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A STOCHASTIC REPRESENTATION FOR MEAN CURVATURE TYPE GEOMETRIC FLOWS

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A smooth solution $\{\Gamma(t)\}_{t\in[0,T]} \subset \mathbb{R}^d$ of a parabolic geometric flow is characterized as the reachability set of a stochastic target problem. In this control problem the controller tries to steer the state process into a given deterministic set $\mathcal{T}$ with probability one. The reachability set, $V(t)$, for the target problem is the set of all initial data x from which the state process $X_x^\nu(t) \in \mathcal{T}$ for some control process ν. This representation is proved by studying the squared distance function to $\Gamma(t)$. For the codimension k mean curvature flow, the state process is $dX(t) = \sqrt{2}P\,dW(t)$, where $W(t)$ is a d-dimensional Brownian motion, and the control P is any projection matrix onto a $(d-k)$-dimensional plane. Smooth solutions of the inverse mean curvature flow and a discussion of non smooth solutions are also given.

1. Introduction. Motivated by pricing problems in mathematical finance [23], the authors introduced in [24] the *stochastic target* problems. In this control problem, the controller tries to steer the state process $X(\cdot)$ into a prescribed deterministic target set $\mathcal{T}$ by a judicious choice of controls. The main object of this problem is the reachability set $V(t)$: all initial points $x = X(0)$ from which the state $X(t)$ at time t can be placed into the target with probability one. It is clear that if initially $x = X(0) \in V(t)$, then the state process remains in the reachability sets for all $s \in [0,t]$: $X(s) \in V(t-s)$. The converse of this statement is also proved in [24]. Namely, if starting at $x = X(0)$ at time s, we can place the state process into the reachability set $V(t-s)$ by an appropriate choice of a control process, then necessarily $x \in V(t)$. Hence, for all $s > 0$, the reachability set $V(t)$ is the collection of all initial positions starting from which, at time s, one can place the state process into the reachability set $V(t-s)$ by some judicious choice of controls. Intuitively, this geometric dynamic programming principle yields a differential equation for the reachability sets. This paper studies the corresponding geometric equation and proves that if there is a smooth solution of the geometric differential equation starting from the target, then the smooth solution is the reachability set.

To describe the control problem mathematically, let U be the set of all possible control actions. In this paper, we only consider state processes which are diffusions. So let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis for a d-dimensional

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Brownian motion W; see Karatzas and Shreve [19] for the definitions. Then, an admissible control process is an $\mathbb{F}$ adapted function of $\Omega \times [0, \infty)$ with values in U. Let $\mathcal{A}$ be the set of all admissible controls. For a given $\nu \in \mathcal{A}$ and an initial point $x \in \mathbb{R}^d$, the corresponding state process $X_x^\nu(\cdot)$ is the unique solution of the stochastic differential equation

$$dX_x^\nu(t) = \mu\big(X_x^\nu(t), \nu(t)\big)\,dt + \sigma\big(X_x^\nu, \nu(t)\big)\,dW(t),$$

with initial data $X_x^\nu(0) = x$. Then the *reachability set* is the following subset of $\mathbb{R}^d$:

$$V(t) := \big\{x \in \mathbb{R}^d : X_x^\nu(t) \in \mathcal{T} \text{ a.s. for some } \nu \in \mathcal{A}\big\}.$$

Observe that the above reachability set is closely connected to the theory of backward and forward-backward stochastic differential equations; see [21] for an overview and [9] for the constrained case.

In [24], under some assumptions, it is shown that the characteristic function v of the complement of $V(t)$ solves

$$v_t - F(x, Dv, D^2 v) = 0 \qquad \text{in the viscosity sense,} \tag{1.1}$$

where for $x, \xi \in \mathbb{R}^d$ and a symmetric matrix A,

$$F(x, \xi, A) := \inf_{u \in \mathcal{N}(x,\xi)} \{f^u(x, \xi, A)\}, \tag{1.2}$$

$$f^u(x, \xi, A) := \mu(x, u)\xi + \tfrac{1}{2}\mathrm{trace}[a(x, u)A] \tag{1.3}$$

and

$$\mathcal{N}(x, \xi) := \{u \in U : a(x, u)\xi = 0\}, \qquad a(x, u) := \sigma(x, u)\sigma^*(x, u).$$

The above dynamic programming equation is geometric, that is, F satisfies

$$F(x, \mu\xi, \mu A + \lambda\xi \otimes \xi) = \mu F(x, \xi, A) \qquad \forall \mu > 0,\ \lambda \in \mathbb{R}^1, \tag{1.4}$$

where $(\xi \otimes \xi)_{ij} = \xi_i \xi_j$. Nonlinear equations with a geometric nonlinearity are known to be related to geometric flows [4]. This is the starting point of our stochastic representation. However analysis of this paper relies only on elementary applications of the Itô's formula, and in particular we do not use the above or any other result from [24].

To demonstrate the connection between the geometric flows and the target problem, we examine the codimension k mean curvature flow which is the driving example of this study. Following a suggestion of DeGiorgi [10], and the earlier work of Chen, Giga and Goto [7] and Evans and Spruck [11] on codimension one flows, Ambrosio and Soner [2] showed that this flow can be characterized as the zero level of the unique viscosity solution of

$$v_t = F_k(Dv, D^2 v)$$

where for $\xi \in \mathbb{R}^d$ and a $d \times d$ symmetric matrix A

$$F_k(\xi, A) = \inf\{\text{trace}[PA] | P \in \mathcal{U}_k, P\xi = 0\}, \tag{1.5}$$

$$\mathcal{U}_k := \{P : P^2 = P, \text{trace}[P] = k\}, \tag{1.6}$$

that is, $\mathcal{U}_k$ is the set of all $d \times d$ projection matrices onto a $(d-k)$-dimensional plane.

A brief discussion of the mean curvature flow and the level set approach to these equations is given in Section 3 below.

An examination of the above equation indicate that the above level set equation coincides with the dynamic programming equation (1.1) with $U = \mathcal{U}_k$ and

$$\mu \equiv 0, \qquad \sigma(x, P) = \sqrt{2}P.$$

So we expect the stochastic target problem with the above choices of the parameters is related to the mean curvature flow. Indeed, more generally we prove that:

THEOREM 1.1. *Let* $\{\Gamma(t)\}_{t\in[0,T]}$ *be a smooth solution of the geometric equation related to the target problem. Then, the reachability set* $V(t)$ *is equal to* $\Gamma(t)$.

An explanation of this result is the following. Intuitively, Brownian motion moving on the tangent plane of a Euclidean manifold, moves away from the manifold in the normal direction with a velocity equal to the half of its mean curvature. Therefore, reversing time, this means that if the Brownian motion (multiplied by $\sqrt{2}$) starts on the solution of the mean curvature flow, $\Gamma(t)$, and diffuses on the tangent plane at all times, then it will arrive $\Gamma(0)$ at time t. On the other hand, Brownian motion moving on some other plane would go away from the manifold, and we could never guarantee that it would return with probability one.

Examples studied in the Section 3 suggest that the corresponding geometric equation related to the target problem is formally (2.1) of Section 2. As stated this equation is not well defined and a careful definition of even smooth flows is needed. Again the examples studied in Section 3 and the work of Professor DeGiorgi suggest a definition in terms of the squared distance function. This definition is given in Section 2.

The above theorem is proved by using the properties of the squared distance function η to the smooth solution $\Gamma(t)$. The chief calculation in this proof is a straightforward application of the Itô's rule to the squared distance function. The following simple example gives insight into the proof.

EXAMPLE. Consider the mean curvature flow in the plane starting from the initial set $\Gamma(0) = R(0)S^1$, where $R(0)$ is some given positive constant and $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit circle in $\mathbb{R}^2$, that is, a circle with radius $R(0)$. Then the solution is $\Gamma(t) = R(t)S^1$ where $R(t)$ is a real-valued function satisfying

$$\frac{d}{dt}R(t) = -\frac{1}{R(t)},$$

that is,

$$R(t) = \sqrt{R(0)^2 - 2t}.$$

Let us show that from any initial point $x \in \Gamma(t)$ we can reach $\Gamma(0)$ in time t. Consider the function $\hat{P}(y) := [I_2 - (y \otimes y)/|y|^2]$ where I_2 is the two-dimensional identity matrix, and $\otimes$ is the standard tensor notation, that is, $(y \otimes y)_{ij} = y_i y_j$. Let $\hat{X}_x$ be the corresponding state process, that is, $\hat{X}_x$ is the unique solution of

$$d\hat{X}_x(s) = \sqrt{2}\hat{P}\big(\hat{X}_x(s)\big)\,dW(s) = \sqrt{2}\bigg(I_2 - \frac{\hat{X}_x(s) \otimes \hat{X}_x(s)}{|\hat{X}_x(s)|^2}\bigg)\,dW(s),$$

with initial data $\hat{X}_x(0) = x$ until the random time at which $\hat{X}_x$ reaches the origin. Then, $\hat{X}_x$ is the controlled process corresponding to the control process $s \mapsto P(\hat{X}_x(s))$. Since this control only depends on the present value of the state process, it is called a feedback Markov control in the control literature.

We directly calculate that

$$d|\hat{X}_x(s)|^2 = 2\hat{X}_x(s)\,d\hat{X}_x(s) + 2\,\mathrm{trace}\big[\hat{P}\big(\hat{X}_x(s)\big)\big]\,ds = 2\,ds.$$

Hence $|\hat{X}_x(t)|^2 = |x|^2 + 2t = R(t)^2 + 2t = R(0)^2$. So $\hat{X}_x$ never hits the origin and thus defined for all time. Moreover, $\hat{X}_x(t) \in \Gamma(0)$.

In conclusion, we showed that from any $x \in \Gamma(t)$ we can reach the target $\Gamma(0)$ in time t. Hence the reachability set $V(t)$ contains $\Gamma(t)$.

To prove the opposite inclusion, let $x \in V(t)$. There there exists a control process $P(s) \in \mathcal{U}_1$ so that the solution $\hat{X}_x$ of

$$\hat{X}_x(s) = x + \int_0^s \sqrt{2}P(u)\,dW(u),$$

satisfies the target condition, that is, $\hat{X}_x(t) \in \Gamma(0)$. Therefore, $|\hat{X}_x(t)|^2 = R(0)^2$. We again use the Itô's rule to obtain

$$|\hat{X}_x(t)|^2 = R(0)^2 = |x|^2 + \int_0^t 2\,\mathrm{trace}(P(s))\,ds + \int_0^t 2\hat{X}(s)\,d\hat{X}(s).$$

Since we are on the plane, it follows from the definition of $\mathcal{U}_1$ that $\mathrm{trace}(P(s)) = 1$, and therefore $|x|^2 = R(0)^2 - 2t$ by taking the expected values in the above equality. This exactly means that $x \in \Gamma(t)$. Hence $V(t)$ is equal to $\Gamma(t)$.

The proof of our main result, Theorem 2.1, uses a similar calculation based not on $|\hat{X}|^2$ but rather the square distance function η. Note that in this simple example $\eta(t,x) = \frac{1}{2}(|x| - R(t))^2$ is closely related to $|x|^2$.

The paper is organized as follows. In the next section, we define smooth geometric flows and then prove that any smooth solution, when it exists, is equal to the reachability set. In Section 3, we briefly recall several properties of the mean curvature flow and the inverse mean curvature flow. Level set equations for these equations will also be introduced in that section. Applications to mean and inverse mean curvature flows are stated as corollaries as well. Level set equation satisfied by the reachability sets is given in Section 4. In the final section, we discuss an alternate definition.

2. Stochastic representation. In this section we prove that if the geometric flow (2.1) has a smooth solution for some time $[0, T]$, then it coincides with the reachability set $V(t)$ of the stochastic target problem in that time interval. This is similar to the verification theorems in classical stochastic optimal control theory; see Fleming and Soner [12].

When there are no smooth solutions, we need to consider weak solutions of (2.1), and this will be discussed in Section 4. This part is also similar to the results in classical stochastic optimal control theory which states that a value function is the viscosity solution of the dynamic programming equation; see [12].

Examples of the next section suggest that the geometric equation related to the dynamic programming equation (1.1) is

$$\vec{v}(t,x) = \inf\{\mu(x,u) + \vec{H}_{a(x,u)} : u \in \mathcal{K}(t,x)\}, \qquad x \in \Gamma(t), \tag{2.1}$$

where $\vec{v}$ is the velocity vector of the moving manifold, $\vec{H}_{a(x,u)}$ is the mean curvature vector at (t,x) using the metric generated by the quadratic form of the matrix $a(x,u)$ and

$$\mathcal{K}(t,x) := \{u \in U : \text{Normal space to } \Gamma(t) \text{ at } x \subset \text{ Kernel } a(x,u)\}.$$

Note that this definition is not rigorous as we take the infimum of vector valued functions. Therefore a careful definition of smooth flows is needed. To motivate the definition first we briefly discuss the properties of the square distance function.

2.1. *The square distance function.* Let $\Gamma = \{\Gamma(t)\}_{t\in[0,T]}$ be a collection of smooth manifolds embedded in the Euclidean space $\mathbb{R}^d$, and parameterized by the time variable t. The chief technical tool to analyze the geometric flows is the squared distance function:

$$\eta(t,x) := \tfrac{1}{2}[\rho(t,x)]^2 \qquad \text{with } \rho(t,x) := \text{distance}(x, \Gamma(t)).$$

As observed by Professor DeGiorgi [10], important geometric quantities of the smooth geometric flow $\{\Gamma(t)\}_{t\in[0,T]}$ are closely connected to the derivatives of η. This was first used by Ambrosio and Soner [2] and later studied by Ambrosio and Mantegazza [1].

The first observation is that although ρ is not differentiable on the manifold $\{\eta = 0\}$, η is smooth in a tubular neighborhood $\{\eta < \delta\}$ of Γ, for some $\delta > 0$. The following is proved in [2].

LEMMA 2.1. *Let k be the codimension of $\Gamma(t) \subset \mathbb{R}^d$. Then in a tubular neighborhood of Γ, $D^2\eta$ has exactly k eigenvalues equal to one and all other eigenvalues are on the order of $\sqrt{\eta}$. In particular, on $\Gamma(t)$, the Hessian $D^2\eta$ is the projection matrix onto the normal space.*

Moreover, for $x \in \Gamma(t)$,

$$\vec{v}(t,x) = -D\eta_t(t,x) \quad \textit{and} \quad \vec{H} = -D\Delta\eta(t,x),$$

where $\vec{v}(t,x)$ is the velocity vector of the moving manifold, and $\vec{H}(t,x)$ is the mean curvature vector of the manifold at $x \in \Gamma(t)$.

The following simple result will be used several times so we formulate it as a lemma.

LEMMA 2.2. *Let $\mathcal{G} := \{\eta < \delta\}$ be a tubular neighborhood of Γ in which η is smooth. Let φ be a $C^1(\mathcal{G}, \mathbb{R})$ function with a bounded Hessian $D^2\varphi$. Assume further that*

$$\varphi(t,x) = 0 \quad \textit{and} \quad D\varphi(t,x) = 0 \qquad \textit{for } x \in \Gamma(t).$$

Then, on $\mathcal{G}$,

$$|\varphi(t,x)| \le C\eta(t,x),$$

for some constant $C > 0$ depending only on the bound on $D^2\varphi$.

PROOF. Fix $x \in \mathcal{G}$, and let $y \in \Gamma$ be such that $|x - y|^2 = 2\eta(t,x)$, that is, y is the closest point on $\Gamma(t)$ to x. Since t is only a parameter, in our notation we drop the dependence on t. Set

$$\rho(x) := |x - y| \quad \text{and} \quad \vec{n}(x) := \rho(x)^{-1}(x - y).$$

Consider the functions $s \mapsto \varphi(y + s\vec{n}(x))$, and $s \mapsto D\varphi(y + s\vec{n}(x)) \cdot \vec{n}$ on $[0, \rho(x)]$. By calculus,

$$\begin{aligned}\varphi(x) &= \int_0^{\rho(x)} D\varphi\big(y + s\vec{n}(x)\big) \cdot \vec{n}(x)\, ds \\ &= \int_0^{\rho(x)} \int_0^s D^2\varphi\big(y + \tau\vec{n}(x)\big)\vec{n}(x) \cdot \vec{n}(x)\, d\tau\, ds.\end{aligned}$$

The result follows from the boundedness of $D^2\varphi$. $\square$

2.2. *A remark on mean curvature flow.* Combining the two results we obtain the following characterization of the mean curvature flow. A family of codimension k smooth manifolds $\{\Gamma(t)\}$ is said to be a mean curvature flow if it satisfies the equation

$$\vec{v} = \vec{H} \qquad \forall x \in \Gamma(t).$$

By Lemma 2.1, in a tubular neighborhood of Γ, k eigenvalues of $D^2\eta(t,x)$ equal to one and the others are smaller than one. Observing that $D\eta(t,x)$ is an eigenvector of $D^2\eta(t,x)$ with unit eigenvalue, we see that, in this neighborhood,

$$F_k(D\eta, D^2\eta) = \Delta\eta - k \quad \text{and} \quad DF_k(D\eta, D^2\eta) = D\Delta\eta,$$

where F_k is as in (1.5). Then, using again Lemma 2.1, it follows that

$$D\eta_t(t,x) = D\Delta\eta(t,x) = DF_k\big(D\eta(t,x), D^2\eta(t,x)\big) \qquad \text{for all } x \in \Gamma(t).$$

Combining this with Lemma 2.2, we obtain the following property of the square distance function to a mean curvature flow.

$$\big|\eta_t - F_k\big(D\eta(t,x), D^2\eta(t,x)\big)\big| \le C\eta \qquad \text{in } \{\eta < \delta\}$$

for some constants C and δ. More details of this calculation is given in Section 3.1.

Lemma 2.1 also yields a remarkable identity for the squared distance function in a tubular neighborhood

$$F_k(D\eta, D^2\eta) = G_k(D^2\eta), \tag{2.2}$$

where

$$G_k(A) = \inf\{\operatorname{trace}[PA] : P \in \mathcal{U}_k\}$$

and $\mathcal{U}_k$ is as in (1.6). Notice that the only difference between the definition of F_k, (1.5) and the above definition for G_k is that in the latter we remove the requirement that $P\xi = 0$. So in general $G_k \le F_k$ but for the square distance function they agree.

2.3. *Definition.* The following definition is a natural extension of DeGiorgi's ideas for the mean curvature flow in this context, and motivated by the calculations of Section 3 and [2]. Let F be as in (1.2).

DEFINITION 2.1. We say that $\{\Gamma(t)\}_{t\in[0,T]}$ is a smooth solution of (2.1) if the square distance function η satisfies the following conditions:

(i) η_t and $D\eta$ are Lipschitz continuous in a tubular neighborhood $\{\eta < \delta\}$.
(ii) On $\Gamma(t)$,

$$D\eta_t = D[F(x, D\eta, D^2\eta)].$$

There are several observations:

1. In view of Lemma 2.2, on a tubular region $\{\eta < \delta\}$,
$$|\eta_t - F(x, D\eta, D^2\eta)| \le C\eta,$$
for some appropriate constant C.
2. Due to the calculations in Section 3 and in [2], smooth solutions of the mean curvature flow as well as the inverse mean curvature flow are smooth solutions in the above sense; see (3.2) and (3.8).
3. In the above definition, the square distance function η is assumed to be sufficiently smooth (in a tubular neighborhood) in order for the generalized Itô's lemma [20] to apply.
4. In the above definition we have not made any assumption on the dimension of the solution. In particular, when either the sign of the curvature or more generally the orientation of the solution is needed to define the flow, it is appropriate to consider solid sets with full dimension as the solution $\Gamma(\cdot)$. An instance of this is the inverse mean curvature flow. In these examples, the boundary of $\Gamma(\cdot)$ is an hypersurface satisfying the corresponding geometric equation. Note that in such cases, η is identically equal to zero inside the solid set and there the conditions on η are trivially satisfied.

   An alternate definition using the dimension information is discussed in Section 5 below. This definition is obtained by using a more complicated nonlinear function. Since both the definition and the proofs with the alternate definition is more technical we chose to use the above definition although the alternate definition seems to be more general.

2.4. *Main result.* Let F, f^u and $\mathcal{N}$ be as in (1.2) and (1.3). Let $\{\Gamma(t)\}_{t\in[0,T]}$ be a smooth solution of (2.1) . Motivated by the mean curvature, in particular by (2.2), we assume that in a tubular neighborhood of the smooth flow

$$|G(x, D\eta, D^2\eta) - F(x, D\eta, D^2\eta)| \le C\eta, \tag{2.3}$$

for some constant C, where

$$G(x, \xi, A) := \inf_{u\in U}\{\mu(x, u)\cdot \xi + \tfrac{1}{2}\operatorname{trace}[a(x, u)A]\}.$$

Notice that as in the definition of G_k in (2.2), in the definition of G above we allow all controls, while in the definition of F controls are restricted to be in $\mathcal{N}(x, \xi)$.

We are now ready for the chief result of this paper.

THEOREM 2.1. *Suppose that* $\{\Gamma(t)\}_{t\in[0,T]}$ *is a smooth solution of* (2.1) *satisfying* (2.3). *Assume that there exists a Lipschitz continuous map*

$$\hat{\nu} : [0, T] \times \mathbb{R}^d \to U,$$

satisfying $\{\eta < \delta\}$, $\hat{\nu}(t,x) \in \mathcal{N}(x, D\eta(t,x))$ *and*

$$f^{\hat{\nu}(t,x)}\big(x, D\eta(t,x), D^2\eta(t,x)\big) = F\big(x, D\eta(t,x), D^2\eta(t,x)\big).$$

Further assume that for any initial data $x \in \mathbb{R}^d$ *and* $t \in [0,T]$, *there is exists a unique solution* $\hat{X}$ *of*

$$\begin{aligned} d\hat{X}(s) = {} & \mu\big(\hat{X}(s), \hat{\nu}\big(t-s, \hat{X}(s)\big)\big)\,ds \\ & + \sigma\big(\hat{X}(s), \hat{\nu}\big(t-s, \hat{X}(s)\big)\big)\,dW(s), \qquad s \in (0,t), \end{aligned}$$

with initial data $\hat{X}(0) = x$. *Then the reachability set* $V(t)$ *of the corresponding stochastic target problem with target set* $\mathcal{T} = \Gamma(0)$ *is equal to the smooth solution* $\Gamma(t)$ *of* (2.1) *for all* $t \in [0,T]$. *Moreover, for* $x \in \Gamma(t)$, $\hat{X}(\cdot)$ *is the state process* $X_x^{\nu^*}$ *corresponding to the Markov control* $\nu^*(\cdot) = \hat{\nu}(t - \cdot, \hat{X}(\cdot))$, *and for* $s \in [0,t]$, $\hat{X}(s) \in \Gamma(t-s)$ *with probability one. In particular, it reaches the target in time* t.

Smooth solutions of both the mean curvature and the inverse mean curvature flows satisfy the assumptions of the theorem. These are stated in Corollaries 3.1 and 3.2.

PROOF OF THEOREM 2.1. We first prove the inclusion $V(t) \subset \Gamma(t)$. Let $x \in V(t)$. Then there is $\nu \in \mathcal{A}$ so that $X_x^\nu(t) \in \Gamma(0)$ with probability one, or equivalently $\eta(0, X_x^\nu(t)) = 0$.

Step 1. By the definition of a smooth flow, η is smooth in a tubular neighborhood $\{\eta < \delta\}$. We extend η smoothly to the whole domain by setting $\hat{\eta}(x,t) := \phi(\eta(x,t))$ for some smooth, nondecreasing ϕ which satisfies $\phi(r) = r$ for r small and $\phi(r)$ is constant for $r \geq \delta$. By the definition of a smooth flow,

$$\big|\hat{\eta}_t - F(x, D\hat{\eta}, D^2\hat{\eta})\big| \leq \beta\hat{\eta} \qquad \text{on } \mathbb{R}^d \times [0,T],$$

and by assumption (2.3),

$$\big|\hat{\eta}_t - G(x, D\hat{\eta}, D^2\hat{\eta})\big| \leq \beta\hat{\eta} \qquad \text{on } \mathbb{R}^d \times [0,T], \tag{2.4}$$

possibly with a different constant β.

Step 2. We apply the Itô's lemma to $\hat{\eta}(t - \cdot, X_x^\nu(\cdot))$. The result is

$$\begin{aligned} \hat{\eta}\big(t-s, X_x^\nu(s)\big) = {} & \hat{\eta}\big(t-\tau, X_x^\nu(\tau)\big) \\ & + \int_\tau^s \big(-\hat{\eta}_t + f^{\nu(r)}\big(X_x^\nu(r), D\hat{\eta}, D^2\hat{\eta}\big)\big)\big(t-r, X_x^\nu(r)\big)\,dr \\ & + \int_\tau^s \sigma\big(X_x^\nu(r), \nu(r)\big)^* D\hat{\eta}\big(t-r, X_x^\nu(r)\big)\,dW(r). \end{aligned} \tag{2.5}$$

By the definition of G,

$$f^{\nu(r)}\big(X_x^\nu(r), D\hat{\eta}, D^2\hat{\eta}\big)\big(t-r, X_x^\nu(r)\big) \geq G\big(X_x^\nu(r), D\hat{\eta}, D^2\hat{\eta}\big)\big(t-r, X_x^\nu(r)\big),$$

almost surely. Set $\alpha(s) := E\hat{\eta}(t-s, X_x^\nu(s))$. We use the above inequality in (2.5) then take the expected value to arrive at

$$\alpha(s) \geq \alpha(\tau) + E\left[\int_\tau^s \left(-\hat{\eta}_t + G\big(X_x^\nu(r), D\hat{\eta}, D^2\hat{\eta}\big)\right)\big(t-r, X_x^\nu(r)\big)\, dr\right].$$

Step 3. By (2.4),

$$\alpha(s) \geq \alpha(\tau) - CE\left[\int_\tau^s \alpha(r)\, dr\right] \qquad \text{for all } \tau \leq s \in [0,t).$$

We now use the Gronwall lemma, to conclude that $E[\hat{\eta}(t-s, X_x^\nu(s))] \geq \hat{\eta}(t,x)e^{-Cs}$ for all $s \in [0,t]$. Since $\hat{\eta}(0, X_x^\nu(t)) = 0$, this proves that $\hat{\eta}(t,x) = 0$. Hence $x \in \Gamma(t)$.

Next we prove the opposite inclusion $\Gamma(t) \subset V(t)$. Let $x \in \Gamma(t)$ so that $\eta(t,x) = 0$. We will construct a control $\nu \in \mathcal{A}$ so that $X_x^\nu(t) \in \Gamma(0)$ with probability one, or equivalently $\eta(0, X_x^\nu(t)) = 0$.

Step 4. Let $\hat{\nu}$, $\hat{X}$ be as in the statement of the theorem. Then, $\hat{X}(\cdot)$ is the state process $X_x^{\nu^*}$ corresponding to the Markov control $\nu^*(\cdot) = \hat{\nu}(t-\cdot, \hat{X}(\cdot))$. Set

$$\theta := t \wedge \theta_\delta \qquad \text{where } \theta_\delta := \inf\big\{s \geq 0 : \eta\big(t-s, \hat{X}(s)\big) > \delta\big\}.$$

By continuity of the state process $\hat{X}$, it follows that,

$$\eta\big(t-\theta_\delta, \hat{X}(\theta_\delta)\big) > 0 \qquad \text{a.s.} \tag{2.6}$$

Step 5. By the definition of θ, $\eta(\cdot, \hat{X}(\cdot))$ is smooth on $[\![t,\theta]\!]$. We apply the Itô's lemma to $\eta(t-\cdot, \hat{X}(\cdot))$ on this interval and use the fact that $\eta(t, \hat{X}(0)) = \eta(t,x) = 0$. The result is

$$\begin{aligned}\eta\big(t-\theta, \hat{X}(\theta)\big) = &\int_0^\theta \left(-\eta_t + f^{\nu^*(s)}\big(\hat{X}(s), D\eta, D^2\eta\big)\right)\big(t-s, \hat{X}(s)\big)\, ds \\ &+ \int_0^\theta \left(\sigma\big(\hat{X}(s), \nu^*(s)\big)^* D\eta\big(t-s, \hat{X}(s)\big)\right) dW(s).\end{aligned}$$

Since $\nu^*(s) = \hat{\nu}(t-s, \hat{X}(s)) \in \mathcal{N}(X_x^\nu(s), D\eta(t-s, \hat{X}(s)))$,

$$\sigma\big(\hat{X}(s), \nu^*(s)\big)^* D\eta\big(t-s, \hat{X}(s)\big) = 0$$

on $[\![0,\theta]\!]$, recall that $a = \sigma\sigma^*$. Again by hypothesis on $[\![0,\theta]\!]$,

$$f^{\nu^*(s)}\big(\hat{X}(s), D\eta, D^2\eta\big)\big(t-s, \hat{X}(s)\big) = F\big(\hat{X}(s), D\eta, D^2\eta\big)\big(t-s, \hat{X}(s)\big).$$

Hence

$$\eta\big(t-\theta, \hat{X}(\theta)\big) = \int_0^\theta \left(-\eta_t + F\big(\hat{X}(s), D\eta, D^2\eta\big)\right)\big(t-s, \hat{X}(s)\big)\, ds.$$

Step 6. By (2.4),

$$\eta\big(t-\theta, \hat{X}(\theta)\big) \leq C\int_0^\theta \eta\big(t-s, \hat{X}(s)\big)\, ds.$$

By Gronwall's lemma, this proves that $\eta(t-\theta, \hat{X}(\theta)) = 0$. Then, it follows from (2.6) that $\theta = t < \theta_\delta$ and $\eta(0, \hat{X}(t)) = 0$. Since $\nu^* \in \mathcal{A}$ and $\hat{X}$ is the state process corresponding to ν^*, this implies that $x \in V(t)$. □

3. Examples of geometric flows. In this section, we briefly discuss the mean curvature, the inverse mean curvature flows and an anisotropic flow from material science. At the end of the section we state a geometric equation related to the general stochastic target problem.

3.1. *Mean curvature flow.* This is a geometric initial value problem in which the solution is a collection of smooth manifolds without boundary $\{\Gamma(t)\}_{t\in[0,T]}$ embedded in the Euclidean space $\mathbb{R}^d$ parametrized by the time variable t. They are said to be a mean curvature flow starting from the initial data $\Gamma(0)$ if they satisfy

$$\vec{v}(t,x) = \vec{H}(t,x) \qquad \text{for all } t \in (0,T],\ x \in \Gamma(t), \tag{3.1}$$

that is, the velocity of $\Gamma(t)$ is given by the mean curvature vector. The mean curvature flow of planar curves is known as the curve shortening equation and it has smooth solutions due to the work of Gage and Hamilton [13] and Grayson [14]. Convex hypersurfaces also flow smoothly in time until they shrink to a point as proved by Huisken [17]. In general the mean curvature flow develops singularities [15]. Starting from the pioneering work of Brakke [6] several weak solutions have been proposed, most notably the level set solutions of Evans and Spruck [11] and Chen, Giga and Goto [7]. We also refer to [2] for the level set approach in arbitrary codimension as well as for other references.

We continue by defining the flow by means of the square distance function η. In view of Lemma 2.1, $\{\Gamma(t)\}_{t\in[0,T]}$ is a mean curvature flow if and only if

$$D\eta_t(t,x) = D\Delta\eta(t,x) \qquad \text{for all } t \in (0,T],\ x \in \Gamma(t).$$

It is convenient to extend this equation to a tubular neighborhood in the following way. For $x \in \Gamma(t)$, it follows from Lemma 2.1 that $\Delta\eta(t,x)$ is equal to the dimension k of the normal space. Moreover, the squared distance function $\eta(t,x)$ is nonnegative everywhere and zero for $x \in \Gamma(t)$. Then, $\eta_t(t,x) = 0$ for $x \in \Gamma(t)$. Hence, for $x \in \Gamma(t)$, we have

$$\eta_t - \Delta\eta + k = 0 \quad \text{and} \quad D(\eta_t - \Delta\eta + k) = 0.$$

By Lemma 2.2, this shows that $\{\Gamma(t)\}_{t\in[0,T]}$ is a smooth mean curvature flow if and only if,

$$|\eta_t - \Delta\eta + k| \le C\eta \qquad \text{in } \{\eta < \delta\}$$

for some appropriate constants C and δ.

In view of Lemma 2.1, in a tubular neighborhood

$$\Delta\eta - k = F_k(D\eta, D^2\eta),$$

where F_k is as in (1.5). Then, we rewrite the above inequality as

$$\left|\eta_t - F_k(D\eta, D^2\eta)\right| \le C\eta \qquad \text{on } \{\eta < \delta\}. \tag{3.2}$$

Hence the smooth solutions of the mean curvature satisfy the definition given in the previous section.

As discussed in the Introduction, the level set equation for the mean curvature flow can be seen as the dynamic programming equation (1.1) with the choices

$$U = \mathcal{U}_k, \qquad \mu \equiv 0, \qquad \sigma(x, P) = \sqrt{2}P,$$

where $\mathcal{U}_k$ is as in (1.6).

Then the corresponding nonlinearity in (1.1) has the form

$$\begin{aligned} F(x, \xi, A) &:= \inf_{u \in \mathcal{N}(x,\xi)} \left\{\mu(x, u) \cdot \xi + \tfrac{1}{2}\,\mathrm{trace}[a(x, u)A]\right\} \\ &= \inf\left\{\mathrm{trace}[PA] : P \in \mathcal{U}_k,\ P\xi = 0\right\} \\ &= F_k(\xi, A). \end{aligned}$$

Hence the dynamic programming equation (1.1) is the level set equation for the codimension k mean curvature flow. The stochastic target problem with the above choices of the parameters is related to the mean curvature flow. Indeed, we have the following representation of the mean curvature flow.

COROLLARY 3.1 (Representation of the mean curvature flow). *Suppose that* $\{\Gamma(t)\}_{t\in[0,T]}$ *are smooth codimension k manifolds satisfying* (3.1). *Then,*

$$\Gamma(t) = \left\{x \in \mathbb{R}^d : X_x^\nu(t) \in \Gamma(0) \text{ a.s. for some } \nu \in \mathcal{A}\right\},$$

where

$$dX_x^\nu(s) = \sqrt{2}\nu(s)\,dW(s),$$

$\mathcal{A}$ *is the set of all adapted maps* $\nu(\cdot)$ *with values in* $\mathcal{U}_k$*—the set of all* $d \times d$ *projection matrices onto a* $d - k$ *plane.*

PROOF. It is sufficient to check that the conditions of Theorem 2.1 are satisfied. Indeed, for a smooth solution of the mean curvature flow, $D^2\eta$ has exactly k eigenvalues equal to one in the tubular region $\{\eta < \delta\}$. In this region, we take $\hat{\nu}(t, x)$ to be the projection on the $(d - k)$-dimensional plane orthogonal to all eigendirections of $D^2\eta$ with eigenvalue one. Then we extend $\hat{\nu}$ smoothly to the whole space. This feedback control satisfies the assumptions of Theorem 2.1. Assumption (2.3) is proved in Section 2.1. □

3.2. *Level set equation.* Although in this paper we only consider smooth flows, the level set equations are useful in guessing the form of the stochastic target problem. So here we briefly recall the derivation of the level set equation for the codimension one mean curvature flow. For codimension one flow, the solutions are hypersurfaces and therefore they can be represented as the zero level set of an auxiliary scalar-valued function ϕ:

$$\Gamma(t) = \{x : \phi(t,x) = 0\}.$$

By calculus, the unit normal vector $\vec{n}$ is given by $\vec{n} = D\phi/|D\phi|$, and $\vec{v}$, $\vec{H}$ are parallel to $\vec{n}$ satisfying

$$\text{normal velocity} = V = \vec{v} \cdot \vec{n} = -\frac{\phi_t}{|D\phi|},$$

$$\begin{aligned}\text{mean curvature} &= \mathbf{H} := -\vec{H} \cdot \vec{n} \\ &= D \cdot \left(\frac{D\phi}{|D\phi|}\right) = \frac{1}{|D\phi|}\left[\Delta\phi - \frac{D^2\phi D\phi \cdot D\phi}{|D\phi|^2}\right] \\ &= \frac{1}{|D\phi|} F_1(D\phi, D^2\phi),\end{aligned} \tag{3.3}$$

where for $\xi \in \mathbb{R}^d$ and a symmetric matrix A, $F_1(\xi, A)$ is defined in (1.5), that is,

$$\begin{aligned}F_1(\xi, A) &= \inf\{\operatorname{trace}[AP] : P \in \mathcal{U}_1, P\xi = 0\} \\ &= \operatorname{trace}\left[A\left(I_d - \frac{\xi \otimes \xi}{|\xi|^2}\right)\right] \qquad \text{for } \xi \neq 0,\end{aligned}$$

where I_d is the d-dimensional identity matrix. Then $\Gamma(t) = \{\phi = 0\}$ is a smooth mean curvature flow, if

$$\phi_t = F_1(D\phi, D^2\phi) \qquad \text{on } \Gamma(t) = \{\phi = 0\}, \tag{3.4}$$

In [7, 11] a weak-viscosity solution of the mean curvature flow is defined by solving (3.4) on all of the space not just on $\{\phi = 0\}$. This corresponds to moving all level sets of ϕ by their curvature not only the zero level set. This approach was extended to mean curvature flows with arbitrary codimension by Ambrosio and Soner [2]. The corresponding level set equation is $\phi_t = F_k(D\phi, D^2\phi)$ where the nonlinear term F_k is as in (1.5).

3.3. *Inverse mean curvature flow.* Another important parabolic flow is the inverse mean curvature flow for hypersurfaces

$$\vec{v} = -\frac{\vec{H}}{\mathbf{H}^2} \qquad \forall t \in (0, T],\ x \in \Gamma(t). \tag{3.5}$$

Recall that the mean curvature $\mathbf{H}$ is defined in (3.3). This equation was recently used by Huisken and Ilmanen [18] to prove the Riemannian Penrose inequality of general relativity. Clearly this flow is not defined when $\mathbf{H} = 0$ and the flow is studied for surfaces with positive mean curvature, $\mathbf{H} > 0$. Even starting from a smooth hypersurface, the inverse mean curvature flow creates singularities and a weak formulation is given in [18]. Here we only consider the smooth flows and follow the preceding arguments to obtain a characterization in terms of the square distance function η. The new ingredient needed is the following identity

$$-\frac{1}{x} = \inf_{\alpha \geq 0} \{\alpha^2 x - 2\alpha\} \qquad \text{for } x > 0. \tag{3.6}$$

We continue by deriving the level set equation for the inverse mean curvature flow. The equation will only be used to guess the stochastic target problem related to this flow. Since for this the flow sign convention for the curvature is important, we are forced to consider sublevel sets instead of level sets. So suppose that $\Gamma(t) = \{\phi \leq 0\}$ for some scalar function. We rewrite (3.5) in a scalar form by taking the inner product of (3.5) with $-\vec{n}$. The result is

$$\frac{\phi_t}{|D\phi|} = -\vec{n} \cdot \vec{v} = \frac{\vec{n} \cdot \vec{H}}{\mathbf{H}^2} = -\frac{\mathbf{H}}{\mathbf{H}^2} = -\frac{1}{\mathbf{H}}.$$

Since $\mathbf{H} > 0$, we may use the identity (3.6) together with (3.3) to arrive at

$$\begin{aligned}
\frac{\phi_t}{|D\phi|} &= \inf_{\alpha \geq 0} \{\alpha^2 \mathbf{H} - 2\alpha\} \\
&= \inf_{\alpha \geq 0} \left\{\alpha^2 \frac{F_1(D\phi, D^2\phi)}{|D\phi|} - 2\alpha\right\} \\
&= \frac{1}{|D\phi|} \inf\{\alpha^2 \operatorname{trace}[AP] - 2\alpha|D\phi| : \alpha \geq 0,\, P \in \mathcal{U}_1 \text{ and } PD\phi = 0\},
\end{aligned}$$

where F_1 and $\mathcal{U}_1$ are as defined in (1.5) and (1.6). Hence the level set equation for the inverse mean curvature flow is

$$\phi_t = F_{\text{inv}}(D\phi, D^2\phi),$$

where for $\xi \in \mathbb{R}^d$ and a symmetric matrix A,

$$F_{\text{inv}}(\xi, A) = \inf\{\alpha^2 \operatorname{trace}[AP] - 2\alpha|\xi| : \alpha \geq 0,\, P \in \mathcal{U}_1 \text{ and } P\xi = 0\}. \tag{3.7}$$

We continue by rewriting the level set equation for the inverse mean curvature flow in the form (1.1). An examination of the nonlinearity (3.7) suggests the choice $U = \mathcal{U}_1 \times [0, \infty)$. Note that any $P \in \mathcal{U}_1$ has the form $P = I_d - \vec{w} \otimes \vec{w}$ for some unit vector $\vec{w} \in S^{d-1}$. So we may use S^{d-1} instead of $\mathcal{U}_1$. We set $U = S^{d-1} \times [0, \infty)$, and for $(\vec{w}, \alpha) \in U$ set $\mu(\vec{w}, \alpha) = -2\alpha\vec{w}$ and $\sigma(\vec{w}, \alpha) = \sqrt{2}\alpha(I_d - \vec{w} \otimes \vec{w})$, so that

$$\mathcal{N}(\xi) = \{(\vec{w}, \alpha) \in U : \sigma(\vec{w}, \alpha)\xi = 0\} = \{(\vec{w}, \alpha) \in U : \vec{w} = \pm\xi\} \qquad \text{for } \xi \neq 0,$$

and for $\xi \neq 0$, the corresponding nonlinearity F has the form

$$\begin{aligned} F(x, \xi, A) &:= \inf_{u \in \mathcal{N}(\xi)} \left\{ \mu(x,u) \cdot \xi + \frac{1}{2} \operatorname{trace}[a(x,u)A] \right\} \\ &= \inf\{ -2\alpha \vec{w} \cdot \xi + 2\alpha^2 (I_d - \vec{w} \otimes \vec{w}) : \alpha \geq 0, \vec{w} = \pm \xi \} \\ &= \inf_{\alpha \geq 0} \left\{ -2\alpha |\xi| + 2\alpha^2 \left(I_d - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right\} \\ &= F_{\mathrm{inv}}(\xi, A). \end{aligned}$$

The following result is the analogue of (3.2) and it proves that smooth solutions of the inverse mean curvature flow satisfy the definition of the previous section.

LEMMA 3.1. *Let the boundary of an open set* $\Gamma(t) = \partial\Omega(t)$ *be a smooth inverse mean curvature flow on* $t \in [0, T]$ *with* $\mathbf{H} > 0$. *Then there are constants* C *and* δ *satisfying*

$$|\eta_t - F_{\mathrm{inv}}(D\eta, D^2\eta)| \leq C\eta \qquad \text{on } \{\eta < \delta\}. \tag{3.8}$$

PROOF. We proceed in several steps.

Step 1. Since $\Gamma(t)$ is an hypersurface we can use the signed distance function d in our calculations:

$$d(t,x) := \begin{cases} \rho(t,x) = \operatorname{distance}(x, \Gamma(t)), & \text{if } x \notin \Omega(t), \\ -\rho(t,x) = -\operatorname{distance}(x, \Gamma(t)), & \text{if } x \in \Omega(t). \end{cases}$$

Then, d is smooth in a tubular neighborhood and $d_t = -v$, $\Delta d = -\mathbf{H}$ on $\Gamma(t)$, where $v = \|\vec{v}\|$ and $\mathbf{H} = \|\vec{H}\|$ are respectively the velocity and the mean curvature of Γ. Hence on $\Gamma(t)$, d solves

$$d_t = -\frac{1}{\Delta d}. \tag{3.9}$$

Step 2. Set

$$\varphi(t,x) := \begin{cases} \eta_t(t,x) - F_{\mathrm{inv}}(D\eta, D^2\eta), & \text{if } x \notin \Gamma(t), \\ 0, & \text{if } x \in \Gamma(t). \end{cases}$$

In the following steps, we will show that φ is continuous and satisfies the inequality (3.8).

Step 3. The continuity of φ is clear away from the boundary. We now prove the continuity of φ by showing that limits from inside and outside of $\Omega(t)$ are both zero. Indeed, since $\Omega(t)$ is open, η_t, $D\eta$ and $D^2\eta$ are all identically zero in $\Omega(t)$ and so is φ. For $x \notin \overline{\Omega}(t)$ but sufficiently close to $\Gamma(t)$, we calculate that

$$F_{\mathrm{inv}}(D\eta, D^2\eta) = -\frac{d}{\Delta d}.$$

Hence

$$\eta_t - F_{\text{inv}}(D\eta, D^2\eta) = d\left[d_t + \frac{1}{\Delta d}\right].$$

So φ is Lipschitz continuous in a tubular neighborhood.

Step 4. Using the above equation we calculate that

$$D\varphi = Dd\left[d_t + \frac{1}{\Delta d}\right] + dD\left[d_t + \frac{1}{\Delta d}\right]$$

for $x \notin \bar{\Omega}(t)$. Hence, by (3.9),

$$\lim_{d \downarrow 0} D\varphi = Dd\left[d_t + \frac{1}{\Delta d}\right] = 0.$$

Therefore $D\varphi$ is Lipschitz continuous with $D\varphi = 0$ on $\Gamma(t)$. We obtain (3.8) by applying Lemma 2.2 to φ. $\square$

We are now ready to state the representation for the inverse mean curvature flow.

COROLLARY 3.2 (Representation for the inverse m.c.f.). *Suppose that* $\{\Gamma(t)\}_{t\in[0,T]}$ *are subsets of* $\mathbb{R}^d$ *with smooth codimension one boundaries. Further assume that* $\partial\Gamma(t)$ *satisfy* (3.5). *Then*,

$$\Gamma(t) = \{x \in \mathbb{R}^d : X_x^\nu(t) \in \Gamma(0) \text{ a.s. for some } \nu \in \mathcal{A}\},$$

where

$$dX_x^\nu(s) = -2\alpha(s)\vec{w}(s)\,ds + \alpha(s)\sqrt{2}\big(I_d - \vec{w}(s) \otimes \vec{w}(s)\big)\,dW(s),$$

$\mathcal{A}$ *is the set of all adapted maps* $\nu(\cdot) = (\alpha(\cdot), \vec{w}(\cdot))$ *with values in* $[0, \infty) \times S^{d-1}$.

PROOF. Let $\hat{w}$ be a smooth extension of the unit normal vector and let $\hat{\alpha}$ be a smooth extension of $1/\mathbf{H}$. Since in (3.6) $\alpha = 1/x$ is the minimizer, $\hat{\nu} = (\hat{\alpha}, \hat{w})$ satisfies the hypothesis of the theorem.

We continue by verifying assumption (2.3). Indeed by Lemma 2.1, for any $\vec{w}$,

$$\operatorname{trace}[D^2\eta(I_d - \vec{w} \otimes \vec{w})] \geq \Delta\eta - 1 \quad \text{and} \quad -\vec{w} \cdot D\eta \geq -|D\eta|.$$

Hence

$$\begin{aligned}
&F_{\text{inv}}(D\eta, D^2\eta)\\
&\quad = \inf_{\alpha>0}\{-2\alpha|D\eta| + \alpha^2[\Delta\eta - 1]\}\\
&\quad \leq \inf\{-2\alpha\vec{w} \cdot D\eta + \alpha^2 \operatorname{trace}[D^2\eta(I_d - \vec{w} \otimes \vec{w})] : \alpha > 0, \vec{w} \in S^{d-1}\}\\
&\quad := G_{\text{inv}}(D\eta, D^2\eta).
\end{aligned}$$

By definition, $G_{\text{inv}} \leq F_{\text{inv}}$. Hence (2.3) holds for the inverse mean curvature flow with $C = 0$. $\square$

3.4. *Gurtin's anisotropic flow.* In materials science geometric equations are often used to model the evolution of a crystal. Due to the underlying crystalline structure these equations often are anisotropic having different speeds for different directions. The following two dimensional equation is derived by Gurtin [16] as a simple model. In this model a planar curve moves according to

$$v = g(\theta)\kappa + c,$$

where as before v is the normal velocity, $\kappa = -\mathbf{H}$ is the curvature of the curve, θ is the angle between the normal and the x-axis, c is the energy difference between the two phases, and g is positive function related to the surface energy $S : S^1 \mapsto \mathbb{R}^1$. Indeed if we extend S to whole of R^2 as a homogeneous of degree one function,

$$S(x) = |x| S\left(\frac{x}{|x|}\right),$$

then for $\vec{n} = (\cos\theta, \sin\theta)$

$$D^2 S(\vec{n}) = g(\theta)[I_2 - \vec{n} \otimes \vec{n}].$$

The level set equation for this flow is

$$\phi_t = F_{\text{gurtin}}(\xi, A),$$

where with $\bar{\xi} := \xi/|\xi|$,

$$F_{\text{gurtin}}(\xi, A) = g(\bar{\xi})\,\text{trace}[(I_2 - \bar{\xi} \otimes \bar{\xi})A] - c|\xi|.$$

In material science, it is natural to assume that both g, c are positive and that g is even. Under these assumptions,

$$F_{\text{gurtin}}(\xi, A) = \inf\{g(\vec{n})\,\text{trace}[(I_2 - \vec{n} \otimes \vec{n})A] + c\vec{n}\xi : \vec{n} = \pm\bar{\xi}\}.$$

Hence the corresponding target problem has the coefficients

$$U = S^1, \qquad \mu(x, \vec{n}) = c\vec{n}, \qquad \sigma(x, \vec{n}) = \sqrt{2g(\vec{n})}[I_2 - \vec{n} \otimes \vec{n}].$$

A stochastic representation for the Gurtin flow can be proved exactly as in the other examples.

4. Weak solutions. In the preceding discussion we always assume the existence of a smooth solution and then proved that this solution is the reachability set of the target problem. However it is well known that most geometric flows, including the mean and the inverse mean curvature flows create singularities in finite time even if the initial data is very smooth. For instance, a dumbell shape in $\mathbb{R}^3$ flowing by its mean curvature would split into two pieces if the connecting tube is sufficiently thin [15].

The connection between the target problem and geometric flows is not restricted to smooth flows. Indeed under some assumptions, the authors in [24] proved that the characteristic function v of the complement of the reachability set is a viscosity solution of the dynamic programming equation (1.1); see the seminal paper of Crandall, Evans and Lions [8] or the book [12] for information on viscosity solutions. Afterwards a similar result for codimension one mean curvature flow was also obtained independently by Buckdahn, Cardaliaguet and Quincampoix [5].

Results of [24] apply to mean curvature flow but not to the inverse mean curvature flow. Indeed it is known that the solutions of the inverse mean curvature flow may not be continuous in time and therefore a careful definition of the controlled process is needed in order to obtain such a characterization. This is discussed in the forthcoming paper of the authors [25].

In view of this result of [24], it is possible to give a representation of the viscosity solutions of the level set equations of the general geometric equation. This characterization and several other properties of the weak solutions are discussed in [25]. Here we briefly discuss this characterization.

Let v be a continuous viscosity solution of the (1.1) with initial data

$$v(0,x) = g(x), \qquad x \in \mathbb{R}^d. \tag{4.1}$$

Then, it is known that any nondecreasing function $\Theta : \mathbb{R}^1 \mapsto \mathbb{R}^1$, $\Theta(v)$ is also a viscosity solution of (1.1) with initial data $\Theta(g)$. This is an immediate consequence of the geometric property (1.4); see, for instance, [7, 4].

For a real number τ set

$$\Theta_\tau(r) = \begin{cases} 1, & \text{if } r > \tau, \\ 0, & \text{if } r \le \tau. \end{cases}$$

Then Θ_τ is the characteristic function of the complement of the τ sublevel set of v: $\{v \le \tau\}$. In view of the results of [25], this construction suggests the following connection between the target problem and v.

For $\tau \in \mathbb{R}^1$ consider the control problem with target $\mathcal{T} = \{g \le \tau\}$ and define

$$\begin{aligned} V_\tau(t) &:= \text{reachability set at time } t \text{ with target } \mathcal{T}_\tau \\ &= \{x \in \mathbb{R}^d : g(X_x^\nu(t)) \le \tau \text{ a.s. for some } \nu \in \mathcal{A}\}. \end{aligned}$$

Clearly

$$V_\tau(t) \subset V_\rho(t), \qquad \tau \le \rho,$$

so that we may define

$$\hat{v}(t,x) := \inf\{\tau : x \in V_\tau(t)\}.$$

Provided that (1.1) has "good" uniqueness properties, this discussion implies that $\hat{v}$ is equal to the unique solution v of (1.1) with initial data g. This is proved in [25].

We close this section with a brief comment on the inverse mean curvature flow. Starting from a thick enough torus, the inverse mean curvature flow develops points with zero mean curvature. Since the flow is defined for mean convex surfaces, such points create singularities. Indeed in [18] solutions are allowed to jump in time should this situation arise. This is seen in the stochastic target problem as the possible singular behavior of the state process. Recall that for the inverse mean curvature flow, the controlled state process is

$$dX_x^\nu(s) = -2\alpha\vec{w}\,ds + \sqrt{2}\alpha(I_d - \vec{w}\otimes\vec{w})\,dW(s),$$

and there is no upper bound for the control parameter α. For smooth flows it is optimal to choose $\alpha = 1/\mathbf{H}$. However for singular surfaces the definition of α has to be revisited. This will be studied in the future.

5. Alternative definition. The nonlinearity and therefore the definition given involve the dimension of the flow. A definition using this information possibly gives a larger class of smooth functions. In this section we discuss such a definition. Let f^u and $\mathcal{N}$ be as in the Introduction. We define a subset of $\mathcal{N}$ as follows.

As before let $\mathcal{U}_k$ be the set of all projection matrices onto a $(d-k)$-dimensional plane. Given $\xi \in \mathbb{R}^d$ and a $d \times d$ symmetric matrix A, let $\mathcal{M}_k(\xi, A)$ be the set of all $P \in \mathcal{U}_k$ satisfying the following:

$$P\xi = 0 \quad \text{and} \quad \operatorname{trace}[AP] = \inf\{\operatorname{trace}[AQ] : Q \in \mathcal{U}_k,\ Q\xi = 0\}.$$

Finally we say that $u \in \mathcal{V}_k(x, \xi, A)$ if there exists $P \in \mathcal{M}_k(\xi, A)$ such that

$$P\sigma(x, u) = \sigma(x, u).$$

The nonlinearity corresponding to a codimension k flow is

$$G_k(x, \xi, A) = \inf\{f^u(x, \xi, A) : u \in \mathcal{V}_k(x, \xi, A)\}.$$

Note that $G_1 = \mathcal{N}$. We then define the smooth flow as in Section 2.3 but by using G_k instead of F. Then, the representation result Theorem 2.1 still holds. However we need to modify the proof. Instead of just using the square distance function η we use k smooth functions, ρ_i such that on $\Gamma(t)$ the vectors $D\rho_i$ form an orthogonal basis for the normal space.

The difference between F and G_k is this. If we want to characterize the flow with one level set function, then we are forced to use a nonnegative function and the zero level set as the solution Γ. In this case, generically almost all level sets of the auxiliary function are codimension one. Then it is not appropriate to use the dimension information and the corresponding level set function is F. If however, we use k auxiliary functions and represent the solution Γ as the intersection of any level sets of this functions, the resulting level set function is G_k.

Note that G_k is also geometric in the sense of (1.4).

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