int_t^1 Z_s \circ dB_s, \\ dZ_t = \Gamma_t dB_t + A_t dt, \end{cases} \quad 0 \le t \le 1, \quad \mathbb{P}_0 - \text{a.s.} \tag{3.8}$$

where h is a deterministic function. Then uniqueness is proved in an appropriate space $\mathcal{Z}$ for Z. The specification of $\mathcal{Z}$ is crucial, and there can be no uniqueness result if the solution is allowed to be a general square integrable process. Indeed, the following "simplest" 2BSDE with $d = 1$ has multiple solutions in the natural square integrable space:

$$\begin{cases} Y_t = \int_t^1 \frac{1}{2} c\Gamma_s ds - \int_t^1 Z_s \circ dB_s, \\ dZ_t = \Gamma_t dB_t + A_t dt, \end{cases} \quad 0 \le t \le 1, \quad \mathbb{P}_0 - \text{a.s.} \tag{3.9}$$

where $c \neq 1$ is a constant. See Example 7.1 below. The reason is that, unless $c = 1$, $\mathbb{P}_0$ is not in $\mathcal{P}_H^\kappa$ for $H(\gamma) := \frac{1}{2}c\gamma$. Also see subsection 3.4 below.

3.3 Connection with $G-$expectations and $G-$martingales

In [15] we established the martingale representation theorem for $G-$martingales, which were introduced by Peng [12]. In our framework, this corresponds to the specification $H = G(\gamma) := \frac{1}{2} \sup_{\underline{a} \le a \le \overline{a}} a : \gamma$, for some $\underline{a}, \overline{a} \in \mathbb{S}_d^{>0}$.

As an extension of [15], and as a special case of our current setting, we set

$$H_t(y, z, \gamma) := G(\gamma) - f_t(y, z). \tag{3.10}$$

Then one can easily check that:

- $D_{F_t} = [\underline{a}, \overline{a}]$ and $F_t(y, z, a) = f_t(y, z)$ for all $a \in [\underline{a}, \overline{a}]$;
- $\mathcal{P}_H^\kappa = \Big\{ \mathbb{P} \in \overline{\mathcal{P}}_S : \underline{a} \le \hat{a} \le \overline{a}, dt \times d\mathbb{P} - \text{a.s. and } \mathbb{E}^{\mathbb{P}}\Big[\big(\int_0^1 |f_t(0,0)|^\kappa dt \big)^{\frac{2}{\kappa}} \Big] < \infty \Big\}.$

In this case (3.1) is reduced to the following 2BSDE:

$$Y_t \;=\; \xi + \int_t^1 f_s(Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t, \quad \mathcal{P}_H^\kappa\text{−q.s.} \tag{3.11}$$

Moreover, we may decompose K into $dK_t = k_t dt + dK_t^0$, where $k \ge 0$ and dK_t^0 is a measure singular to the Lebesgue measure dt. One can easily check that there exists process Γ such that $G(\Gamma_t) - \frac{1}{2}\hat{a}_t : \Gamma_t = k_t$. Then (3.11) becomes

$$Y_t = \xi + \int_t^1 \Big(\frac{1}{2}\hat{a}_s : \Gamma_s - G(\Gamma_s) + f_s(Y_s, Z_s) \Big) ds - \int_t^1 Z_s dB_s + K_1^0 - K_t^0, \; \mathcal{P}_H^\kappa\text{−q.s.} \tag{3.12}$$

The wellposedness of the latter $G-$BSDE (with $K^0 = 0$ and $\kappa = 2$) was left by Peng as an open problem. We remark that, although the above two forms are equivalent, we prefer (3.11) than (3.12) because the component Γ of the solution is not unique, and we have no appropriate norm for the process Γ.

3.4 Connection with the standard BSDE

If H is linear in γ:

$$H_t(y,z,\gamma) = \frac{1}{2}a_t^0 : \gamma - f_t(y,z), \tag{3.13}$$

where $a^0 : [0,1]\times\Omega \to \mathbb{S}_d^{>0}$ is $\mathbb{F}-$progressively measurable and has uniform lower and upper bounds. We remark that in this case we do not need to assume that a^0 and f are uniformly continuous in ω. Then, under obvious extension of notations, we have

$$D_{F_t(\omega)} = \{a_t^0(\omega)\} \quad \text{and} \quad \hat{F}_t(y,z) = f_t(y,z).$$

Assume further that there exists $\mathbb{P} \in \overline{\mathcal{P}}_S$ such that $\hat{a} = a^0$, $\mathbb{P}-$a.s. and $\mathbb{E}^{\mathbb{P}}\big[\int_0^1 |f_t(0,0)|^2 dt\big] < \infty$, then $\mathcal{P}_H^\kappa = \mathcal{P}_H^2 = \{\mathbb{P}\}$. In this case, the minimum condition (3.3) implies

$$0 = K_0 = \mathbb{E}^{\mathbb{P}}[K_1] \quad \text{and thus} \quad K = 0, \quad \mathbb{P}-\text{a.s.}$$

Hence, the 2BSDE (3.1) is equivalent to the following standard BSDE:

$$Y_t = \xi - \int_t^1 f_s(Y_s,Z_s)ds - \int_t^1 Z_s dB_s, \quad 0 \le t \le 1, \quad \mathbb{P}-\text{a.s.} \tag{3.14}$$

We note that, by Remark 2.6, in this case we have

$$\mathcal{L}_H^{2,\kappa} = \mathbb{L}_H^{2,\kappa} = L_H^{2,\kappa} = \mathbb{L}^2(\mathbb{P}) \quad \text{for all} \quad 1 \le \kappa < 2.$$

4 Wellposedness of 2BSDEs

Throughout this paper Assumption 2.1 and the following assumption will always be in force.

Assumption 4.1 (i) *$\mathcal{P}_H^\kappa$ is not empty.*
(ii) *The process $\hat{F}^0$ satisfies the following stronger integrability:*

$$\|\hat{F}^0\|^2_{\mathbb{H}_H^{2,\kappa}} < \infty \quad \textit{where} \quad \|\hat{F}^0\|^p_{\mathbb{H}_H^{p,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{0\le t\le 1}^{\mathbb{P}} \big(\mathbb{E}_t^{H,\mathbb{P}}\big[\int_0^1 |\hat{F}_s^0|^\kappa ds\big]\big)^{\frac{p}{\kappa}}\Big]. \tag{4.1}$$

(iii) *There exists a constant C such that for all $(y,z_1,z_2) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$:*

$$\big|\hat{F}_t(y,z_1) - \hat{F}_t(y,z_2)\big| \le C|\hat{a}_t^{1/2}(z_1-z_2)|, \; dt\times d\mathbb{P} - a.s. \tag{4.2}$$

Clearly the norm $\|\hat{F}^0\|_{\mathbb{H}^{p,\kappa}_H}$ is motivated by the norm $\|\xi\|_{\mathbb{L}^{p,\kappa}_H}$ in (2.10), and it satisfies

$$\sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^1 |\hat{F}^0_t|dt\Big)^2\Big] \le \|\hat{F}^0\|^2_{\mathbb{H}^{2,\kappa}_H}. \tag{4.3}$$

Here we abuse the notation $\mathbb{H}^{p,\kappa}_H$ slightly by noting that, unlike the elements in $\mathbb{H}^p_H$, $\hat{F}^0$ is $1-$dimensional and the norm in (4.1) does not contain the factor $\hat{a}^{1/2}$.

Remark 4.2 Assumption 4.1 (ii) is implied by the following condition on H:

$$|H_t(y,z_1,\gamma) - H_t(y,z_2,\gamma)| \quad \le \quad C\big|\hat{a}_t^{1/2}(z_1-z_2)\big|, \ dt\times d\mathbb{P}-\text{a.s.}$$

for some constant C which does not depend on (t,ω,y,γ). □

For any $\mathbb{P}\in\mathcal{P}^\kappa_H$, $\mathbb{F}-$stopping time τ, and $\mathcal{F}_\tau-$measurable random variable $\xi\in\mathbb{L}^2(\mathbb{P})$, let $(\mathcal{Y}^{\mathbb{P}},\mathcal{Z}^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(\tau,\xi),\mathcal{Z}^{\mathbb{P}}(\tau,\xi))$ denote the solution to the following standard BSDE:

$$\mathcal{Y}^{\mathbb{P}}_t = \xi - \int_t^\tau \hat{F}_s(\mathcal{Y}^{\mathbb{P}}_s,\mathcal{Z}^{\mathbb{P}}_s)ds - \int_t^\tau \mathcal{Z}^{\mathbb{P}}_s dB_s, \ \ 0\le t\le\tau, \ \mathbb{P}-\text{a.s.} \tag{4.4}$$

We have the following result which is slightly stronger than the standard ones in the literature. The proof is provided in subsection 7.2 of the Appendix for completeness.

Lemma 4.3 *Under the Lipschitz conditions* (2.7) *and* (4.2)*, for each* $\mathbb{P}\in\mathcal{P}^\kappa_H$*, the BSDE* (4.4) *has a unique solution satisfying the following estimates:*

$$|\mathcal{Y}^{\mathbb{P}}_t|^2 \le C_\kappa\Big(\mathbb{E}^{\mathbb{P}}_t\Big[|\xi|^\kappa + \int_t^1|\hat{F}^0_s|^\kappa ds\Big]\Big)^{\frac{2}{\kappa}}, \ 0\le t\le 1, \ \mathbb{P}-a.s. \tag{4.5}$$

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}^{1/2}_t\mathcal{Z}^{\mathbb{P}}_t|^2dt\Big] \le C_\kappa\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}\Big(\mathbb{E}^{\mathbb{P}}_t\Big[|\xi|^\kappa + \int_0^1|\hat{F}^0_s|^\kappa ds\Big]\Big)^{\frac{2}{\kappa}}\Big]. \tag{4.6}$$

We note that in above lemma, and in all subsequent results, we shall denote by C a generic constant which may vary from line to line and depends only on the dimension d and the Lipschitz constants in Assumptions 2.1 and 4.1. We shall also denote by C_κ a generic constant which may depend on κ as well. We emphasize that, due to Assumption 4.1, the constants C and C_κ in the estimates will not depend on the bounds $\underline{a}_{\mathbb{P}}$ and $\overline{a}_{\mathbb{P}}$ in (2.9).

4.1 Representation and uniqueness of the solution

Theorem 4.4 *Let Assumptions 2.1 and 4.1 hold. Assume that* $\xi\in\mathbb{L}^{2,\kappa}_H$ *and that* $(Y,Z)\in\mathbb{D}^{2,\kappa}_H\times\mathbb{H}^{2,\kappa}_H$ *is a solution to 2BSDE* (3.1)*. Then, for any* $\mathbb{P}\in\mathcal{P}^\kappa_H$ *and* $0\le t_1<t_2\le 1$,

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^\kappa_H(t_1,\mathbb{P})} \mathcal{Y}^{\mathbb{P}'}_{t_1}(t_2,Y_{t_2}), \quad \mathbb{P}-a.s. \tag{4.7}$$

Consequently, the 2BSDE (3.1) *has at most one solution in* $\mathbb{D}^{2,\kappa}_H\times\mathbb{H}^{2,\kappa}_H$.

Proof. First, if (4.7) holds, then

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathcal{Y}_t^{\mathbb{P}'}(1,\xi), \quad \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}_H^\kappa, t \in [0,1], \tag{4.8}$$

and thus is unique. Note that $d\langle Y, B\rangle_t = Z_t d\langle B\rangle_t$, $\mathcal{P}_H^\kappa-$q.s. then Z is also unique.

It remains to prove (4.7).

(i) Fix $0 \le t_1 < t_2 \le 1$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$. For any $\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})$, note that

$$Y_t = Y_{t_2} - \int_t^{t_2} \hat{F}_s(Y_s, Z_s) ds - \int_t^1 Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, \quad 0 \le t \le t_2, \ \mathbb{P}'-\text{a.s.}$$

and that $K^{\mathbb{P}'}$ is nondecreasing, $\mathbb{P}'-$a.s. By (2.7) and (4.2), and applying the comparison principle for standard BSDE under $\mathbb{P}$, we have $Y_{t_1} \ge \mathcal{Y}_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$, $\mathbb{P}'-$a.s. Since $\mathbb{P}' = \mathbb{P}$ on $\mathcal{F}_t^+$, we get $Y_{t_1} \ge \mathcal{Y}_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$, $\mathbb{P}-$a.s. and thus

$$Y_{t_1} \ge \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1,\mathbb{P})} \mathcal{Y}_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \quad \mathbb{P}-\text{a.s.} \tag{4.9}$$

(ii) We now prove the other direction of the inequality. Fix $\mathbb{P} \in \mathcal{P}_H^\kappa$. We shall prove in Step 3 below that

$$C_{t_1}^{\mathbb{P}} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}'}\Big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2\Big] \;\;<\;\; \infty, \quad \mathbb{P}-\text{a.s.} \tag{4.10}$$

For every $\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})$, denote:

$$\delta Y := Y - \mathcal{Y}^{\mathbb{P}'}(t_2, Y_{t_2}) \quad \text{and} \quad \delta Z := Z - \mathcal{Z}^{\mathbb{P}'}(t_2, Y_{t_2}).$$

By the Lipschitz conditions (2.7) and (4.2), there exist bounded processes λ, η such that

$$\delta Y_t = \int_t^{t_2} \big(\lambda_s \delta Y_s + \eta_s \hat{a}_s^{1/2} \delta Z_s\big) ds - \int_t^{t_2} \delta Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, \ t \le t_2, \mathbb{P}'-\text{a.s.} \tag{4.11}$$

Define:

$$M_t := \exp\Big(-\int_0^t \eta_s \hat{a}_s^{-1/2} dB_s - \int_0^t (\lambda_s + \frac{1}{2}|\eta_s|^2) ds\Big), \quad 0 \le t \le t_2, \ \mathbb{P}'-\text{a.s.} \tag{4.12}$$

By Itô's formula, we have:

$$d\big(M_t \delta Y_t\big) = M_t\big(\delta Z_t - \delta Y_t \eta_t \hat{a}_t^{-1/2}\big) dB_t - M_t dK_t^{\mathbb{P}'}, \quad t_1 \le t \le t_2, \ \mathbb{P}'-\text{a.s.} \tag{4.13}$$

Then, since $\delta Y_{t_2} = 0$, using standard localization arguments if necessary, we compute that:

$$Y_{t_1} - \mathcal{Y}_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}) = \delta Y_{t_1} = M_{t_1}^{-1} \mathbb{E}_{t_1}^{\mathbb{P}'}\Big[\int_{t_1}^{t_2} M_t dK_t^{\mathbb{P}'}\Big] \le \mathbb{E}_{t_1}^{\mathbb{P}'}\Big[\sup_{t_1 \le t \le t_2} (M_{t_1}^{-1} M_t)(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})\Big]$$

by the non-decrease of $K^{\mathbb{P}'}$. By the boundedness of λ, η, for every $p \geq 1$ we have,

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\Big[\sup_{t_1 \leq t \leq t_2}(M_{t_1}^{-1}M_t)^p + \sup_{t_1 \leq t \leq t_2}(M_{t_1}M_t^{-1})^p\Big] \leq C_p, \quad t_1 \leq t \leq t_2, \ \mathbb{P}' - \text{a.s.} \tag{4.14}$$

Then it follows from the Hölder inequality that:

$$\begin{aligned} Y_{t_1} - \mathcal{Y}_t^{\mathbb{P}'}(t_2, Y_{t_2}) &\leq \Big(\mathbb{E}_{t_1}^{\mathbb{P}'}\Big[\sup_{t_1 \leq t \leq t_2}(M_{t_1}^{-1}M_t)^3\Big]\Big)^{1/3}\Big(\mathbb{E}_{t_1}^{\mathbb{P}'}\big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^{3/2}\big]\Big)^{2/3} \\ &\leq C\Big(\mathbb{E}_{t_1}^{\mathbb{P}'}\big[K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\big]\mathbb{E}_{t_1}^{\mathbb{P}'}\big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2\big]\Big)^{1/3} \\ &\leq C^4(C_{t_1}^{\mathbb{P}})^{1/3}\Big(\mathbb{E}_{t_1}^{\mathbb{P}'}\big[K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'}\big]\Big)^{1/3}, \quad \mathbb{P} - \text{a.s.} \end{aligned}$$

By the minimum condition (3.3) and the arbitrariness of $\mathbb{P}' \in \mathcal{P}_H^\kappa(t, \mathbb{P})$, this implies that (4.9) holds with equality.

(iii) It remains to show that the estimate (4.10) holds. By the definition of the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ we have:

$$\sup_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1, \mathbb{P})} \mathbb{E}^{\mathbb{P}'}\Big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2\Big] \leq C\Big(\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \|\hat{F}^0\|^2_{\mathbb{H}_H^{2,\kappa}}\Big) < \infty. \tag{4.15}$$

We next use the definition of the essential supremum, see e.g. Neveu [10] to see that

$$\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1, \mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}'}\Big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2\Big] = \sup_{n \geq 1}\mathbb{E}_{t_1}^{\mathbb{P}_n}\Big[(K_{t_2}^{\mathbb{P}_n} - K_{t_1}^{\mathbb{P}_n})^2\Big], \quad \mathbb{P} - \text{a.s.} \tag{4.16}$$

for some sequence $(\mathbb{P}_n)_{n \geq 1} \subset \mathcal{P}_H^\kappa(t_1, \mathbb{P})$. Observe that for $\mathbb{P}'_1, \mathbb{P}'_2 \in \mathcal{P}_H^\kappa(t_1, \mathbb{P})$, there exists $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1, \mathbb{P})$ such that

$$\mathbb{E}_{t_1}^{\mathbb{P}'}\Big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2\Big] \geq \mu_{t_1} := \max\Big\{\mathbb{E}_{t_1}^{\mathbb{P}'_1}\Big[(K_{t_2}^{\mathbb{P}'_1} - K_{t_1}^{\mathbb{P}'_1})^2\Big], \mathbb{E}_{t_1}^{\mathbb{P}'_1}\Big[(K_{t_2}^{\mathbb{P}'_1} - K_{t_1}^{\mathbb{P}'_1})^2\Big]\Big\}. \tag{4.17}$$

Indeed, considering the subsets of $\mathcal{F}_{t_1}$:

$$E_1 := \Big\{\mu_{t_1} = \mathbb{E}_{t_1}^{\mathbb{P}'_1}\Big[(K_{t_2}^{\mathbb{P}'_1} - K_{t_1}^{\mathbb{P}'_1})^2\Big]\Big\} \quad \text{and} \quad E_2 := \Omega \setminus E_1,$$

we may define the probability measure $\mathbb{P}'$ by:

$$\mathbb{P}'[E] := \mathbb{P}'[E \cap E_1] + \mathbb{P}'[E \cap E_2] \quad \text{for all} \quad E \in \mathcal{F}_1.$$

Then $\mathbb{P}'$ satisfies (4.17) with equality, by definition, and

$$\mathbb{P}' \in \mathcal{P}_H^\kappa(t, \mathbb{P}) \tag{4.18}$$

is proved in subsection 7.3 in the Appendix. With this property, we can re-write (4.16), without loss of generality, into:

$$\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1, \mathbb{P})} \mathbb{E}_{t_1}^{\mathbb{P}'}\Big[(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^2\Big] = \lim_{n \to \infty}\mathbb{E}_{t_1}^{\mathbb{P}_n}\Big[(K_{t_2}^{\mathbb{P}_n} - K_{t_1}^{\mathbb{P}_n})^2\Big].$$

It follows from (4.15) that

$$\mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1,\mathbb{P})}\mathbb{E}_{t_1}^{\mathbb{P}'}\Big[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^2\Big]\Big]<\infty,$$

and we then deduce the required estimate (4.10) immediately. $\square$

As an immediate consequence of the representation formula (4.8), together with the comparison principle for BSDEs, we have the following comparison principle for 2BSDEs.

Corollary 4.5 *Let Assumptions 2.1 and 4.1 hold. Assume* $\xi^i \in \mathbb{L}_H^{2,\kappa}$ *and* $(Y^i, Z^i) \in \mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$ *is a corresponding solution of the 2BSDE* (3.1)*,* $i = 1, 2$*. If* $\xi^1 \le \xi^2$*,* $\mathcal{P}_H^\kappa-$*q.s. then* $Y^1 \le Y^2$*,* $\mathcal{P}_H^\kappa-$*q.s.*

4.2 A priori estimates and existence of the solution

Theorem 4.6 *Let Assumptions 2.1 and 4.1 hold.*
(i) *Assume that* $\xi \in \mathbb{L}_H^{2,\kappa}$ *and that* $(Y, Z) \in \mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$ *is a solution to 2BSDE* (3.1)*. Then there exist a constant* C_κ *such that*

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}}+\|Z\|^2_{\mathbb{H}_H^{2,\kappa}}+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}[|K_1^{\mathbb{P}}|^2] \;\le\; C_\kappa\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}}+\|\hat{F}^0\|^2_{\mathbb{H}_H^{2,\kappa}}\big). \tag{4.19}$$

(ii) *Assume that* $\xi^i \in \mathbb{L}_H^{2,\kappa}$ *and that* $(Y^i, Z^i) \in \mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$ *is a corresponding solution to 2BSDE* (3.1)*,* $i = 1, 2$*. Denote* $\delta\xi := \xi^1-\xi^2$*,* $\delta Y := Y^1-Y^2$*,* $\delta Z := Z^1-Z^2$*, and* $\delta K^{\mathbb{P}} := K^{1,\mathbb{P}}-K^{2,\mathbb{P}}$*. Then there exists a constant* C_κ *such that*

$$\begin{aligned}\|\delta Y\|_{\mathbb{D}_H^{2,\kappa}} &\le C_\kappa\|\delta\xi\|_{\mathbb{L}_H^{2,\kappa}},\\ \|\delta Z\|^2_{\mathbb{H}_H^{2,\kappa}}+\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}|\delta K_t^{\mathbb{P}}|^2\Big] &\le C_\kappa\|\delta\xi\|_{\mathbb{L}_H^{2,\kappa}}\Big(\|\xi^1\|_{\mathbb{L}_H^{2,\kappa}}+\|\xi^2\|_{\mathbb{L}_H^{2,\kappa}}+\|\hat{F}^0\|_{\mathbb{H}_H^{2,\kappa}}\Big).\end{aligned} \tag{4.20}$$

Proof. (i) First, by Lemma 4.3 we have:

$$|\mathcal{Y}_t^{\mathbb{P}}(1,\xi)|^2\le C_\kappa\Big(\mathbb{E}_t^{\mathbb{P}}\Big[|\xi|^\kappa+\int_t^1|\hat{F}_s^0|^\kappa ds\Big]\Big)^{2/\kappa},\ \mathbb{P}-\text{a.s. for all } \mathbb{P}\in\mathcal{P}_H^\kappa, t\in[0,1].$$

By the representation formula (4.8), this provides

$$|Y_t|^2\le C_\kappa\Big(\mathbb{E}_t^{H,\mathbb{P}}\Big[|\xi|^\kappa+\int_t^1|\hat{F}_s^0|^\kappa ds\Big]\Big)^{2/\kappa},\ \mathbb{P}-\text{a.s. for all } \mathbb{P}\in\mathcal{P}_H^\kappa, t\in[0,1],$$

and, by the definition of the norms, we get

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}}\le C_\kappa\left(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}}+\|\hat{F}^0\|^2_{\mathbb{H}_H^{2,\kappa}}\right). \tag{4.21}$$

Next, under each $\mathbb{P} \in \mathcal{P}_H^\kappa$, applying Itô's formula to $|Y|^2$, it follows from the Lipschitz conditions (2.7) and (4.2) that:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}_s^{1/2} Z_s|^2 ds\Big] &\leq \mathbb{E}^{\mathbb{P}}\Big[|Y_0|^2 + \int_0^1 |\hat{a}_s^{1/2} Z_s|^2 ds\Big] \\
&\leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^2 + \int_0^1 |Y_t|\big(|\hat{F}_t^0| + |Y_t| + |\hat{a}_t^{1/2} Z_t|\big) ds + \int_0^1 |Y_t| dK_t^{\mathbb{P}}\Big] \\
&\leq C\varepsilon^{-1}\mathbb{E}^{\mathbb{P}}\Big[|\xi|^2 + \sup_{0\leq t\leq 1} |Y_t|^2 + \Big(\int_0^1 |\hat{F}_t^0| dt\Big)^2\Big] \\
&\quad + \varepsilon\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt + |K_1^{\mathbb{P}}|^2\Big]
\end{aligned}$$

for any $\varepsilon \in (0,1]$. By the definition of $K^{\mathbb{P}}$, one gets immediately that

$$\mathbb{E}^{\mathbb{P}}[|K_1^{\mathbb{P}}|^2] \leq C_0\mathbb{E}^{\mathbb{P}}\Big[|\xi|^2 + \sup_{0\leq t\leq 1} |Y_t|^2 + \int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt + \Big(\int_0^1 |\hat{F}_t^0| dt\Big)^2\Big], \tag{4.22}$$

for some constant C_0 independent of ε. Then,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}_s^{1/2} Z_s|^2 ds\Big] &\leq C\varepsilon^{-1}\mathbb{E}^{\mathbb{P}}\Big[|\xi|^2 + \sup_{0\leq t\leq 1} |Y_t|^2 + \Big(\int_0^1 |\hat{F}_t^0| dt\Big)^2\Big] \\
&\quad + (1 + C_0)\varepsilon\, \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\Big].
\end{aligned}$$

By setting $\varepsilon := [2(1 + C_0)]^{-1}$, this provides

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}_s^{1/2} Z_s|^2 ds\Big] \leq C\mathbb{E}^{\mathbb{P}}\Big[|\xi|^2 + \sup_{0\leq t\leq 1} |Y_t|^2 + \Big(\int_0^1 |\hat{F}_t^0| dt\Big)^2\Big].$$

By (4.21) and noting that $\|\hat{F}^0\|_{\mathbb{H}_H^{2,1}} \leq \|\hat{F}^0\|_{\mathbb{H}_H^{2,\kappa}}$ for $\kappa > 1$, we have

$$\|Z\|^2_{\mathbb{H}_H^{2,\kappa}} \leq C\big(\|\xi\|^2_{\mathbb{L}_H^{2,\kappa}} + \|\hat{F}^0\|^2_{\mathbb{H}_H^{2,\kappa}}\big). \tag{4.23}$$

This, together with (4.21) and (4.22), proves (4.19).

(ii) First, following the same arguments in Lemma 4.3, we have

$$\big|\mathcal{Y}_t^{\mathbb{P}}(1,\xi_1) - \mathcal{Y}_t^{\mathbb{P}}(1,\xi_2)\big| \leq C\Big(\mathbb{E}_t^{\mathbb{P}}\Big[|\delta\xi|^\kappa\Big]\Big)^{2/\kappa}, \ \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}_H^\kappa, t \in [0,1].$$

Then, following similar arguments as in (i) we have

$$\|\delta Y\|_{\mathbb{D}_H^{2,\kappa}} \leq C\|\delta\xi\|_{\mathbb{L}_H^{2,\kappa}}. \tag{4.24}$$

Next, under each $\mathbb{P}\in\mathcal{P}_H^\kappa$, applying Itô's formula to $|\delta Y|^2$ we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat a_s^{1/2}\delta Z_s|^2 ds\Big] &\le \mathbb{E}^{\mathbb{P}}\Big[|\delta Y_0|^2+\int_0^1 |\hat a_s^{1/2}\delta Z_s|^2 ds\Big]\\
&\le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^2+\int_0^1 |\delta Y_t|\big(|\delta Y_t|+|\hat a_t^{1/2}\delta Z_t|\big)ds+\Big|\int_0^1 \delta Y_t d(\delta K_t^{\mathbb{P}})\Big|\Big]\\
&\le C\mathbb{E}^{\mathbb{P}}\Big[|\delta\xi|^2+\sup_{0\le t\le 1}|\delta Y_t|^2+\sup_{0\le t\le 1}|\delta Y_t|[K_1^{1,\mathbb{P}}+K_1^{2,\mathbb{P}}]\Big]\\
&\quad +\frac{1}{2}\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat a_t^{1/2}\delta Z_t|^2 dt\Big].
\end{aligned}$$

Then, by (4.24) and (4.19),

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat a_s^{1/2}\delta Z_s|^2 ds\Big] &\le C_\kappa\|\delta\xi\|^2_{\mathbb{L}^{2,\kappa}_H}+C_\kappa\|\delta\xi\|_{\mathbb{L}^{2,\kappa}_H}\Big(\mathbb{E}^{\mathbb{P}}\big[|K_1^{1,\mathbb{P}}|^2+|K_1^{2,\mathbb{P}}|^2\big]\Big)^{1/2}\\
&\le C_\kappa\|\delta\xi\|^2_{\mathbb{L}^{2,\kappa}_H}+C_\kappa\|\delta\xi\|_{\mathbb{L}^{2,\kappa}_H}\Big(\|\xi^1\|_{\mathbb{L}^{2,\kappa}_H}+\|\xi^2\|_{\mathbb{L}^{2,\kappa}_H}+\|\hat F^0\|_{\mathbb{H}^{2,\kappa}_H}\Big).
\end{aligned}$$

The estimate for $\delta K^{\mathbb{P}}$ is obvious now. □

We are now ready to state the main result of this paper.

Theorem 4.7 *Let Assumptions 2.1 and 4.1 hold. Then for any $\xi\in\mathcal{L}_H^{2,\kappa}$, the 2BSDE* (3.1) *has a unique solution $(Y,Z)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$.*

Proof. (i) We first assume $\xi\in\mathrm{UC}_b(\Omega)$. In this case, by Step 4 of the proof of Theorem 4.5 in [16], there exist $(Y,Z)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$ such that $Y_1=\xi$, $\mathcal{P}_H^\kappa-$q.s. and the $K^{\mathbb{P}}$ defined by (3.2) is nondecreasing, $\mathbb{P}-$a.s. Moreover, the representation (4.8) holds:

$$Y_t=\operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})}{}^{\mathbb{P}}\ \mathcal{Y}_t^{\mathbb{P}'}(1,\xi),\quad \mathbb{P}-\text{a.s. for all } \mathbb{P}\in\mathcal{P}_H^\kappa, t\in[0,1], \tag{4.25}$$

It remains to check the minimum condition (3.3). We follow the arguments in the proof of Theorem 4.4. For $t\in[0,1]$, $\mathbb{P}\in\mathcal{P}_H^\kappa$, and $\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})$, we denote $\delta Y:=Y-\mathcal{Y}^{\mathbb{P}'}(1,\xi)$, $\delta Y:=Z-\mathcal{Z}^{\mathbb{P}'}(1,\xi)$, and we introduce the process M of (4.12). Then, it follows from the non-decrease of $K^{\mathbb{P}'}$ that

$$Y_t-\mathcal{Y}_t^{\mathbb{P}'}(1,\xi)=\delta Y_t=\mathbb{E}_t^{\mathbb{P}'}\Big[\int_t^1 M_s dK_s^{\mathbb{P}'}\Big]\ge\mathbb{E}_t^{\mathbb{P}'}\Big[\big(\inf_{t\le s\le 1}M_t^{-1}M_s\big)\big(K_1^{\mathbb{P}'}-K_t^{\mathbb{P}'}\big)\Big]. \tag{4.26}$$

On the other hand, by (4.14) and (4.26), we estimate by the Hölder inequality that

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{P}'}\big[K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}\big] \\
= \; & \mathbb{E}_t^{\mathbb{P}'}\Big[\big(\inf_{t\le s\le 1} M_t^{-1}M_s\big)^{1/3}\big(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}\big)^{1/3}\big(\inf_{t\le s\le 1} M_t^{-1}M_s\big)^{-1/3}\big(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}\big)^{2/3}\Big] \\
\le \; & \Big(\mathbb{E}_t^{\mathbb{P}'}\big[\big(\inf_{t\le s\le 1} M_t^{-1}M_s\big)\big(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}\big)\big]\mathbb{E}_t^{\mathbb{P}'}\big[\sup_{t\le s\le 1} M_t M_s^{-1}\big]\mathbb{E}_t^{\mathbb{P}'}\big[\big(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}\big)^2\big]\Big)^{1/3} \\
\le \; & C\Big(\mathbb{E}_t^{\mathbb{P}'}\big[\big(K_1^{\mathbb{P}'}\big)^2\big]\mathbb{E}_t^{\mathbb{P}'}\big[\big(\inf_{t\le s\le 1} M_t^{-1}M_s\big)\big(K_1^{\mathbb{P}'} - K_t^{\mathbb{P}'}\big)\big]\Big)^{1/3} \\
\le \; & C\Big(\mathbb{E}_t^{\mathbb{P}'}\big[\big(K_1^{\mathbb{P}'}\big)^2\big]\Big)^{1/3}\big(\delta Y_t\big)^{1/3}.
\end{aligned}
$$

By following the argument of the proof of Theorem 4.4 (ii) and (iii), we then deduce that the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ inherits the minimum condition (3.3) from (4.25).

(ii) In general, for $\xi \in \mathcal{L}_H^{2,\kappa}$, by the definition of the space $\mathcal{L}_H^{2,\kappa}$ there exist $\xi_n \in \mathrm{UC}_b(\Omega)$ such that $\lim_{n\to\infty}\|\xi_n - \xi\|_{\mathbb{L}_H^{2,\kappa}} = 0$. Then it is clear that

$$
\sup_{n\ge 1}\|\xi_n\|_{\mathbb{L}_H^{2,\kappa}} < \infty \quad \text{and} \quad \lim_{n,m\to\infty}\|\xi_n - \xi_m\|_{\mathbb{L}_H^{2,\kappa}} = 0. \tag{4.27}
$$

Let $(Y^n, Z^n) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ be the solution to 2BSDE (3.1) with terminal condition ξ_n, and

$$
K_t^{n,\mathbb{P}} := Y_0^n - Y_t^n + \int_0^t \hat{F}_s(Y_s^n, Z_s^n)ds + \int_0^t Z_s^n dB_s, \quad 0 \le t \le 1, \quad \mathbb{P}-\text{a.s.} \tag{4.28}
$$

By Theorem 4.6, as $n, m \to \infty$ we have

$$
\begin{aligned}
& \|Y^n - Y^m\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n - Z^m\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}|K_t^{n,\mathbb{P}} - K_t^{m,\mathbb{P}}|^2\Big] \\
\le \; & C_\kappa\|\xi_n - \xi_m\|^2_{\mathbb{L}_H^{2,\kappa}} + C_\kappa\big(\|\xi_n\|_{\mathbb{L}_H^{2,\kappa}} + \|\xi_m\|_{\mathbb{L}_H^{2,\kappa}} + \|\hat{F}^0\|_{\mathbb{H}_H^{2,\kappa}}\big)\|\xi_n - \xi_m\|_{\mathbb{L}_H^{2,\kappa}} \to 0.
\end{aligned}
$$

Then by otherwise choosing a subsequence, we may assume without loss of generality that,

$$
\|Y^n - Y^m\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n - Z^m\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}|K_t^{n,\mathbb{P}} - K_t^{m,\mathbb{P}}|^2\Big] \;\le\; 2^{-n}, \tag{4.29}
$$

for all $m \ge n \ge 1$. This implies that, for every $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $m \ge n \ge 1$,

$$
\mathbb{P}\Big[\sup_{0\le t\le 1}\big[|Y_t^n - Y_t^m|^2 + |K_t^{n,\mathbb{P}} - K_t^{m,\mathbb{P}}|^2\big] + \int_0^1 |Z_t^n - Z_t^m|^2 dt > \frac{1}{n}\Big] \;\le\; Cn2^{-n}. \tag{4.30}
$$

Define

$$
Y := \overline{\lim_{n\to\infty}} Y^n, \quad Z := \overline{\lim_{n\to\infty}} Z^n, \quad K^{\mathbb{P}} := \overline{\lim_{n\to\infty}} K^{n,\mathbb{P}}, \tag{4.31}
$$

where the $\overline{\lim}$ for Z is taken componentwise. It is clear that $Y, Z, K^{\mathbb{P}}$ are all $\mathbb{F}^+-$progressively measurable. By (4.30), it follows from the Borel-Cantelli Lemma that

$$\lim_{n\to\infty}\Big[\sup_{0\le t\le 1}\big\{|Y_t^n - Y_t|^2 + |K_t^{n,\mathbb{P}} - K_t^{\mathbb{P}}|^2\big\} + \int_0^1 |Z_t^n - Z_t|^2 dt\Big] = 0, \quad \mathbb{P}-\text{a.s. for all } \mathbb{P}\in\mathcal{P}_H^\kappa.$$

Since $Y^n, K^{n,\mathbb{P}}$ are càdlàg and $K^{n,\mathbb{P}}$ is nondecreasing, this implies that Y is càdlàg, $\mathcal{P}_H^\kappa-$q.s. and $K^{\mathbb{P}}$ is càdlàg and nondecreasing, $\mathbb{P}-$a.s. Moreover, for every $\mathbb{P}\in\mathcal{P}_H^\kappa$ and $n\ge 1$, send $m\to\infty$ in (4.29) and apply Fatou's lemma under $\mathbb{P}$, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}\big\{|Y_t^n - Y_t|^2 + |K_t^{n,\mathbb{P}} - K_t^{\mathbb{P}}|^2\big\} + \int_0^1 |Z_t^n - Z_t|^2\Big] \le 2^{-n},$$

This implies that

$$\|Y^n - Y\|^2_{\mathbb{D}_H^{2,\kappa}} + \|Z^n - Z\|^2_{\mathbb{H}_H^{2,\kappa}} + \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa}\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}|K_t^{n,\mathbb{P}} - K_t^{\mathbb{P}}|^2\Big] \quad\le\quad 2^{-n}\to 0, \quad \text{as } n\to\infty.$$

Then it is clear that $(Y,Z)\in\mathbb{D}_H^{2,\kappa}\times\mathbb{H}_H^{2,\kappa}$.

Finally, since $(Y^n, Z^n, K^{n,\mathbb{P}})$ satisfy (3.4) and (3.3), the limit $(Y, Z, K^{\mathbb{P}})$ also satisfy (3.4) and (3.3). Hence (Y,Z) is a solution to 2BSDE (3.1). □

5 Connection with fully nonlinear PDE

5.1 The Markovian setup

In this section we consider the case:

$$H_t(\omega, y, z, \gamma) \quad=\quad h(t, B_t(\omega), y, z, \gamma),$$

where $h : [0,1]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times D_h\to\mathbb{R}$ is a deterministic map. Then the corresponding conjugate and bi-conjugate functions become

$$f(t,x,y,z,a) \quad:=\quad \sup_{\gamma\in D_h}\Big\{\frac{1}{2}a:\gamma - h(t,x,y,z,\gamma)\Big\}, \quad a\in\mathbb{S}_d^{>0}, \tag{5.1}$$

$$\hat{h}(t,x,y,z,\gamma) \quad:=\quad \sup_{a\in\mathbb{S}_d^{>0}}\Big\{\frac{1}{2}a:\gamma - f(t,x,y,z,a)\Big\}, \quad \gamma\in\mathbb{R}^{d\times d}. \tag{5.2}$$

Notice that $\hat{h}\le h$, in general, and $\hat{h}=h$ if and only if h is concave and nondecreasing in γ. The following is a slight strengthening of Assumption 2.1 to the present context.

Assumption 5.1 *h is continuous in t, uniformly in γ, lower-semicontinuous in γ, and*

$$|h(t,x,y,z,\gamma) - h(t,x',y',z',\gamma)| \le \rho(|x-x'|) + C\big(|y-y'| + |z-z'|\big), \tag{5.3}$$

for some uniform constant C and modulus of continuity ρ with polynomial growth.

Then clearly, f and $\hat{h}$ are Lebesgue measurable in all variables, $-\infty < \hat{h} \le h$, $\hat{h}$ is nondecreasing and convex in γ, the domain D_{f_t} of f in a is independent of (x, y, z), and the domain $D_{\hat{h}}$ of $\hat{h}$ contains D_h and can be larger in general. Moreover, in their domains,

$$\begin{array}{c} f \text{ (resp. } \hat{h}) \text{ is uniformly continuous in } t, \text{ uniformly in } a \text{ (resp. } \gamma), \\ f \text{ and } \hat{h} \text{ are uniformly continuous in } x, \text{ with the same modulus of continuity } \rho, \\ f \text{ and } \hat{h} \text{ are uniformly Lipschitz in } (y, z). \end{array} \tag{5.4}$$

We also reformulate Assumption (4.2) in the present context.

Assumption 5.2 *There exists a constant C such that*

$$\big|f(t, x, y, z_1, a) - f(t, x, y, z_2, a)\big| \le C\big|a^{1/2}(z_1 - z_2)\big|, \quad t \in [0, 1], a \in D_{f_t}, x, z_1, z_2 \in \mathbb{R}^d, y \in \mathbb{R}.$$

Next, let $g : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable function. In this section we shall always consider the 2BSDE (3.1) in this Markovian setting with terminal condition $\xi = g(B_1)$:

$$Y_t = g(B_1) - \int_t^1 f(s, B_s, Y_s, Z_s, \hat{a}_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t, \quad 0 \le t \le 1, \quad \mathcal{P}_H^\kappa\text{−q.s.} \tag{5.5}$$

Our main objective is to establish the connection $Y_t = v(t, B_t)$, $t \in [0, 1]$, $\mathcal{P}_H^\kappa$−q.s. where v solves, in some sense, the following fully nonlinear PDE:

$$\begin{cases} \mathcal{L}v(t, x) := \partial_t v(t, x) + \hat{h}\big(t, x, v(t, x), Dv(t, x), D^2 v(t, x)\big) = 0, & 0 \le t < 1, \\ v(1, x) = g(x). \end{cases} \tag{5.6}$$

We remark that the nonlinearity of the above PDE is the nondecreasing and convex envelope $\hat{h}$, not the original h. This is illustrated by the following example.

Example 5.3 The problem of hedging under gamma constraints in dimension $d = 1$, as formulated by Cheridito, Soner and Touzi [3], leads to the specification

$$h(t, x, y, z, \gamma) := \frac{1}{2}\gamma \text{ if } \gamma \in [\underline{\Gamma}, \overline{\Gamma}], \quad \text{and} \quad \infty \text{ otherwise},$$

where $\underline{\Gamma} < 0 < \overline{\Gamma}$ are given constants. Then, direct calculation leads to

$$\begin{aligned} f(a) &= \frac{1}{2}\big(\overline{\Gamma}(a-1)^+ - \underline{\Gamma}(a-1)^-\big), \quad a > 0, \\ \hat{h}(\gamma) &= \frac{1}{2}(\gamma \vee \underline{\Gamma}) \text{ if } \gamma \le \overline{\Gamma}, \text{ and } \infty \text{ otherwise.} \end{aligned}$$

□

We will discuss further this case in Example 5.12 below, in order to obtain the nonlinearity appearing in the PDE characterization of [3] for the superhedging problem under gamma constraints. Indeed, equation (5.6) needs to be reformulated in some appropriate sense if $D_h \ne \mathbb{R}^{d\times d}$, because then $\hat{h}$ may take infinite values, and the meaning of (5.6) is not clear anymore. This leads typically to a boundary layer and requires the interpretation of the equation in the relaxed boundary value sense of viscosity solutions, see, e.g. [5].

5.2 A nonlinear Feynman-Kac representation formula

Theorem 5.4 *Let Assumptions 5.1 and 5.2 hold true. Suppose further that $\hat{h}$ is continuous in γ in its domain, D_f is independent of t and is bounded both from above and away from 0, and $g(B_1) \in \mathbb{L}_H^{2,\kappa}$. If $v \in C^{1,2}([0,1), \mathbb{R}^d)$ is a classical solution of* (5.6)*, then:*

$$Y_t := v(t, B_t),\ \ Z_t := Dv(t, B_t),\ \ K_t := \int_0^t k_s ds$$
$$\text{with } k_t := \hat{h}\left(t, B_t, Y_t, Z_t, \Gamma_t\right) - \tfrac{1}{2}\hat{a}_t : \Gamma_t + f\left(t, B_t, Y_t, Z_t, \hat{a}_t\right) \ \text{ and } \Gamma_t := D^2 v(t, B_t)$$

is the unique solution of the 2BSDE (5.5)*.*

Proof. By definition $Y_1 = g(B_1)$ and (5.5) is verified by immediate application of Itô's formula. It remain to prove the minimum condition:

$$\operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^1 k_t dt\right] = 0 \quad \text{for all} \quad t \in [0,1],\ \mathbb{P} \in \mathcal{P}_H^\kappa, \tag{5.7}$$

by which we can conclude that (Y, Z, K) is a solution of the 2BSDE (5.5). Since $g(B_1) \in \mathbb{L}_H^{2,\kappa}$, the uniqueness follows from Theorems 4.4 and 4.6 (i).

To prove (5.7), we follow the same argument as in the proof of Lemma 3.1 in [8]. For every $\varepsilon > 0$, notice that the set

$$\left\{a \in D_f :\ \hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \le \frac{1}{2} a : \Gamma_t - f(t, B_t, Y_t, Z_t, a) + \varepsilon\right\}$$

is not empty. Then it follows from a measurable selection argument that there exists a predictable process a^ε taking values in D_f such that

$$\hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \quad \le \quad \frac{1}{2} a_t^\varepsilon : \Gamma_t - f(t, B_t, Y_t, Z_t, a_t^\varepsilon) + \varepsilon.$$

We note that this in particular implies that $\Gamma_t \in D_{\hat{h}}$.

We now fix an arbitrary $\mathbb{P} := \mathbb{P}^\alpha \in \mathcal{P}_H$ and $t_0 \in [0,1]$. Let $\tau_0^\varepsilon := t_0$, and define:

$$\tau_{n+1}^\varepsilon := \inf\left\{t \ge \tau_n^\varepsilon : \hat{h}(t, B_t, Y_t, \Gamma_t) \quad \ge \quad \frac{1}{2} a_{\tau_n^\varepsilon}^\varepsilon : \Gamma_t - f(t, B_t, Y_t, Z_t, a_{\tau_n^\varepsilon}^\varepsilon) + 2\varepsilon\right\} \wedge 1,$$

for $n \ge 0$. Note that B, Y, Z, Γ are all continuous in t, and thus, for any fixed ω, are uniformly continuous in t. This also implies that $\Gamma_t(\omega)$ is bounded for $t \in [0,1]$. By the uniform continuity of $\hat{h}$ and f in (5.4), we know that, for $\mathcal{P}_H^\kappa$−q.s. $\omega \in \Omega$,

$$\hat{h}(t, B_t(\omega), Y_t(\omega), Z_t(\omega), \Gamma_t(\omega)) - \frac{1}{2} a_{\tau_n^\varepsilon}^\varepsilon(\omega) : \Gamma_t(\omega) + f(t, B_t(\omega), Y_t(\omega), Z_t(\omega), a_{\tau_n^\varepsilon}^\varepsilon(\omega))$$

is uniformly continuous in t for $t \in [\tau_n^\varepsilon(\omega), 1]$. Then $\tau_{n+1}^\varepsilon(\omega) - \tau_n^\varepsilon(\omega) \ge \delta(\varepsilon, \omega) > 0$ whenever $\tau_{n+1}^\varepsilon(\omega) < 1$, where the constant $\delta(\varepsilon, \omega)$ does not depend on n. This implies that $\tau_n^\varepsilon(\omega) = 1$ for n large enough. Applying the arguments in Example 4.5 of [14] on $[t_0, 1]$, one can easily see that there exists an $\mathbb{F}-$progressively measurable process α^ε taking values in D_f such that

$$\alpha_t^\varepsilon = \alpha_t \ \text{ for } t \in [0, t_0] \quad \text{and} \quad \hat{a}_t = \sum_{n=0}^{\infty} a_{\tau_n^\varepsilon}^\varepsilon \mathbf{1}_{[\tau_n^\varepsilon, \tau_{n+1}^\varepsilon)}(t), dt \times d\mathbb{P}^{\alpha^\varepsilon} - \text{a.s. on } [t_0, 1] \times \Omega.$$

This implies that

$$\hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) \le \frac{1}{2}\hat{a}_t : \Gamma_t - f(t, B_t, Y_t, Z_t, \hat{a}_t) + 2\varepsilon, \ dt \times d\mathbb{P}^{\alpha^\varepsilon} - \text{a.s. on } [t_0, 1] \times \Omega,$$

Under our conditions it is obvious that $\mathbb{P}^{\alpha^\varepsilon} \in \mathcal{P}_H^\kappa$, then $\mathbb{P}^{\alpha^\varepsilon} \in \mathcal{P}_H^\kappa(t_0, \mathbb{P})$, and therefore,

$$\underset{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_0, \mathbb{P})}{\text{ess inf}}{}^{\mathbb{P}} \ \mathbb{E}_{t_0}^{\mathbb{P}'}\left[\int_{t_0}^1 k_t dt\right] \ \le \ \mathbb{E}_{t_0}^{\mathbb{P}^{\alpha^\varepsilon}}\left[\int_{t_0}^1 k_t dt\right] \ \le \ 2\varepsilon(1 - t_0), \ \ \mathbb{P} - \text{a.s.}$$

By the arbitrariness of $\varepsilon > 0$, and the nonnegativity of k, this provides (5.7). □

5.3 Markovian solution of the 2BSDE

Following the classical terminology in the BSDE literature, we say that the solution of the 2BSDE is Markovian if it can be represented by means of a determinitic function of (t, B_t). In this subsection we construct a deterministic function u, by using a probabilistic representation in the spirit of (4.8), and show its connection with 2BSDE (5.5). In next subsection we establish the connection between u and the PDE (5.6).

Following [16], we introduce the shifted probability spaces. For $0 \le t \le 1$, denote by $\Omega^t := \{\omega \in C([t, 1], \mathbb{R}^d) : \omega(t) = 0\}$ the shifted canonical space; B^t the shifted canonical process on Ω^t; $\mathbb{P}_0^t$ the shifted Wiener measure; $\mathbb{F}^t$ the shifted filtration generated by B^t, $\overline{\mathcal{P}}_S^t$ the corresponding collection of martingales measures induced by strong formulation, and $\hat{a}^t$ the universal quadratic variation density of B^t. In light of Definition 2.2, we define

Definition 5.5 *For $t \in [0, 1]$, let $\mathcal{P}_h^{\kappa,t}$ denote the collection of all those $\mathbb{P} \in \overline{\mathcal{P}}_S^t$ such that*

$$\begin{aligned} &\underline{a}_{\mathbb{P}} \le \hat{a}^t \le \overline{a}_{\mathbb{P}}, \quad ds \times d\mathbb{P} - a.s. \ on \ [t, 1] \times \Omega^t, \quad for\ some \ \underline{a}_{\mathbb{P}}, \overline{a}_{\mathbb{P}} \in \mathbb{S}_d^{>0}, \\ &and \ \ \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_t^1 |f_s^{t,0})|^\kappa ds\Big)^{2/\kappa}\Big] < \infty, \quad where \ \ f_s^{t,0} := f(s, 0, 0, 0, \hat{a}_s^t). \end{aligned} \tag{5.8}$$

By (5.4), the polynomial growth of ρ, and the first part of (5.8), it is clear that

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_t^1 |f_s^{t,0}|^\kappa ds\Big)^{2/\kappa}\Big] < \infty \text{ if and only if } \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_t^1 |f(s, B_s^t, 0, 0, \hat{a}_s^t)|^\kappa ds\Big)^{2/\kappa}\Big] < \infty,$$

and thus $\mathcal{P}_h^\kappa := \mathcal{P}_h^{\kappa,0} = \mathcal{P}_H^\kappa$ as defined in Definition 2.2.

We next define a similar notation to (4.4). For any $(t, x) \in [0, 1] \times \mathbb{R}^d$, denote

$$B_s^{t,x} := x + B_s^t \quad \text{for all} \quad s \in [t, 1].$$

For any $\mathbb{F}^t$−stopping time τ, any $\mathbb{P} \in \mathcal{P}_h^{\kappa,t}$, and any $\mathcal{F}_\tau^t$−measurable r.v. η satisfying $\mathbb{E}^{\mathbb{P}}[|\eta|^2] < \infty$, we denote by $\big(\mathcal{Y}^{\mathbb{P}}, \mathcal{Z}^{\mathbb{P}}\big) := \big(\mathcal{Y}^{t,x,\mathbb{P}}(\tau,\eta), \mathcal{Z}^{t,x,\mathbb{P}}(\tau,\eta)\big)$ of the following BSDE:

$$\mathcal{Y}_s^{\mathbb{P}} = \eta - \int_t^\tau f(r, B_r^{t,x}, \mathcal{Y}_r^{\mathbb{P}}, \mathcal{Z}_r^{\mathbb{P}}, \hat{a}_r)dr - \int_s^\tau \mathcal{Z}_r^{\mathbb{P}} dB_r,\ t \le s \le \tau,\ \mathbb{P}-\text{a.s.} \tag{5.9}$$

Similar to (4.4), under our assumptions the above BSDE has a unique solution. We now introduce the value function:

$$u(t,x) := \sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \mathcal{Y}_t^{t,x,\mathbb{P}}\big(1, g(B_1^{t,x})\big), \quad \text{for} \quad (t,x) \in [0,1]\times\mathbb{R}^d. \tag{5.10}$$

By the Blumenthal zero-one law (2.5), it follows that $\mathcal{Y}_t^{t,x,\mathbb{P}}\big(1, g(B_1^{t,x})\big)$ is a constant and thus $u(t,x)$ is deterministic.

Assumption 5.6 *$\mathcal{P}_h^\kappa$ is not empty, g has polynomial growth, and there exists a continuous positive function $\Lambda(t,x)$ such that, for any (t,x):*

$$\sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \mathbb{E}^{\mathbb{P}}\Big[|g(B_1^{t,x})|^\kappa + \int_t^1 |f(s, B_s^{t,x}, 0, 0, \hat{a}_s^t)|^\kappa ds\Big] \le \Lambda^\kappa(t,x), \tag{5.11}$$

$$\sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\le s\le 1} \Lambda^2(s, B_s^{t,x})\Big] < \infty. \tag{5.12}$$

Remark 5.7 By Lemma 6.1 below, $\mathcal{P}_h^\kappa \neq \emptyset$ implies that $\mathcal{P}_h^{\kappa,t} \neq \emptyset$ for all $t \in [0,1]$. □

Remark 5.8 There are two typical sufficient conditions for the existence of such Λ:
(i) f and g are bounded. In this case one can choose Λ to be a constant.
(ii) D_f is bounded and $\sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^1 |f_s^{t,0}|^\kappa ds\Big] \le C$ for all t. In this case one can choose Λ to be a polynomial of $|x|$. □

Theorem 5.9 *Let Assumptions 5.1, 5.2, and 5.6 hold true, and $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ denote the unique solution to 2BSDE* (5.5)*. If u is uniformly continuous in x and right continuous in t, then $Y_t = u(t, B_t)$.*

Proof. The regularity of u implies that $u(t,B_t)$ is right continuous in t. Moreover, the uniform continuity of u in x leads to the uniform continuity of g. Then it follows from the proof of Theorem 4.5 of [16] that

$$Y_t = \overline{\lim_{r\downarrow t, r\in\mathbb{Q}\cap(t,1]}} u(r, B_r) = u(t, B_t).$$

□

We next provide sufficient conditions for the regularity of u.

Proposition 5.10 *Let Assumptions 5.1, 5.2, and 5.6 hold true. Then*
(i) $|u| \le \Lambda$.
(ii) *If g is lower-semicontinuous, then u is lower-semicontinuous in (t,x).*
(iii) *If g is uniformly continuous, then u is uniformly continuous in x.*
(iv) *If g is uniformly continuous and D_f is bounded, then u is right continuous in t.*

Proof. (i) is a direct consequence of Lemma 4.3. (iii) follows from Lemma 4.6 of [16]; alternatively one can follow the proof of Lemma 4.3 applied to the difference of two solutions. Finally, (ii) and (iv) are closely related to the Dynamic Programming Principle, and we postpone their proofs to subsection 6.4. □

5.4 The viscosity solution property

We shall make use of the classical notations in the theory of viscosity solutions:

$$\begin{aligned} u_*(\theta) &:= \underline{\lim}_{\theta'\to\theta} u(\theta) \quad \text{and} \quad u^*(\theta) := \overline{\lim}_{\theta'\to\theta} u(\theta'), \quad \text{for} \quad \theta = (t,x);\\ \hat h_*(\theta) &:= \underline{\lim}_{\theta'\to\theta} \hat h(\theta') \quad \text{and} \quad \hat h^*(\theta) := \overline{\lim}_{\theta'\to\theta} \hat h(\theta'), \quad \text{for} \quad \theta = (t,x,y,z,\gamma). \end{aligned} \tag{5.13}$$

Theorem 5.11 *Let Assumptions 5.1, 5.2 and 5.6 hold true. Then:*
(i) *u is a viscosity subsolution of*

$$-\partial_t u^* - \hat h^*(\cdot, u^*, Du^*, D^2u^*) \le 0 \quad on \quad [0,1)\times\mathbb{R}^d. \tag{5.14}$$

(ii) *Assume further that g is lower-semicontinuous and D_f is independent of t, then u is a viscosity supersolution of*

$$-\partial_t u_* - \hat h_*(\cdot, u_*, Du_*, D^2u_*) \ge 0 \quad on \quad [0,1)\times\mathbb{R}^d. \tag{5.15}$$

Example 5.12 Let us illustrate the role of $\hat h^*$ and $\hat h_*$ in the context of Example 5.3. In this case, one can check immediately that

$$\hat h_* = \hat h \quad \text{and} \quad \hat h^*(\gamma) = \frac{1}{2}(\gamma\vee\underline{\Gamma})\mathbf{1}_{\{\gamma<\bar\Gamma\}} + \infty\mathbf{1}_{\{\gamma\ge\bar\Gamma\}}.$$

Then the above viscosity properties are equivalent to

$$\begin{aligned} \min\big\{-\partial_t u^* - \frac{1}{2}(D^2u^*\vee\underline{\Gamma}),\ \bar\Gamma - D^2u^*\big\} \le 0,\\ \min\big\{-\partial_t u_* - \frac{1}{2}(D^2u_*\vee\underline{\Gamma}),\ \bar\Gamma - D^2u_*\big\} \ge 0, \end{aligned}$$

which is exactly the nonlinearity obtained in [3]. □

Remark 5.13 (i) If u is continuous and $D_{\hat{h}} = \mathbb{R}^{d\times d}$, then by Theorem 5.11 u is a viscosity solution to PDE (5.6) in the standard sense.
(ii) If the comparison principle for the following relaxed boundary value fully nonlinear PDE (5.14)-(5.15) with boundary condition holds:

$$\begin{aligned} \max\Big\{\big(-\partial_t v - \hat{h}_*(\cdot, v, Dv, D^2 v)\big)(T,.), v(T,.) - g\Big\} &\geq 0 \\ \min\Big\{\big(-\partial_t v - \hat{h}^*(\cdot, v, Dv, D^2 v)\big)(T,.), v(T,.) - g\Big\} &\leq 0 \end{aligned} \tag{5.16}$$

then u is continuous and is the unique viscosity solution to the above problem. We refer to Crandal, Ishii and Lions [5] for the notion of relaxed boundary problems. □

The viscosity property is a consequence of the following dynamic programming principle.

Proposition 5.14 *Let g be lower-semicontinuous, $t \in [0,1]$, and $\{\tau^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_h^{\kappa,t}\}$ be a family of $\mathbb{F}^t$−stopping times. Then, under Assumptions 5.1, 5.2, 5.6:*

$$u(t,x) = \sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \mathcal{Y}_t^{t,x,\mathbb{P}}\big(\tau^{\mathbb{P}}, u(\tau^{\mathbb{P}}, B^{t,x}_{\tau^{\mathbb{P}}})\big).$$

The proof of Proposition 5.14 is reported in subsections 6.2 and 6.4.

Proof of Theorem 5.11. (i) We argue by contradiction, and we aim for a contradiction of the dynamic programming principle. Assume to the contrary that

$$0 = (u^* - \varphi)(t_0, x_0) > (u^* - \varphi)(t,x) \quad \text{for all} \quad (t,x) \in ([0,1]\times\mathbb{R}^d) \setminus \{(t_0,x_0)\} \tag{5.17}$$

for some $(t_0, x_0) \in [0,1) \times \mathbb{R}^d$ and

$$\big(-\partial_t\varphi - \hat{h}^*(.,\varphi, D\varphi, D^2\varphi)\big)(t_0,x_0) > 0, \tag{5.18}$$

for some smooth function φ. By Proposition 5.10, without loss of generality we may assume $|\varphi| \leq \Lambda$. We note that (5.18) implies that $D^2\varphi(t_0,x_0) \in D_{\hat{h}}$. Since $\hat{h}^*$ is upper-semicontinuous and φ is smooth, there exists an open ball $O_r(t_0,x_0)$, centered at (t_0,x_0) with radius r, such that

$$-\partial_t\varphi - \hat{h}(.,\varphi, D\varphi, D^2\varphi) \geq 0, \quad \text{on} \quad O_r(t_0,x_0).$$

Then, we deduce from the definition of $\hat{h}$ that

$$-\partial_t\varphi - \frac{1}{2}\alpha : D^2\varphi + f(.,\varphi, D\varphi, \alpha) \geq 0 \text{ on } O_r(t_0,x_0) \quad \text{for all} \quad \alpha \in \mathbb{S}_d^{>0}(\mathbb{R}). \tag{5.19}$$

By the strict maximum property (5.17), we notice that

$$\eta := -\max_{\partial O_r(t_0,x_0)} (u^* - \varphi) > 0. \tag{5.20}$$

Let (t_n, x_n) be a sequence of $O_r(t_0, x_0)$ such that

$$(t_n, x_n) \longrightarrow (t_0, x_0) \quad \text{and} \quad u(t_n, x_n) \longrightarrow u^*(t_0, x_0),$$

and define the stopping time $\tau_n := \inf\{s > t_n : (s, B_s^{t_n,x_n}) \notin O_r(t_0, x_0)\}$. Without loss of generality we may assume $r < 1 - t_0$, then $\tau_n < 1$ and thus $(\tau_n, B_{\tau_n}^{t_n,x_n}) \in O_r(t_0, x_0)$. With this construction we have

$$c_n := (\varphi - u)(t_n, x_n) \to 0 \quad \text{and} \quad u^*(\tau_n, B_{\tau_n}^{t_n,x_n}) \le \varphi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \eta, \tag{5.21}$$

by the continuity of the coordinate process.

For any $\mathbb{P}^n \in \mathcal{P}_h^{\kappa,t_n}$, we now compute by the comparison result for BSDEs and classical estimates that

$$\begin{aligned}
\mathcal{Y}_{t_n}^{t_n,x_n,\mathbb{P}^n}\big(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})\big) \;&-\; u(t_n, x_n) \\
&\le \mathcal{Y}_{t_n}^{t_n,x_n,\mathbb{P}^n}\big(\tau_n, \varphi(\tau_n, B_{\tau_n}^{t_n,x_n}) - \eta\big) - \varphi(t_n, x_n) + c_n \\
&\le \mathcal{Y}_{t_n}^{t_n,x_n,\mathbb{P}^n}\big(\tau_n, \varphi(\tau_n, B_{\tau_n}^{t_n,x_n})\big) - \varphi(t_n, x_n) + c_n - \eta'
\end{aligned} \tag{5.22}$$

for some positive constant η' independent of n. Set

$$(Y^n, Z^n) := \big(\mathcal{Y}^{t_n,x_n,\mathbb{P}^n}, \mathcal{Z}^{t_n,x_n,\mathbb{P}^n}\big)\big(\tau_n, \varphi(\tau_n, B_{\tau_n}^{t_n,x_n})\big),$$

$$\delta Y_s^n := Y_s^n - \varphi(s, B_s^{t_n,x_n}), \text{ and } \delta Z_s^n := Z_s^n - D\varphi(s, B_s^{t_n,x_n}).$$

It follows from Itô's formula together with the Lipschitz properties of f that, $\mathbb{P}^n$−a.s.

$$\begin{aligned}
d\delta Y_s^n &= \Big(-\partial_t\varphi - \frac{1}{2}\hat{a}_s : D^2\varphi + f(., Y_s^n, Z_s^n, \hat{a}_s)\Big)(s, B_s^{t_n,x_n})ds + \delta Z_s^n dB_s \\
&= (\phi_s^n + \lambda_s \delta Y_s^n + \delta Z_s^n \bar{\alpha}^{1/2}\beta_s)ds + \delta Z_s^n dB_s
\end{aligned}$$

where λ and β are bounded progressively measurable processes, and

$$\phi_s^n := \Big(-\partial_t\varphi - \frac{1}{2}\hat{a}_s : D^2\varphi + f(., \varphi, D\varphi, \hat{a}_s)\Big)(s, B_s^{t_n,x_n}) \ge 0 \quad \text{for} \quad s \in [t_n, \tau_n],$$

by (5.25) and the definition of τ_n. Let M be defined by (4.12), but staring from t_n and under $\mathbb{P}^n$. Then

$$\mathcal{Y}_{t_n}^{t_n,x_n,\mathbb{P}^n}\big(\tau_n, \varphi(\tau_n, B_{\tau_n}^{t_n,x_n})\big) - \varphi(t_n, x_n) = \delta Y_{t_n}^n \le \mathbb{E}^{\mathbb{P}^n}\big[M_{\tau_n}\delta Y_{\tau_n}^n\big] = 0.$$

Plugging this in (5.22), we get

$$\mathcal{Y}_{t_n}^{t_n,x_n,\mathbb{P}^n}\big(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})\big) - u(t_n, x_n) \quad \le \quad c_n - \eta'.$$

Note that $\mathbb{P}^n \in \mathcal{P}_h^{\kappa,t_n}$ is arbitrary and c_n does not depend on $\mathbb{P}^n$. Then

$$\sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t_n}} \mathcal{Y}_{t_n}^{t_n,x_n,\mathbb{P}}\big(\tau_n, u^*(\tau_n, B_{\tau_n}^{t_n,x_n})\big) - u(t_n, x_n) \quad \le \quad c_n - \eta' < 0,$$

for large n. This is in contradiction with the dynamic programming principle of Proposition 5.14 (or, more precisely, Lemma 6.2 below to avoid the condition that g is lower-semicontinuous) .

(ii) We again argue by contradiction, aiming for a contradiction of the dynamic programming principle of Proposition 5.14. Assume to the contrary that

$$0 = (u_* - \varphi)(t_0, x_0) < (u_* - \varphi)(t, x) \quad \text{for all} \quad (t, x) \in ([0, 1] \times \mathbb{R}^d) \setminus \{(t_0, x_0)\} \tag{5.23}$$

for some $(t_0, x_0) \in [0, 1) \times \mathbb{R}^d$ and

$$\big(- \partial_t \varphi - \hat{h}_*(., \varphi, D\varphi, D^2\varphi)\big)(t_0, x_0) < 0,$$

for some smooth function φ. By Proposition 5.10, without loss of generality we may assume again that $|\varphi| \leq \Lambda$. Note that $\hat{h}_* \leq \hat{h}$. Then

$$\big(- \partial_t \varphi - \hat{h}(., \varphi, D\varphi, D^2\varphi)\big)(t_0, x_0) < 0.$$

If $D^2\varphi(t_0, x_0) \in D_{\hat{h}}$, then it follows from the definition of $\hat{h}$ that

$$\Big(- \partial_t \varphi - \frac{1}{2} \bar{\alpha} : D^2\varphi + f(., \varphi, D\varphi, \bar{\alpha})\Big)(t_0, x_0) \leq 0 \tag{5.24}$$

for some $\bar{\alpha} \in \mathbb{S}_d^{>0}$. In particular, this implies that $\bar{\alpha} \in D_f$. If $D^2\varphi(t_0, x_0) \notin D_{\hat{h}}$, since $\partial_t \varphi(t_0, x_0)$ is finite, we still have $\bar{\alpha} \in D_f$ so that (5.24) holds. Now by the smoothness of φ and (5.4), and recalling that D_f is independent of t, there exists an open ball $O_r(t_0, x_0)$ with $0 < r < 1 - t_0$ such that

$$-\partial_t \varphi - \frac{1}{2} \bar{\alpha} : D^2\varphi + f(., \varphi, D\varphi, \bar{\alpha}) \leq 0 \text{ on } O_r(t_0, x_0). \tag{5.25}$$

By the strict minimum property (5.23), we notice that

$$\eta \;:=\; \min_{\partial B_r(t_0, x_0)} (u_* - \varphi) \;>\; 0. \tag{5.26}$$

As in (i), we consider a sequence (t_n, x_n) of $O_r(t_0, x_0)$ such that

$$(t_n, x_n) \longrightarrow (t_0, x_0) \quad \text{and} \quad u(t_n, x_n) \longrightarrow u_*(t_0, x_0),$$

and we define the stopping time $\tau_n := \inf\{s > t_n : (s, B_s^{t_n, x_n}) \notin O_r(t_0, x_0)\}$, so that

$$c_n := (u - \varphi)(t_n, x_n) \to 0 \quad \text{and} \quad u_*(\tau_n, B_{\tau_n}^{t_n, x_n}) \geq \varphi(\tau_n, B_{\tau_n}^{t_n, x_n}) + \eta. \tag{5.27}$$

For each n, let $\bar{\mathbb{P}}^n := \mathbb{P}^{\bar{\alpha}} \in \overline{\mathcal{P}}_S^{t_n}$ be the local martingale measure induced by the constant diffusion $\bar{\alpha}$. By (5.4), one can easily see that $\bar{\mathbb{P}}^n \in \mathcal{P}_H^{\kappa, t_n}$. We then follow exactly the same line of argument as in (i) to see that

$$u(t_n, x_n) - \mathcal{Y}_{t_n}^{t_n, x_n, \bar{\mathbb{P}}^n}\big(\tau_n, u_*(\tau_n, B_{\tau_n}^{t_n, x_n})\big) \leq c_n - \eta', \quad \bar{\mathbb{P}} - \text{a.s.}$$

where η' is a positive constant independent of n. For large n, we have $c_n - \eta' < 0$, and this is in contradiction with the dynamic programming principle. □

6 The dynamic programming principle

In this section we prove Proposition 5.14, and the remaining regularity properties (ii) and (iv) of Proposition 5.10.

6.1 Regular conditional probability distributions

The key tool to prove the dynamic programming principle is the regular conditional probability distributions (r.c.p.d.), introduced by Stroock-Varadhan [17]. We adopt the notations in our paper [16]. For $0 \le t \le s \le 1$, $\omega \in \Omega^t$, $\tilde\omega \in \Omega^s$, and $\mathcal{F}^t_1-$measurable random variable ξ, define:

$$\xi^{s,\omega}(\tilde\omega) := \xi(\omega \otimes_s \tilde\omega) \text{ where } (\omega \otimes_s \tilde\omega)(r) := \omega_r \mathbf{1}_{[t,s)}(r) + (\omega_s + \tilde\omega_r)\mathbf{1}_{[s,1]}(r),\ r \in [t,1]. \quad (6.1)$$

In particular, for any $\mathbb{F}^t-$stopping time τ, one can choose $s = \tau(\omega)$ and simplify the notation: $\omega \otimes_\tau \tilde\omega := \omega \otimes_{\tau(\omega)} \tilde\omega$. Clearly $\omega \otimes_\tau \tilde\omega \in \Omega^t$ and, for each $\omega \in \Omega^t$, $\xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}$ is $\mathcal{F}^{\tau(\omega)}_1-$measurable. For each probability measure $\mathbb{P}$ on $(\Omega^t, \mathcal{F}^t_1)$, by Stroock-Varadhan [17] there exist r.c.p.d. $\mathbb{P}^{\tau,\omega}$ for all $\omega \in \Omega^t$ such that $\mathbb{P}^{\tau,\omega}$ is a probability measure on $(\Omega^{\tau(\omega)}, \mathcal{F}^{\tau(\omega)}_1)$, and for all $\mathcal{F}^t_1-$measurable $\mathbb{P}-$integrable random variable ξ:

$$\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}^t_\tau](\omega) = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}], \quad \text{for} \quad \mathbb{P} - \text{a.e. } \omega \in \Omega^t. \quad (6.2)$$

In particular, this implies that the mapping $\omega \mapsto \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}]$ is $\mathcal{F}^t_\tau-$measurable. Moreover, following the arguments in Lemmas 4.1 and 4.3 of [16], one can easily show that:

Lemma 6.1 *Let $t \in [0,1]$, τ an $\mathbb{F}^t-$stopping time, and $\mathbb{P} \in \mathcal{P}^{\kappa,t}_h$. Then:*

$$\text{for } \mathbb{P}-a.e.\ \omega \in \Omega^t: \quad \mathbb{P}^{\tau,\omega} \in \mathcal{P}^{\kappa,\tau(\omega)}_h \text{ and } (\hat a^t)^{\tau,\omega}_r = \hat a^{\tau(\omega)}_r, \quad dr \times d\mathbb{P}^{\tau,\omega} \text{ on } [\tau(\omega),1] \times \Omega^{\tau(\omega)}.$$

6.2 A weak partial dynamic programming principle

In this section, we prove the following result adapted from [2].

Lemma 6.2 *Under Assumptions 5.1, 5.2 and 5.6, for any (t,x) and arbitrary $\mathbb{F}^t-$stopping times $\{\tau^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}^{\kappa,t}_h\}$:*

$$u(t,x) \quad \le \quad \sup_{\mathbb{P} \in \mathcal{P}^{\kappa,t}_h} \mathcal{Y}^{t,x,\mathbb{P}}_t\big(\tau^{\mathbb{P}}, u^*(\tau^{\mathbb{P}}, B^{t,x}_{\tau^{\mathbb{P}}})\big).$$

Proof. We shall prove the slightly stronger result:

$$\begin{gathered} \mathcal{Y}^{t,x,\mathbb{P}}_t\big(1, g(B^{t,x}_1)\big) \le \mathcal{Y}^{t,x,\mathbb{P}}_t\big(\tau^{\mathbb{P}}, \varphi(\tau^{\mathbb{P}}, B^{t,x}_{\tau^{\mathbb{P}}})\big) \\ \text{for any } \mathbb{P} \in \mathcal{P}^{\kappa,t}_h \text{ and any Lebesgue measurable function } \varphi \ge u. \end{gathered} \quad (6.3)$$

Fix $\mathbb{P}$ and φ. For notation simplicity, we omit the dependence of $\tau^{\mathbb{P}}$ on $\mathbb{P}$. We first note that, by Proposition 5.10 (i), without loss of generality we may assume $|\varphi| \le \Lambda$. Then Assumption 5.6 implies that $\mathcal{Y}_t^{t,x,\mathbb{P}}\big(\tau, \varphi(\tau, B_\tau^{t,x})\big)$ is well defined. By (6.2), one can easily show that

$$\mathcal{Y}_t^{t,x,\mathbb{P}}\big(1, g(B_1^{t,x})\big) = \mathcal{Y}_t^{t,x,\mathbb{P}}\Big(\tau, \mathcal{Y}_\tau^{\tau(\omega),B_\tau^{t,x}(\omega),\mathbb{P}^{\tau,\omega}}\big(1, g(B_1^{\tau(\omega),B_\tau^{t,x}(\omega)})\big)\Big)$$

By Lemma 6.1, $\mathbb{P}^{\tau,\omega} \in \mathcal{P}_h^{\kappa,\tau(\omega)}$, $\mathbb{P}-$a.e. $\omega \in \Omega^t$. Then

$$\mathcal{Y}_\tau^{\tau(\omega),B_\tau^{t,x}(\omega),\mathbb{P}^{\tau,\omega}}\big(1, g(B_1^{\tau(\omega),B_\tau^{t,x}(\omega)})\big) \le u\big(\tau(\omega), B_\tau^{t,x}(\omega)\big) \le \varphi\big(\tau(\omega), B_\tau^{t,x}(\omega)\big), \ \mathbb{P} - \text{a.e. } \omega \in \Omega^t.$$

It follows from the comparison result for BSDEs that

$$\mathcal{Y}_t^{t,x,\mathbb{P}}\big(1, g(B_1^{t,x})\big) \quad \le \quad \mathcal{Y}_t^{t,x,\mathbb{P}}\Big(\tau, \varphi\big(\tau, B_\tau^{t,x}\big)\Big).$$

This implies (6.3), and by the arbitrariness of $\mathbb{P}$, Lemma 6.2 is proved. □

6.3 Concatenation of probability measures

In preparation to the proof of Proposition 5.14, we introduce the concatenation of probability measures. For any $0 \le t_0 \le t \le 1$ and $\omega \in \Omega^{t_0}$, denote $\omega^t \in \Omega^t$ by $\omega_s^t := \omega_s - \omega_t$, $s \in [t,1]$. For any $\mathbb{P}_1 = \mathbb{P}^{\alpha^1} \in \mathcal{P}_h^{\kappa,t_0}$, $\mathbb{P}_2 = \mathbb{P}^{\alpha^2} \in \mathcal{P}_h^{\kappa,t}$, let $\mathbb{P} := \mathbb{P}_1 \otimes_t \mathbb{P}_2$ denote the probability measure $\mathbb{P}^\alpha$, where

$$\alpha_s(\omega) := \alpha_s^1(\omega)\mathbf{1}_{[t_0,t]}(s) + \alpha_s^2(\omega^t)\mathbf{1}_{[t,1]}(s), \quad \omega \in \Omega^{t_0}.$$

Lemma 6.3 *Let $\mathbb{P} := \mathbb{P}_1 \otimes_t \mathbb{P}_2$ be defined above. Then, under Assumption 5.1,*

$$\mathbb{P} \in \mathcal{P}_h^{\kappa,t_0}, \quad \mathbb{P} = \mathbb{P}_1 \ \text{ on } \ \mathcal{F}_t^{t_0}, \quad \text{and} \quad \mathbb{P}^{t,\omega} = \mathbb{P}_2 \ \text{ for } \ \mathbb{P}_1 - a.e. \ \omega \in \Omega^{t_0}. \tag{6.4}$$

Proof. First by (5.8), we have $\underline{a}_{\mathbb{P}_i} \le \alpha^i \le \overline{a}_{\mathbb{P}_i}$, $i = 1, 2$. Then $\underline{a}_{\mathbb{P}_1} \wedge \underline{a}_{\mathbb{P}_2} \le \alpha \le \overline{a}_{\mathbb{P}_1} \vee \overline{a}_{\mathbb{P}_1}$. In particular, this implies that $\int_{t_0}^1 |\alpha_s| ds < \infty$. Then $\mathbb{P} \in \overline{\mathcal{P}}_S^{t_0}$ and $\underline{a}_{\mathbb{P}_1} \wedge \underline{a}_{\mathbb{P}_2} \le \hat{a} \le \overline{a}_{\mathbb{P}_1} \vee \overline{a}_{\mathbb{P}_2}$, $\mathbb{P}-$a.s. The two last claims in (6.4) are obvious, and imply that:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[\big(\int_{t_0}^1 |f_s^{t_0,0}|^\kappa ds\big)^{2/\kappa}\Big] &\le C_\kappa \mathbb{E}^{\mathbb{P}}\Big[\big(\int_{t_0}^t |f_s^{t_0,0}|^\kappa ds\big)^{2/\kappa} + \big(\int_t^1 |f_s^{t_0,0}|^\kappa ds\big)^{2/\kappa}\Big] \\
&= C_\kappa \Big(\mathbb{E}^{\mathbb{P}_1}\Big[\big(\int_{t_0}^t |f_s^{t_0,0}|^\kappa ds\big)^{2/\kappa}\Big] + \mathbb{E}^{\mathbb{P}_1}\Big[\mathbb{E}^{\mathbb{P}_2}\big[\big(\int_t^1 |f_s^{t,0}|^\kappa ds\big)^{2/\kappa}\big]\Big]\Big) \\
&= C_\kappa \Big(\mathbb{E}^{\mathbb{P}_1}\Big[\big(\int_{t_0}^t |f_s^{t_0,0}|^\kappa ds\big)^{2/\kappa}\Big] + \mathbb{E}^{\mathbb{P}_2}\Big[\big(\int_t^1 |f_s^{t,0}|^\kappa ds\big)^{2/\kappa}\Big]\Big) < \infty.
\end{aligned}$$

This implies that $\mathbb{P} \in \mathcal{P}_h^{\kappa,t_0}$. □

6.4 Dynamic programming and regularity

We first prove the dynamic programming principle of Proposition 5.14 for stopping times taking countably many values. From this, we will deduce the lower-semicontinuity of u stated in Proposition 5.10 (ii), which in turn provides Proposition 5.14 by passing to limits. The right-continuity property of Proposition 5.10 (iv) is also proved using the dynamic programming principle.

Lemma 6.4 *Proposition 5.14 holds true under the additional condition that each $\tau^{\mathbb{P}}$ takes countable many values.*

Proof. (i) We first observe that the lower semicontinuity of g implies that

$$x \longmapsto \mathcal{Y}_t^{t,x,\mathbb{P}}(1, g(B_1^{t,x})) \quad \text{is lower-semicontinuous for all} \quad \mathbb{P} \in \mathcal{P}_h^{\kappa,t}. \tag{6.5}$$

This is a direct consequence of the stability and comparison principle of BSDEs. Then, for all fixed (t,x), and all sequence $(x_n)_{n\geq 1}$ converging to x, it follows that:

$$u(t,x) = \sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \mathcal{Y}_t^{t,x,\mathbb{P}}\big(1, g(X_1^{t,x})\big) \leq \sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t}} \varliminf_{n\to\infty} \mathcal{Y}_t^{t,x_n,\mathbb{P}}\big(1, g(X_1^{t,x_n})\big) \leq \varliminf_{n\to\infty} u(t,x_n).$$

Hence $u(t,.)$ is lower-semicontinuous, and therefore measurable.
(ii) We now fix (t_0,x_0) and prove the result at this point. Let τ be an $\mathbb{F}^{t_0}-$stopping time with values in $\{t_k, k\geq 1\} \subset [t_0,1]$. Since $u(t_k,.)$ is measurable, we deduce that $u(\tau, B_\tau^{t_0,x_0}) = \sum_{k\geq 1} u(t_k, B_{t_k}^{t_0,x_0})\mathbf{1}_{\{\tau=t_k\}}$ is $\mathcal{F}_\tau-$measurable. Then, it follows from (6.3) that

$$u(t_0,x_0) \quad \leq \quad \sup_{\mathbb{P}\in\mathcal{P}_h^{\kappa,t_0}} \mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\big(\tau^{\mathbb{P}}, u(\tau^{\mathbb{P}}, B_{\tau^{\mathbb{P}}}^{t_0,x_0})\big).$$

(iii) To complete the proof, we fix $\mathbb{P} \in \mathcal{P}_h^{\kappa,t_0}$, denote $\tau := \tau^{\mathbb{P}}$, and proceed in four steps to show that

$$\mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big) \quad \leq \quad u(t_0,x_0). \tag{6.6}$$

Step 1. We first fix $t \in (t_0,1]$, and show that,

$$\mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}(t, \varphi(B_t^{t_0,x_0})) \leq u(t_0,x_0), \tag{6.7}$$

for any continuous function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $-\Lambda(t,\cdot) \leq \varphi(\cdot) \leq u(t,\cdot)$. Indeed, for any $\mathbb{P}^t \in \mathcal{P}_h^{\kappa,t}$, by the lower-semicontinuity property (6.5), we may argue exactly as in Step 2 of the proof of Theorem 3.1 in [2] to deduce that, for every $\varepsilon > 0$, there exist sequences $(x_i,r_i)_{i\geq 1} \subset \mathbb{R}^d \times (0,1]$ and $\mathbb{P}_i \in \mathcal{P}_h^{\kappa,t}$, $i \geq 1$ such that

$$\mathcal{Y}_t^{t,\cdot,\mathbb{P}_i}(1, g(B_1^{t,\cdot})) \geq \varphi(t,\cdot) - \varepsilon \text{ on } Q_i := \{x' \in \mathbb{R}^d : |x'-x_i| < r_i\}, \quad \text{and} \quad \cup_{i\geq 1} Q_i = \mathbb{R}^d.$$

This provides a disjoint partition $(A_i)_{i\geq 1}$ of $\mathbb{R}^d$ defined by $A_i := Q_i \setminus \cup_{j<i} Q_j$. Set

$$E_i := \{B_t^{t_0,x_0} \in A_i\}, \quad i \geq 1, \quad \text{and} \quad \bar{E}_n := \cup_{i>n} E_i, \quad n \geq 1.$$

Then $\{E_i, 1 \leq i \leq n\}$ and $\bar{E}^n$ form a partition of Ω and $\lim_{n\to\infty} \mathbb{P}(\bar{E}_n) = 0$. Define

$$\bar{\mathbb{P}}^n(E) := \sum_{i=1}^n (\mathbb{P} \otimes_t \mathbb{P}_i)(E \cap E_i) + \mathbb{P}(E \cap \bar{E}_n) \;\text{ for all }\; E \in \mathcal{F}_1^{t_0}. \tag{6.8}$$

Combining the arguments for (4.18) and Lemma 6.3, one can easily show that

$$\bar{\mathbb{P}}^n \in \mathcal{P}_h^{\kappa,t_0}(t,\mathbb{P}) \quad \text{and} \quad (\bar{\mathbb{P}}^n)^{t,\omega} = \mathbb{P}_i, \quad \mathbb{P}-\text{a.e. } \omega \in E_i, \quad 1 \leq i \leq n. \tag{6.9}$$

This implies that, for $1 \leq i \leq n$ and $\mathbb{P}-$a.e. $\omega \in E_i$,

$$\mathcal{Y}_t^{t_0,x_0,\bar{\mathbb{P}}^n}(1, g(B_1^{t_0,x_0}))(\omega) = \mathcal{Y}_t^{t,B_t^{t_0,x_0}(\omega),\mathbb{P}_i}(1, g(B_1^{t,B^{t_0,x_0}(\omega)_t})) \geq \varphi\big(B_t^{t_0,x_0}(\omega)\big) - \varepsilon,$$

and, by the comparison result for BSDEs:

$$\begin{aligned}
u(t_0,x_0) &\geq \mathcal{Y}_{t_0}^{t_0,x_0,\bar{\mathbb{P}}^n}(1, g(B_1^{t_0,x_0})) = \mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\Big(t, \mathcal{Y}_t^{t_0,x_0,\bar{\mathbb{P}}^n}(1, g(B_1^{t_0,x_0}))\Big) \\
&\geq \mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\Big(t, \big(\varphi(B_t^{t_0,x_0}) - \varepsilon\big)\mathbf{1}_{(\bar{E}_n)^c} + \mathcal{Y}_t^{t_0,x_0,\bar{\mathbb{P}}^n}(1, g(B_1^{t_0,x_0}))\mathbf{1}_{\bar{E}_n}\Big).
\end{aligned}$$

By the stability of BSDEs and the arbitrariness of $\varepsilon > 0$, this proves (6.7).
Step 2. Since $u(t,\cdot)$ is lower semi-continuous, there exist continuous functions $\{\varphi_n, n \geq 1\}$ such that $\varphi_n \uparrow u(t,\cdot)$. Without loss of generality we may assume $\varphi_n \geq -\Lambda$. Since (6.7) holds for each φ_n, we obtain (6.6) for $\tau = t$ by monotone convergence.
Step 3. Assume τ takes finitely many values $t_0 < t_1 < \cdots < t_n \leq 1$. Note that, $\mathbb{P}-$a.s.

$$\begin{aligned}
&\mathcal{Y}_{\tau\wedge t_{n-1}}^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big) \\
= \;& \mathcal{Y}_\tau^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big)\mathbf{1}_{\{\tau\leq t_{n-1}\}} + \mathcal{Y}_{t_{n-1}}^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big)\mathbf{1}_{\{\tau> t_{n-1}\}} \\
= \;& u(\tau, B_\tau^{t_0,x_0})\mathbf{1}_{\{\tau\leq t_{n-1}\}} + \mathcal{Y}_{t_{n-1}}^{t_{n-1},B_{t_{n-1}}^{t_0,x_0}(\omega),(\mathbb{P})^{t_{n-1},\omega}}\big(t_n, u(t_n, B_{t_n}^{t_{n-1},B_{t_{n-1}}^{t_0,x_0}(\omega)})\big)\mathbf{1}_{\{\tau> t_{n-1}\}}
\end{aligned}$$

By Lemma 6.1, $(\mathbb{P})^{t_{n-1},\omega} \in \mathcal{P}_h^{\kappa,t_{n-1}}$, $\mathbb{P}-$a.s. Then by Step 2 we have

$$\begin{aligned}
\mathcal{Y}_{\tau\wedge t_{n-1}}^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big) &\leq u(\tau, B_\tau^{t_0,x_0})\mathbf{1}_{\{\tau\leq t_{n-1}\}} + u(t_{n-1}, B_{t_{n-1}}^{t_0,x_0})\mathbf{1}_{\{\tau> t_{n-1}\}} \\
&= u(\tau\wedge t_{n-1}, B_{\tau\wedge t_{n-1}}^{t_0,x_0}).
\end{aligned}$$

Then, by the comparison principle of BSDE,

$$\begin{aligned}
\mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big) &= \mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\Big(\tau\wedge t_{n-1}, \mathcal{Y}_{\tau\wedge t_{n-1}}^{t_0,x_0,\mathbb{P}}\big(\tau, u(\tau, B_\tau^{t_0,x_0})\big)\Big) \\
&\leq \mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\Big(\tau\wedge t_{n-1}, u(\tau\wedge t_{n-1}, B_{\tau\wedge t_{n-1}}^{t_0,x_0})\Big).
\end{aligned}$$

Continuing this backward induction provides (6.6).

Step 4. Now assume τ takes countable many values $\{t_k, k \geq 1\}$. Denote $\tau_n := \sum_{k=1}^n t_k \mathbf{1}_{\{\tau = t_k\}} + \mathbf{1}_{\{\tau \neq t_k, 1 \leq k \leq n\}}$. Clearly τ_n is still an $\mathbb{F}^{t_0}-$stopping time. By Step 3,

$$\mathcal{Y}_{t_0}^{t_0,x_0,\mathbb{P}}\big(\tau_n, u(\tau_n, B_{\tau_n}^{t_0,x_0})\big) \quad \leq \quad u(t_0, x_0).$$

For each $\omega \in \Omega^{t_0}$, we have $\tau_n(\omega) = \tau(\omega)$, for sufficiently large n. Then $u(\tau_n(\omega), B_{\tau_n}^{t_0,x_0}(\omega)) = u(\tau(\omega), B_\tau^{t_0,x_0}(\omega))$, and (6.6) follows from the stability of BSDEs. $\square$

As a consequence of Lemma 6.4, we can now prove that u is lower-semicontinuous.

Proof of Proposition 5.10 (ii) Recall the $\mathcal{Y}^{\mathbb{P}}(\tau, \xi)$ defined in (4.4), and define

$$J(t, x, \mathbb{P}) := \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_t^{\mathbb{P}}(1, g(x + B_1 - B_t))\Big] \quad \text{for all} \quad t, x, \text{ and } \mathbb{P} \in \mathcal{P}_h^\kappa. \tag{6.10}$$

(i) We first prove that

$$u(t, x) = \sup_{\mathbb{P} \in \mathcal{P}_h^\kappa} J(t, x, \mathbb{P}). \tag{6.11}$$

To see this, we first observe that, for any $\mathbb{P} \in \mathcal{P}_h^\kappa$, it follows from Lemma 6.1 that

$$\mathcal{Y}_t^{\mathbb{P}}(1, g(x + B_1 - B_t))(\omega) = \mathcal{Y}_t^{t,x,\mathbb{P}^{t,\omega}}(1, g(B_1^{t,x})) \leq u(t, x) \quad \text{for} \quad \mathbb{P}-\text{a.e. } \omega \in \Omega.$$

Then $J(t, x, \mathbb{P}) \leq u(t, x)$ for any $\mathbb{P} \in \mathcal{P}_h^\kappa$.

On the other hand, for any $\mathbb{P}_2 \in \mathcal{P}_h^{\kappa,t}$, choose arbitrary $\mathbb{P}_1 \in \mathcal{P}_h^\kappa$ and let $\mathbb{P} := \mathbb{P}_1 \otimes_t \mathbb{P}_2$. Then $\mathbb{P} \in \mathcal{P}_h^\kappa$ and, by (6.4),

$$\mathcal{Y}_t^{\mathbb{P}}(1, g(x + B_1 - B_t))(\omega) = \mathcal{Y}_t^{t,x,\mathbb{P}^{t,\omega}}(1, g(B_1^{t,x})) = \mathcal{Y}_t^{t,x,\mathbb{P}_2}(1, g(B_1^{t,x})) \quad \text{for} \quad \mathbb{P}-\text{a.e. } \omega \in \Omega.$$

This implies that $J(t, x, \mathbb{P}) = \mathcal{Y}_t^{t,x,\mathbb{P}_2}(1, g(B_1^{t,x}))$ and thus $u(t, x) \leq \sup_{\mathbb{P} \in \mathcal{P}_h^\kappa} J(t, x, \mathbb{P})$.

(ii) We now prove that the lower-semicontinuity of g implies that:

$$(t, x) \longmapsto J(t, x, \mathbb{P}) \quad \text{is lower-semicontinuous for any} \quad \mathbb{P} \in \mathcal{P}_h^\kappa. \tag{6.12}$$

which obviously implies the lower-semicontinuity of u in view of (6.10).

For $(t, x) \in [0, 1] \times \mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}_h^\kappa$, let $(t_n, x_n)_{n \geq 1}$ be a sequence in $[0, 1] \times \mathbb{R}^d$ such that $(t_n, x_n) \longrightarrow (t, x)$. Denote, for each n,

$$\xi_n := \inf_{k \geq n} g(x_k + B_1 - B_{t_k}), \qquad f_s^n(y, z) := \inf_{k \geq n} f(s, x_k + B_s - B_{t_k}, y, z, \hat{a}_s),$$
$$\xi_\infty := \lim_{n \to \infty} \xi_n, \qquad f^\infty := \lim_{n \to \infty} f^n,$$

and, for $1 \leq n \leq \infty$, let $(\mathcal{Y}^n, \mathcal{Z}^n)$ denote the solution to the following BSDE:

$$\mathcal{Y}_s^n = \xi_n - \int_s^1 f_r^n(\mathcal{Y}_r^n, \mathcal{Z}_r^n) dr - \int_s^1 \mathcal{Z}_r^n dB_r, \quad t \leq s \leq 1, \quad \mathbb{P}-\text{a.s.}$$

By Assumptions 5.1 and 5.6, g and the modulus of continuity ρ of f have polynomial growth in x. Then there exist some constants C and p such that

$$\sup_{n\geq 1}\Big\{|\xi_n|+|f^n_t(0,0)|\Big\} \leq |f^{0,0}_r| + C\Big(\sup_{k\geq 1}|x_k|^p + \sup_{0\leq t\leq 1}|B_t|^p\Big). \tag{6.13}$$

Moreover, $\hat{a}$ has upper bound $\overline{a}_{\mathbb{P}}$, $\mathbb{P}-$a.s. then it follows from the Lipschitz conditions of f that the above BSDE has a unique solution for each n, and

$$\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}}[\mathcal{Y}^n_t] = \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^\infty_t].$$

By the lower semi-continuity of g and the uniform continuity of f in x in (5.4), we have $\xi_\infty \geq g(x+B_1-B_t)$ and $f^\infty_s(y,z) = f(s,x+B_1-B_s,y,z,\hat{a}_s)$, $\mathbb{P}-$a.s. Then by the comparison principle of BSDEs one can easily see that

$$\underline{\lim}_{n\to\infty} J(t_n,x_n,\mathbb{P}) \geq \lim_{n\to\infty}\mathbb{E}^{\mathbb{P}}[\mathcal{Y}^n_t] = \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^\infty_t] \geq \mathbb{E}^{\mathbb{P}}[\mathcal{Y}^{\mathbb{P}}_t\big(1,g(x+B_1-B_t)\big)\Big] = J(t,x,\mathbb{P}).$$

This proves the lower-semicontinuity of J for any fixed $\mathbb{P}\in\mathcal{P}^\kappa_h$. □

We now can prove the dynamic programming principle for arbitrary stopping times.

Proof of Proposition 5.14 For any (t,x), $\mathbb{P}\in\mathcal{P}^{\kappa,t}_h$, $\mathbb{F}^t-$stopping time τ, and any n, denote

$$\tau_n := \sum_{i=1}^n \frac{i}{n}\mathbf{1}_{[i-\frac{1}{n},\frac{i}{n})}(\tau) + \mathbf{1}_{\{\tau=1\}}.$$

Then τ_n is an $\mathbb{F}^t-$stopping time, $\tau_n \geq \tau$, and $\tau_n \to \tau$. By Lemma 6.4, together with Proposition 5.10 (ii), we have

$$\mathcal{Y}^{t,x,\mathbb{P}}_t\big(\tau_n, u(\tau_n, B^{t,x}_{\tau_n})\big) \quad \leq \quad u(t,x)$$

Since u is lower-semicontinuous, $\underline{\lim}_{n\to\infty} u(\tau_n,B^{t,x}_{\tau_n}) \geq u(\tau,B^{t,x}_\tau)$. Then it follows from the comparison and the stability of BSDEs that

$$u(t,x) \quad \geq \quad \mathcal{Y}^{t,x,\mathbb{P}}_t\left(\tau, u(\tau,B^{t,x}_\tau)\right).$$

Finally, by the lower-semicontinuity of u, Lemma 6.2 provides the opposite inequality. □

We conclude this section by the

Proof of Proposition 5.10 (iv) Let $t_n\downarrow t$ and $x\in\mathbb{R}^d$, by Proposition 5.14 we have

$$u(t,x) \quad = \quad \sup_{\mathbb{P}\in\mathcal{P}^{\kappa,t}_h}\mathcal{Y}^{t,x,\mathbb{P}}_t\big(t_n, u(t_n, B^{t,x}_{t_n})\big).$$

For each $\mathbb{P} \in \mathcal{P}_h^{\kappa,t}$ and $n \geq 1$, denote

$$\mathcal{Y}^{\mathbb{P},n} := \mathcal{Y}^{t,x,\mathbb{P}}(t_n, u(t_n, X_{t_n}^{t,x})),\ \mathcal{Z}^{\mathbb{P},n} := \mathcal{Z}^{t,x,\mathbb{P}}(t_n, u(t_n, X_{t_n}^{t,x})),\ \text{and } \delta\mathcal{Y}^{\mathbb{P},n} := \mathcal{Y}^{\mathbb{P},n} - u(t_n, x).$$

Then, $\mathbb{P}-$a.s. on $[t, t_n]$,

$$d(\delta\mathcal{Y}_s^{\mathbb{P},n}) \;=\; f\big(s, B_s^{t,x}, \delta\mathcal{Y}_s^{\mathbb{P},n} + u(t_n, x), \mathcal{Z}_s^{\mathbb{P},n}, \hat{a}_s^t\big)ds + \mathcal{Z}_s^{\mathbb{P},n} dB_s^t.$$

Set $\kappa' := \frac{1+\kappa}{2} \in (1, \kappa)$, and apply Lemma 4.3 with κ':

$$|\delta\mathcal{Y}_t^{\mathbb{P},n}|^{\kappa'} \;\leq\; C_\kappa \mathbb{E}^{\mathbb{P}}\Big[|u(t_n, B_{t_n}^{t,x}) - u(t_n, x)|^{\kappa'} + \int_t^{t_n} \big|f\big(s, B_s^{t,x}, u(t_n, x), 0, \hat{a}_s^t\big)\big|^{\kappa'} ds\Big].$$

Then, noting that f is uniformly Lipschitz in y and uniformly continuous in x with modulus of continuity ρ,

$$\begin{aligned}
|\delta\mathcal{Y}_t^{\mathbb{P},n}|^{\kappa} \;&\leq\; C_\kappa \mathbb{E}^{\mathbb{P}}\Big[|u(t_n, B_{t_n}^{t,x}) - u(t_n, x)|^{\kappa} + \Big(\int_t^{t_n} \big|f\big(s, B_s^{t,x}, u(t_n, x), 0, \hat{a}_s^t\big)\big|^{\kappa'} ds\Big)^{\kappa/\kappa'}\Big] \\
&\leq\; C_\kappa \mathbb{E}^{\mathbb{P}}\Big[|u(t_n, B_{t_n}^{t,x}) - u(t_n, x)|^{\kappa} \\
&\qquad + (t_n - t)^{\kappa/\kappa' - 1} \int_t^{t_n} \big(|f_s^{t,0}|^{\kappa} + |\rho(B_s^{t,x})|^{\kappa} + |u(t_n, x)|^{\kappa}\big) ds\Big] \\
&\leq\; C_\kappa \mathbb{E}^{\mathbb{P}}\Big[|u(t_n, B_{t_n}^{t,x}) - u(t_n, x)|^{\kappa} + (t_n - t)^{\kappa/\kappa' - 1} \int_t^{t_n} |\rho(B_s^{t,x})|^{\kappa} ds\Big] \\
&\qquad + (t_n - t)^{\kappa/\kappa' - 1} \Lambda^{\kappa}(t, x) + (t_n - t)^{\kappa/\kappa'} \Lambda^{\kappa}(t_n, x)\Big].
\end{aligned}$$

Since Λ is continuous, we get $\sup_{n \geq 1} \Lambda(t_n, x) < \infty$. Now by the uniform continuity of u in x, together with the boundedness of D_f, we have

$$\lim_{n \to \infty} \sup_{\mathbb{P} \in \mathcal{P}_h^{\kappa,t}} \mathbb{E}^{\mathbb{P}}\Big[|u(t_n, B_{t_n}^{t,x}) - u(t_n, x)|^{\kappa} + (t_n - t)^{\kappa/\kappa' - 1} \int_t^{t_n} |\rho(B_s^{t,x})|^{\kappa} ds\Big] = 0.$$

Then

$$|u(t, x) - u(t_n, x)| = \Big|\sup_{\mathbb{P} \in \mathcal{P}_h^{\kappa,t}} \delta\mathcal{Y}_t^{\mathbb{P},n}\Big| \leq \sup_{\mathbb{P} \in \mathcal{P}_h^{\kappa,t}} |\delta\mathcal{Y}_t^{\mathbb{P},n}| \longrightarrow 0, \quad \text{as } n \to \infty.$$

The proof is complete. □

7 Appendix

7.1 Non-uniqueness in $\mathbb{L}^2(\mathbb{P}_0)$ of the 2BSDE (3.9)

In this section, we provide an example which shows the importance of the constraints imposed in [4] to obtain uniqueness.

Example 7.1 *Consider the following 2 dimensional forward SDEs:*

$$\begin{cases} Y_t = -\int_0^t \frac{3Y_s}{1-s}ds + \int_0^t \frac{X_s}{\sqrt{1-s}}dB_s, \\ X_t = 1 - \int_0^t \frac{3(1+c^2)X_s}{2c^2(1-s)}ds + \int_0^t \frac{3Y_s}{c\sqrt{1-s}}dB_s, \end{cases} \quad \mathbb{P}_0 - a.s. \tag{7.1}$$

Clearly, (7.1) is well-posed on $[0,1)$*. Denote*

$$Z_t := \frac{X_t}{\sqrt{1-t}}; \quad \Gamma_t \triangleq \frac{3Y_t}{c(1-t)}; \quad A_t := -\big(\frac{3}{2c^2}+1\big)\frac{X_t}{(1-t)^{3/2}}.$$

Then (Y, Z, Γ, A) *is a nonzero solution to 2BSDE (3.9).*

Proof. First, applying Itô's formula one can check straightforwardly that (Y, Z, Γ, A) satisfies the SDEs in (3.9). Notice that

$$R_t := \frac{3}{c^2}Y_t^2 + X_t^2 \quad \text{satisfies} \quad dR_t = -\frac{3R_t}{1-t}dt + (\cdots)dB_t,$$

by Itô's formula. Since $R_0 = 1$,

$$\mathbb{E}^{\mathbb{P}_0}[R_t] = 1 - 3\int_0^t \frac{\mathbb{E}^{\mathbb{P}_0}[R_s]}{1-s}ds \quad \text{and thus} \quad \mathbb{E}^{\mathbb{P}_0}[R_t] = (1-t)^3, \quad \text{for all } 0 \le t < 1.$$

Then one can easily see that,

$$\sup_{0\le t<1} \mathbb{E}^{\mathbb{P}_0}\big[|\Gamma_t|^2 + |A_t|^2\big] \le C\mathbb{E}^{\mathbb{P}_0}\Big[\frac{|Y_t|^2}{(1-t)^2} + \frac{|X_t|^2}{(1-t)^3}\Big] \le C,$$

which, together with (3.9), also implies that

$$\mathbb{E}^{\mathbb{P}_0}\Big[\sup_{0\le t<1} [|Y_t|^2 + |Z_t|^2\Big] \le C.$$

Finally, we prove that

$$\lim_{t\uparrow 1} Y_t = 0, \ \mathbb{P}_0 - \text{a.s.} \tag{7.2}$$

In fact, for any $t < T < 1$, by Burkholder-Davis-Gundy Inequality we have

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\le s\le T} |Y_s|^2\Big] &\le C\mathbb{E}\Big[|Y_t|^2 + \int_t^T \frac{|Y_s|^2}{(1-s)^2}ds + \int_t^T \frac{|X_s|^2}{1-s}ds\Big] \\ &\le C\Big((1-t)^3 + \int_t^T \big((1-s)+(1-s)^2\big)ds\Big) \le C(1-t)^2. \end{aligned}$$

Let $T \uparrow 1$ and apply Monotone Convergence Theorem, we get

$$\mathbb{E}\Big[\sup_{t\le s<1} |Y_s|^2\Big] \le C(1-t)^2.$$

Then $\sup_{t\le s<1} |Y_s|^2 \downarrow 0$, as $t \uparrow 1$, $\mathbb{P}_0$−a.s. by the decrease of $\sup_{t\le s<1} |Y_s|^2$ in t, and we deduce (7.2). □

7.2 Proof of Lemma 4.3

If the a priori estimates (4.5) and (4.6) hold, then by the martingale representation property (2.5), the Lipschitz conditions (2.7) and (4.2), and the integrability assumption of $\hat{F}^0$ in (2.9), following the standard arguments one can easily show that BSDE (4.4) has a unique solution.

We now prove (4.5) and (4.6). For notational simplicity in the proof we drop the superscripts $\mathbb{P}$ in $(\mathcal{Y}^{\mathbb{P}}, \mathcal{Z}^{\mathbb{P}})$. By the Lipschitz conditions (2.7) and (4.2), there exist bounded processes λ, η such that

$$\mathcal{Y}_t = \xi + \int_t^1 \big(\hat{F}_s^0 + \lambda_s \mathcal{Y}_s + \eta_s \hat{a}_s^{1/2} \mathcal{Z}_s\big) ds - \int_t^1 \mathcal{Z}_s dB_s, \ 0 \le t \le 1, \ \mathbb{P} - \text{a.s.} \tag{7.3}$$

Define M by (4.12). By Itô's formula, we have:

$$d\big(M_t \mathcal{Y}_t\big) = -M_t \hat{F}_t^0 dt + M_t\big(\mathcal{Z}_t - \mathcal{Y}_t \eta_t \hat{a}_t^{-1/2}\big) dB_t, \quad 0 \le t \le 1, \ \mathbb{P} - \text{a.s.} \tag{7.4}$$

Then, using standard localization arguments if necessary:

$$\mathcal{Y}_t = M_t^{-1} \mathbb{E}_t^{\mathbb{P}}\Big[M_1 \xi + \int_t^1 M_s \hat{F}_s^0 ds\Big], \quad 0 \le t \le 1, \ \mathbb{P} - \text{a.s.}$$

It follows from (4.14) that, for $1 < \kappa \le 2$,

$$\begin{aligned} |\mathcal{Y}_t| &\le \mathbb{E}_t^{\mathbb{P}}\Big[\sup_{t \le s \le 1} (M_t^{-1} M_s)\big(|\xi| + \int_t^1 |\hat{F}_s^0| ds\big)\Big] \\ &\le C_\kappa \Big(\mathbb{E}_t^{\mathbb{P}}\Big[|\xi|^\kappa + \int_t^1 |\hat{F}_s^0|^\kappa ds\big)\Big]\Big)^{1/\kappa}, \quad 0 \le t \le 1, \ \mathbb{P} - \text{a.s.} \end{aligned}$$

This proves (4.5).

Finally, applying Itô's formula on $\mathcal{Y}_t^2$ and following standard arguments we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 |\hat{a}_t^{1/2} \mathcal{Z}_t|^2 dt\Big] \le C \mathbb{E}^{\mathbb{P}}\Big[|\xi|^2 + \sup_{0 \le t \le 1} |\mathcal{Y}_t| \int_0^1 |\hat{F}_t^0| dt\Big] \\ \le \ & C \mathbb{E}^{\mathbb{P}}\Big[\sup_{0 \le t \le 1} |\mathcal{Y}_t|^2 + \Big(\int_0^1 |\hat{F}_t^0| dt\Big)^2\Big] \le C \mathbb{E}^{\mathbb{P}}\Big[\sup_{0 \le t \le 1} |\mathcal{Y}_t|^2 + \Big(\int_0^1 |\hat{F}_t^0|^\kappa dt\Big)^{2/\kappa}\Big]. \end{aligned}$$

This, combing with (4.5), proves (4.6). □

7.3 Proof of (4.18)

By the definition of $\mathcal{P}_H^\kappa$, we have $\mathbb{P} = \mathbb{P}^\alpha$, $\mathbb{P}_1' = \mathbb{P}^{\alpha^1}$, and $\mathbb{P}_2' = \mathbb{P}^{\alpha^2}$ for $\mathbb{F}-$progressively measurable processes $\alpha, \alpha^1, \alpha^2$ taking values in $\mathbb{S}_d^{>0}$. Since $\mathbb{P}, \mathbb{P}_1', \mathbb{P}_2' \in \mathcal{P}_H^\kappa$, by (2.9) there exist $\underline{\alpha}, \overline{\alpha}, \underline{\alpha}^i, \overline{\alpha}^i \in \mathbb{S}_d^{>0}$ such that

$$\underline{\alpha} \le \alpha \le \overline{\alpha}, \quad \underline{\alpha}^i \le \alpha^i \le \overline{\alpha}^i, \quad dt \times d\mathbb{P}_0 - \text{a.s.}$$

Since $\mathbb{P}'_i \in \mathcal{P}_H^\kappa(t,\mathbb{P})$, it is clear that $\alpha = \alpha^i$, $ds \times d\mathbb{P}_0-$a.s. on $[0,t]\times\Omega$. Recall (2.3), then

$$\alpha^*_s(\omega) := \alpha_s(\omega)\mathbf{1}_{[0,t)}(s) + \Big(\alpha^1_s(\omega)\mathbf{1}_{\{X^\alpha\in E_1\}}(\omega) + \alpha^2_s(\omega)\mathbf{1}_{\{X^\alpha\in E_2\}}(\omega)\Big)\mathbf{1}_{[t,1]}(s),\ \ s\in[0,1],$$

is $\mathbb{F}-$progressively measurable and satisfies:

$$0 < \underline{\alpha}\wedge\underline{\alpha}^1\wedge\underline{\alpha}^2 \le \alpha^* \le \overline{\alpha}\vee\overline{\alpha}^1\vee\overline{\alpha}^2.$$

Following a line by line analogy of the proof of Claim 4.19 in [16], which in turn uses the arguments in the proof of Lemma 4.1 in [16], we see that $\mathbb{P}' = \mathbb{P}^{\alpha^*}\in\overline{\mathcal{P}}_S$. Moreover,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}'}\Big[\int_0^1|\hat F^0_s|^2ds\Big] &= \mathbb{E}^{\mathbb{P}}\Big[\int_0^t|\hat F^0_s|^2ds\Big] + \mathbb{E}^{\mathbb{P}'_1}\Big[\int_t^1|\hat F^0_s|^2ds\mathbf{1}_{E_1}\Big] + \mathbb{E}^{\mathbb{P}'_2}\Big[\int_t^1|\hat F^0_s|^2ds\mathbf{1}_{E_2}\Big]\\ &\le \mathbb{E}^{\mathbb{P}}\Big[\int_0^1|\hat F^0_s|^2ds\Big] + \mathbb{E}^{\mathbb{P}'_1}\Big[\int_0^1|\hat F^0_s|^2ds\mathbf{1}_{E_1}\Big] + \mathbb{E}^{\mathbb{P}'_2}\Big[\int_0^1|\hat F^0_s|^2ds\mathbf{1}_{E_2}\Big] < \infty.\end{aligned}$$

Then $\mathbb{P}'\in\mathcal{P}_H^\kappa$. Obviously, $\mathbb{P}'=\mathbb{P}$ on $\mathcal{F}_t$. This proves that $\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})$. □

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