SIAM J. CONTROL OPTIM.
Vol. 41, No. 2, pp. 404–424

STOCHASTIC TARGET PROBLEMS, DYNAMIC PROGRAMMING, AND VISCOSITY SOLUTIONS*

H. METE SONER† AND NIZAR TOUZI‡

Abstract. In this paper, we define and study a new class of optimal stochastic control problems which is closely related to the theory of backward SDEs and forward-backward SDEs. The controlled process (X^ν, Y^ν) takes values in $\mathbb{R}^d \times \mathbb{R}$ and a given initial data for $X^\nu(0)$. Then the control problem is to find the minimal initial data for Y^ν so that it reaches a stochastic target at a specified terminal time T. The main application is from financial mathematics, in which the process X^ν is related to stock price, Y^ν is the wealth process, and ν is the portfolio.

We introduce a new dynamic programming principle and prove that the value function of the stochastic target problem is a discontinuous viscosity solution of the associated dynamic programming equation. The boundary conditions are also shown to solve a first order variational inequality in the discontinuous viscosity sense. This provides a unique characterization of the value function which is the minimal initial data for Y^ν.

Key words. stochastic control, dynamic programming, discontinuous viscosity solutions, forward-backward SDEs

AMS subject classifications. Primary, 49J20, 60J60; Secondary, 49L20, 35K55

PII. S0363012900378863

1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T > 0$, and let $\{W(t), 0 \le t \le T\}$ be a d-dimensional Brownian motion whose P-completed natural filtration is denoted by $\mathbb{F}$. Given a control process $\nu = \{\nu(t), 0 \le t \le T\}$ with values in the control set $\mathcal{U}$, we consider the controlled process $Z_y^\nu = (X_y^\nu, Y_y^\nu) \in \mathbb{R}^d \times \mathbb{R}$ satisfying

$$dZ(t) = \alpha\left(t, Z(t), \nu(t)\right)\, dt + \beta\left(t, Z(t), \nu(t)\right)\, dW(t), \qquad 0 \le t < T, \tag{1.1}$$

together with the initial data $Z^\nu(0) = (X(0), y)$.

For a given real-valued function g, the stochastic target control problem is to minimize the initial data y while satisfying the random constraint $Y_y^\nu(T) \ge g(X_y^\nu(T))$ with probability one, i.e.,

$$v(0, X(0)) := \inf\left\{y \in \mathbb{R} \ : \ \exists\, \nu \in \mathcal{U}, \quad Y_y^\nu(T) \ge \ g\left(X_y^\nu(T)\right) \quad P-\text{a.s.}\right\},$$

which we call the stochastic target problem.

The chief goal of this paper is to obtain a characterization of the value function v as a discontinuous viscosity solution of an associated Hamilton–Jacobi–Bellman (HJB) second order PDE with suitable boundary conditions. We do not address the important uniqueness issue associated to the HJB equation in this paper. We simply refer to Crandall, Ishii, and Lions [5] for some general uniqueness results.

The main step in the derivation of the above-mentioned PDE characterization is a nonclassical dynamic programming principle. To the best of our knowledge, this

*Received by the editors September 26, 2000; accepted for publication (in revised form) November 29, 2001; published electronically June 18, 2002.

http://www.siam.org/journals/sicon/41-2/37886.html

†Department of Mathematics, Koç University, Rumelifeneri Yolu, Sariyer 80910, Istanbul, Turkey (msoner@ku.edu.tr). The work of this author was partially supported by National Science Foundation grant DMS-98-17525. Part of this work was completed during this author's visit to the Feza Gursey Institute for Basic Sciences in Istanbul.

‡CREST, 15 Bd Gabriel Peri, 92245 Malakoff, France (touzi@ensae.fr).

dynamic programming is new; it was only partially used by the authors in a previous paper [23].

This dynamic programming principle is closely related to the theory of viscosity solutions. In the derivation of the supersolution property of the HJB equation, the notion of viscosity solutions is only used to handle the lack of a priori regularity of the value function. However, the use of the notion of viscosity solutions seems necessary in order to derive the subsolution property from our dynamic programming principle, even if the value function were known to be smooth.

This study is mainly motivated by applications to financial mathematics. Indeed, a special specification of the coefficients α and β (see section 6) leads to the so-called superreplication problem; see, e.g., El Karoui and Quenez [11], Cvitanić and Karatzas [6], Broadie, Cvitanić, and Soner [4], Cvitanić, Pham, and Touzi [9], and Cvitanić and Ma [8].

In the financial mathematics literature, the superreplication problem is usually solved via convex duality. In this approach, a classical optimal control problem is derived by first applying the duality; see Jouini and Kallal [15], El Karoui and Quenez [11], Cvitanić and Karatzas [6], and Föllmer and Kramkov [13]. Then, one may use classical dynamic programming to obtain the PDE characterization of the value function v. However, this method cannot be applied to the general stochastic target problem because of the presence of the control ν in the diffusion part of the state process X^ν. The methodology developed in this paper precisely allows us to avoid this step and to obtain the PDE characterization directly from the initial (nonclassical) formulation of the problem without using the duality.

The stochastic target problem is also closely related to the theory of backward SDEs and forward-backward SDEs; see Antonelli [1], Cvitanić, Karatzas, and Soner [7], Hu and Peng [16], Ma, Protter, and Yong [18], Ma and Yong [19], Pardoux [20], and Pardoux and Tang [21]. Indeed, an alternative formulation of the problem is this: find a triple of $\mathbb{F}$-adapted processes (X, Y, ν) satisfying

$$(1.2)\quad (X,Y)\ \text{ solves }\ (1.1)\ \text{ with }\ \nu\in\mathcal{U}, X(0)\ \text{ fixed,}\quad Y(T)+A(T)=g(X(T))$$

for some nondecreasing $\mathbb{F}$-adapted process A with $A(0) = 0$ as well as the minimality condition

$$(\tilde{X},\tilde{Y},\tilde{\nu},\tilde{A})\ \text{ satisfies }\ (1.2) \Longrightarrow Y(.)\ \le\ \tilde{Y}(.)\quad P-\text{a.s.}$$

Notice that the nondecreasing process A is involved in the above definition to account for possible constraints on the control ν; see [7]. In financial applications, this connection has been observed by Cvitanić and Ma [8] and El Karoui, Peng, and Quenez [12].

The paper is organized as follows: the definition of the stochastic target problem is formulated in section 2. In section 3, we state the dynamic programming principle. Section 4 studies the HJB equation satisfied by the value function v in the discontinuous viscosity sense. In section 5, the terminal condition of the problem is characterized by a first order variational inequality again in the discontinuous viscosity sense. Finally, in section 6, we apply our results to the problem of superreplication under portfolio constraints in a large investor financial market.

2. Stochastic target problem. In this section, we define a nonstandard stochastic control problem.

Let $T > 0$ be the finite time horizon, and let $W = \{W(t),\ 0 \le t \le T\}$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

We denote by $\mathbb{F} = \{\mathcal{F}(t),\, 0 \le t \le T\}$ the P-augmentation of the filtration generated by W.

We assume that the control set U is a convex compact subset of $\mathbb{R}^d$ with a nonempty interior, and we denote by $\mathcal{U}$ the set of all progressively measurable processes $\nu = \{\nu(t),\, 0 \le t \le T\}$ with values in U.

The state process is defined as follows: given the initial datum $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}$, an initial time $t \in [0, T]$, and a control process $\nu \in \mathcal{U}$, let the controlled process $Z^\nu_{t,z} = (X^\nu_{t,x}, Y^\nu_{t,z})$ be the solution of the SDE

$$dX^\nu_{t,x}(u) \;=\; \mu\left(u, X^\nu_{t,x}(u), \nu(u)\right) du + \sigma^*\left(u, X^\nu_{t,x}(u), \nu(u)\right) dW(u), \qquad u \in (t, T),$$

$$dY^\nu_{t,x,y}(u) \;=\; b\left(u, Z^\nu_{t,z}(u), \nu(u)\right) du + a^*\left(u, Z^\nu_{t,z}(u), \nu(u)\right) dW(u), \qquad u \in (t, T),$$

with initial data

$$X^\nu_{t,x}(t) \;=\; x, \qquad\qquad Y^\nu_{t,x,y}(t) \;=\; y,$$

where M^* denotes the transpose of the matrix M, and μ, σ, b, a are bounded functions on $[0, T] \times \mathbb{R}^k \times U$ ($k = d$ or $d + 1$) satisfying the usual conditions in order for the process $Z^\nu_{t,z}$ to be well defined.

Throughout the paper, we assume that the matrix $\sigma(t, x, r)$ is invertible and the function

$$r \mapsto \sigma^{-1}(t, x, r)a(t, x, y, r)$$

is one to one for all (t, x, y). Let ψ be its inverse; i.e.,

$$\sigma^{-1}(t, x, r)a(t, x, y, r) \;=\; p \quad \Longleftrightarrow \quad r \;=\; \psi(t, x, y, p). \tag{2.1}$$

This is a crucial assumption which enables us to match the stochastic parts of the X and the Y processes by a judicial choice of the control process ν. Similar assumptions were also utilized in the backward-forward SDEs. See also Remark 2.2.

Now we are in a position to define the "stochastic target" control problem. Let g be a real-valued measurable function defined on $\mathbb{R}^d$. We shall denote by $\mathcal{E}pi(g) := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \ge g(x)\}$ the epigraph of g. Let

$$v(t, x) := \inf\left\{y \in \mathbb{R} \;:\; \exists\, \nu \in \mathcal{U}\,,\; Z^\nu_{t,x,y}(T) \;\in\; \mathcal{E}pi(g) \;\; P-\text{a.s.}\right\}. \tag{2.2}$$

In some cases, it is possible to find initial datum and a control so that $Y^\nu_{t,x,y}(T) = g(X^\nu_{t,x}(T))$. In that case, this problem is equivalent to a backward-forward SDE; see the discussion in our introduction. In particular, when $U = \mathbb{R}^d$, the corresponding backward-forward SDE has a solution (see, e.g., [21]), and it is equal to v. However, when the control set U is bounded, in general there is no solution of the backward-forward equation, and v is the natural generalization of the backward-forward SDE. An alternative generalization can be obtained by involving a nondecreasing process, as discussed in the introduction; see [7].

We conclude this section by introducing several sets to simplify the notation. Let

$$\mathcal{A}(t, x, y) := \left\{\nu \in \mathcal{U} \;:\; Z^\nu_{t,x,y}(T) \;\in\; \mathcal{E}pi(g) \;\; P-\text{a.s.}\right\}.$$

Note that $\mathcal{A}(t, x, y)$ may be empty for some initial datum (t, x, y). Next we define

$$\mathcal{Y}(t, x) := \{y \in \mathbb{R} \;:\; \mathcal{A}(t, x, y) \neq \emptyset\}.$$

Then the stochastic target problem can be written as

$$v(t,x) \;=\; \inf \mathcal{Y}(t,x) = \inf \{y \in \mathbb{R} \;:\; y \in \mathcal{Y}(t,x)\}\,.$$

Remark 2.1. The set $\mathcal{Y}(t,x)$ satisfies the following important property:

$$\text{for all } \; y \in \mathbb{R},\; y \in \mathcal{Y}(t,x) \implies [y,\infty) \subset \mathcal{Y}(t,x).$$

This follows from the facts that $X^{\nu}_{t,x}$ is independent of y and $Y^{\nu}_{t,x,y}(T)$ is nondecreasing in y.

Remark 2.2. A more general formulation of this problem, as discussed in our accompanying paper [24], is obtained by defining the *reachability set* of the deterministic target $\mathcal{E}pi(g)$:

$$V(t) := \left\{ z \in \mathbb{R}^{d+1} \;:\; Z^{\nu}_{t,z}(T) \in \mathcal{E}pi(g) \;\; P-\text{a.s. for some } \nu \in \mathcal{A} \right\}.$$

From the previous remark, the set $V(t)$ is "essentially" characterized as the epigraph of the scalar function $v(t,.)$. A standing assumption in [24] is

$$\mathcal{N}(t,z,p) := \left\{ \nu \in \mathbb{R}^d \;:\; [\sigma|a](t,z,\nu) \begin{bmatrix} p \\ -1 \end{bmatrix} = 0 \right\} \;\neq\; \emptyset;$$

i.e., since we wish to hit the deterministic target $\mathcal{E}pi(g)$ with probability one, the diffusion process has to degenerate along certain directions captured by the kernel $\mathcal{N}$. This degeneracy assumption is directly related to our condition (2.1).

3. Dynamic programming. In this section, we introduce a new dynamic programming equation for the stochastic target problem. This will allow us to characterize the value function of the stochastic target problem as a viscosity solution of a nonlinear PDE. For the classical stochastic control problem, this connection between the dynamic programming principle and the PDEs is well known (see, e.g., [14]). The chief goal of this paper is to develop the same tools for this nonstandard target control problem. Namely, we will formulate an appropriate dynamic programming principle and then derive the corresponding nonlinear PDE as a consequence of it.

A discussion of general dynamic programming of this type is the subject of an accompanying paper by the authors [24].

THEOREM 3.1. *Let $(t,x) \in [0,T] \times \mathbb{R}^d$.*

(DP1) *For any $y \in \mathbb{R}$, set $z := (x,y)$. Suppose that $\mathcal{A}(t,z) \neq \emptyset$. Then, for all $\nu \in \mathcal{A}(t,z)$ and a $[t,T]$-valued stopping time θ,*

$$Y^{\nu}_{t,x,y}(\theta) \geq v\left(\theta, X^{\nu}_{t,x}(\theta)\right) \qquad P-a.s.$$

(DP2) *Set $y^* := v(t,x)$. Let θ be an arbitrary $[t,T]$-valued stopping time. Then, for all $\nu \in \mathcal{U}$ and $\eta > 0$,*

$$P\left[Y^{\nu}_{t,x,y^*-\eta}(\theta) \;>\; v\left(\theta, X^{\nu}_{t,x}(\theta)\right) \right] < 1.$$

Proof. We provide only the main idea of the proof. We refer to [24] for the complete argument. Let $z = (x,y)$ and ν be as in the statement of (DP1). By the definition of $\mathcal{A}(t,z)$, $Z^{\nu}_{t,z}(T) \in \mathcal{E}pi(g)$. Since $Z^{\nu}_{t,z}(T) = Z^{\nu}_{\theta, Z^{\nu}_{t,z}(\theta)}(T)$, it follows that

$$\nu(\cdot) \in \mathcal{A}\left(\theta(w), Z^{\nu}_{t,z}(t+\theta(w))\right) \qquad\qquad \text{for } P \text{ almost every } w \in \Omega\,.$$

Then, again for P almost every $w \in \Omega$, $Y^{\nu}_{t,z}(\theta(w)) \in \mathcal{Y}\left(\theta(w), X^{\nu}_{t,x}(\theta(w))\right)$, and, by the definition of the value function, $v\left(\theta(w), X^{\nu}_{t,x}(\theta(w))\right) \le Y^{\nu}_{t,z}(\theta(w))$.

We prove (DP2) by contraposition. So, toward a contradiction, suppose that there exists a $[t,T)$-valued stopping time θ such that

$$Y^{\nu}_{t,x,y^*-\eta}(\theta) \; > \; v\left(\theta, X^{\nu}_{t,x}(\theta)\right) \qquad P-\text{a.s.}$$

In view of Remark 2.1, this proves that $Y^{\nu}_{t,x,y^*-\eta}(\theta) \in \mathcal{Y}\left(\theta, X^{\nu}_{t,x}(\theta)\right)$. Then there exists a control $\hat{\nu} \in \mathcal{U}$ such that

$$Y^{\hat{\nu}}_{\theta, Z^{\nu}_{t,x,y^*-\eta}(\theta)}(T) \ge g\big(X^{\hat{\nu}}_{\theta, X^{\nu}_{t,x}(\theta)}(T)\big) \qquad P-\text{a.s.}$$

Since the process $(X^{\hat{\nu}}_{\theta, X^{\nu}_{t,x}(\theta)}, Y^{\hat{\nu}}_{\theta, Z^{\nu}_{t,x,y^*-\eta}(\theta)})$ depends on $\hat{\nu}$ only through its realizations in the stochastic interval $[t,\theta]$, we may chose $\hat{\nu}$ so that $\hat{\nu} = \nu$ on $[t,\theta]$. (This is the difficult part of this proof.) Then $Z^{\hat{\nu}}_{\theta, Z^{\nu}_{t,x,y^*-\eta}(\theta)}(T) = Z^{\hat{\nu}}_{t,x,y^*-\eta}(T)$, and therefore $y^* - \eta \in \mathcal{Y}(t,x)$; hence $y^* - \eta \le v(t,x)$. Recall that, by definition, $y^* = v(t,x)$ and $\eta > 0$. $\square$

The dynamic programming principle stated in Theorem 3.1 does not require all of the assumptions made in the first section. Namely, the control set U does not need to be convex or compact, and the function $\sigma^{-1}(t,x,r)a(t,x,y,r)$ is not required to be one to one in the r variable.

For completeness, we mention that the statement of Theorem 3.1 is equivalent to the following, apparently stronger but more natural, dynamic programming principle.

COROLLARY 3.1. *For all $(t,x) \in [0,T) \times \mathbb{R}^d$ and a $[t,T]$-valued stopping time θ, we have*

$$v(t,x) = \inf\left\{y \in \mathbb{R} \;:\; \exists\, \nu \in \mathcal{U},\; Y^{\nu}_{t,x,y}(\theta) \;\ge\; v\left(\theta, X^{\nu}_{t,x}(\theta)\right) \;\; P-a.s.\right\}.$$

4. Viscosity property. In this section, we use the dynamic programming principle stated in Theorem 3.1 to prove that the value function of the stochastic target control problem (2.2) is a discontinuous viscosity solution to the corresponding dynamic programming equation.

Following the convention in the viscosity literature, let v_* (resp., v^*) be the lower (resp., upper) semicontinuous envelope of v; i.e.,

$$v_*(t,x) \;:=\; \liminf_{(t',x')\to(t,x)} v(t',x') \quad \text{and} \quad v^*(t,x) \;:=\; \limsup_{(t',x')\to(t,x)} v(t',x').$$

Let δ_U be the support function of the closed convex set U:

$$\delta_U(\zeta) := \sup_{\nu\in U}\, (\nu^*\zeta), \qquad \zeta \in \mathbb{R}^d.$$

We shall denote by $\tilde{U}$ the effective domain of δ_U and by $\tilde{U}_1$ the restriction of $\tilde{U}$ to the unit circle:

$$\tilde{U} \;=\; \left\{\zeta \in \mathbb{R}^d \;:\; \delta_U(\zeta) \in \mathbb{R}\right\} \text{ and } \tilde{U}_1 \;=\; \left\{\zeta \in \tilde{U} \;:\; |\zeta| = 1\right\}$$

so that $\tilde{U}$ is the closed cone generated by $\tilde{U}_1$. Under our assumptions, since U is a bounded subset of $\mathbb{R}^d$,

$$\tilde{U} \;=\; \mathbb{R}^d \text{ and } \tilde{U}_1 \;=\; \left\{\zeta \in \mathbb{R}^d \;:\; |\zeta| = 1\right\}.$$

Remark 4.1. The compactness of U is only needed in order to establish some results which require us to extract convergent subsequences from sequences in U. Therefore, many results contained in this paper hold for a general closed convex subset U. For this reason, we shall keep using the notation $\tilde{U}$ and $\tilde{U}_1$.

Remark 4.2. For later reference, note that the closed convex set U can be characterized in terms of $\tilde{U}$ (see, e.g., [22]):

$$\begin{aligned}\nu \in U \text{ iff } &\inf_{\zeta\in\tilde{U}} \left(\delta_U(\zeta) - \zeta^*\nu\right) \ \geq\ 0,\\ \text{iff } &\inf_{\zeta\in\tilde{U}_1} \left(\delta_U(\zeta) - \zeta^*\nu\right) \ \geq\ 0;\end{aligned}$$

the second characterization follows from the facts that $\tilde{U}$ is the closed cone generated by $\tilde{U}_1$ and δ_U is positively homogeneous.

Remark 4.3. We shall also use the following characterization of $\mathrm{int}(U)$ in terms of $\tilde{U}_1$:

$$\nu \in \mathrm{int}(U) \text{ iff } \inf_{\zeta\in\tilde{U}_1} \left(\delta_U(\zeta) - \zeta^*\nu\right) \ >\ 0.$$

To see this, suppose that the right-hand side infimum is zero. Then, for all $\varepsilon > 0$, there exists some $\zeta_0 \in \tilde{U}_1$ such that $0 \leq \delta_U(\zeta_0) - \zeta_0^*\nu \leq \varepsilon/2$. Then $\delta_U(\zeta_0) - \zeta_0^*(\nu + \varepsilon\zeta_0) < 0$, and therefore $\nu + \varepsilon\zeta_0 \notin U$ by the previous remark. Since $\varepsilon > 0$ is arbitrary, this proves that $\nu \notin \mathrm{int}(U)$. Conversely, suppose that $\ell := \inf_{\zeta\in\tilde{U}_1} \left(\delta_U(\zeta) - \zeta^*\nu\right) > 0$. Then, by the Cauchy–Schwarz inequality and the characterization of the previous remark, it is easily checked that the ball around ν with radius ℓ is included in U.

Remark 4.4. Let f be the function defined on $\mathbb{R}^d$ by

$$f(\nu) := \inf_{\zeta\in\tilde{U}_1} \left(\delta_U(\zeta) - \zeta^*\nu\right).$$

Then f is continuous. Indeed, since $\tilde{U}_1$ is a compact subset of $\mathbb{R}^d$, the infimum in the above definition of $f(\nu)$ is attained, say, at $\hat{\zeta}(\nu) \in \tilde{U}_1$. Then, for all $\nu, \nu' \in \mathbb{R}^d$,

$$f(\nu') \leq \delta_U(\hat{\zeta}(\nu)) - \hat{\zeta}(\nu)^*\nu + \hat{\zeta}(\nu)^*(\nu - \nu') \ =\ f(\nu) + \hat{\zeta}(\nu)^*(\nu - \nu') \ \leq\ f(\nu) + |\nu - \nu'|$$

by the Cauchy–Schwarz inequality. By symmetry, this proves that f is a contracting mapping.

Finally, we introduce the Dynkin second order differential operator associated to the process X^ν:

$$\mathcal{L}^\nu u(t,x) := \frac{\partial u}{\partial t}(t,x) + \mu(t,x,\nu)^* Du(t,x) + \frac{1}{2}\mathrm{Trace}\left(D^2u(t,x)\sigma^*(t,x,\nu)\sigma(t,x,\nu)\right),$$

where Du and D^2u denote, respectively, the gradient and the Hessian matrix of u with respect to the x variable.

THEOREM 4.1. *Assume that μ, σ, a, b are all bounded and satisfy the usual Lipschitz conditions and that v^*, v_* are finite everywhere. Further assume* (2.1) *and that U has a nonempty interior. Then the value function v of the stochastic target problem is a discontinuous viscosity solution of the equation on $[0,T) \times \mathbb{R}^d$,*

$$\min\left\{-\mathcal{L}^{\nu_0}u(t,x) + b\left(t,x,u(t,x),\nu_0\right);\ H(t,x,u(t,x),Du(t,x))\right\} = 0, \tag{4.1}$$

where

$$\nu_0(t,x) := \psi\left(t,x,u(t,x),Du(t,x)\right), \tag{4.2}$$

$$H(t,x,u(t,x),Du(t,x)) = \inf_{\zeta\in\tilde{U}_1} \left(\delta_U(\zeta) - \zeta^*\nu_0(t,x)\right); \tag{4.3}$$

i.e., v_* and v^* are, respectively, viscosity supersolution and subsolution of (4.1).

Remark 4.5. In view of Remark 4.2, $H \geq 0$ iff $\nu_0 \in U$. Since U has a nonempty interior, it follows from Remark 4.3 that $H > 0$ iff $\nu_0 \in \mathrm{int}(U)$.

The proof of Theorem 4.1 will be completed in the following two subsections. The supersolution part of the claim follows from (DP1) and a classical argument in the viscosity theory which is due to P.-L. Lions. We shall take advantage of the fact that the inequality in (DP1) is in the a.s. sense. This allows for suitable change of measure before taking expectations. The subsolution part is obtained from (DP2) by means of a contraposition argument.

The above result will be completed in Theorem 5.1 by the description of the boundary condition. The reader who is not interested in the technical proof of Theorem 4.1 can go directly to section 5.

4.1. Proof of the viscosity supersolution property. Fix $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$, and let φ be a $C^2([0,T]\times\mathbb{R}^d)$ function satisfying

$$0 \;=\; (v_*-\varphi)(t_0,x_0) = \min_{(t,x)\in[0,T)\times\mathbb{R}^d} (v_*-\varphi)\,.$$

Observe that $v \geq v_* \geq \varphi$ on $[0,T)\times\mathbb{R}^d$.

Step 1. Let $(t_n,x_n)_{n\geq 1}$ be a sequence in $[0,T)\times\mathbb{R}^d$ such that

$$(t_n,x_n) \;\to\; (t_0,x_0) \text{ and } v(t_n,x_n) \;\to\; v_*(t_0,x_0).$$

Set $y_n := v(t_n,x_n)+(1/n)$ and $z_n := (x_n,y_n)$. Then, by the definition of the stochastic target control problem, the set $\mathcal{A}(t_n,z_n)$ is not empty. Let ν_n be any element of $\mathcal{A}(t_n,z_n)$.

For any $[0,T-t_n)$-valued stopping time θ_n (to be chosen later), (DP1) yields

$$Y^{\nu_n}_{t_n,z_n}(t_n+\theta_n) \geq v\left(t_n+\theta_n, X_{t_n,x_n}(t_n+\theta_n)\right) \qquad P-\text{a.s.}$$

Set $\beta_n := y_n - \varphi(t_n,x_n)$. Since, as n tends to infinity, $y_n \to v_*(t_0,x_0)$ and $\varphi(t_n,x_n) \to \varphi(t_0,x_0) = v_*(t_0,x_0)$,

$$\beta_n \to 0\,.$$

Further, since $v \geq v_* \geq \varphi$, we have $v\left(t_n+\theta_n, X_{t_n,x_n}(t_n+\theta_n)\right) \geq \varphi(t_n+\theta_n, X_{t_n,x_n}(t_n+\theta_n))$ P-a.s. Then

$$\beta_n + \left[Y^{\nu_n}_{t_n,z_n}(t_n+\theta_n) - y_n\right] - \left[\varphi\left(t_n+\theta_n, X_{t_n,x_n}(t_n+\theta_n)\right) - \varphi(t_n,x_n)\right] \geq 0 \qquad P-\text{a.s.}$$

By Itô's lemma,

$$\begin{aligned} 0 \leq \beta_n \;+\; &\int_{t_n}^{t_n+\theta_n} \left[b\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \mathcal{L}^{\nu_n(s)}\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right]ds \\ +&\int_{t_n}^{t_n+\theta_n} \left[a\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right)\right. \\ &\qquad\qquad \left. - \sigma\left(s, X^{\nu_n}_{t_n,x_n}(s), \nu_n(s)\right) D\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right]^* dW(s). \end{aligned} \tag{4.4}$$

Step 2. For some large constant C, set

$$\theta_n := \inf\left\{s > t_n \ :\ |X^{\nu_n}_{t_n,x_n}(s)| \ge C\right\}.$$

Since U is bounded in $\mathbb{R}^d$ and $(t_n, x_n) \longrightarrow (t_0, x_0)$, one can easily show that

$$\liminf_{n\to\infty} t \wedge \theta_n \ > \ t_0 \text{ for all } t > t_0. \tag{4.5}$$

For $\xi \in \mathbb{R}$, we introduce the probability measure P^ξ_n equivalent to P defined by the density process

$$M^\xi_n(t) := \mathcal{E}\left(-\xi \int_{t_n}^{t\wedge\theta_n} (a - \sigma D\varphi)\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) dW(s)\right), \quad t \ge t_n\ ,$$

where $\mathcal{E}(.)$ is the Doléans–Dade exponential operator. We shall denote by E^ξ_n the conditional expectation with respect to $\mathcal{F}_{t_n}$ under P^ξ_n.

We take the conditional expectation with respect to $\mathcal{F}_{t_n}$ under P^ξ_n in (4.4). The result is

$$\begin{aligned}0 \le \beta_n &+ E^\xi_n\left[\int_{t_n}^{t_n+h\wedge\theta_n} \left(b\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \mathcal{L}^{\nu_n(s)}\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right) ds\right]\\ &- \xi\ E^\xi_n\left[\int_{t_n}^{t_n+h\wedge\theta_n} \left|a\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right)\right.\right.\\ &\qquad\qquad \left.\left. - \sigma\left(s, X^{\nu_n}_{t_n,x_n}(s), \nu_n(s)\right) D\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right|^2 ds\right]\end{aligned}$$

for all $h > 0$. We now consider two cases:

- Suppose that the set $\{n \ge 1 : \beta_n = 0\}$ is finite. Then there exists a subsequence, renamed $(\beta_n)_{n\ge1}$, such that $\beta_n \ne 0$ for all $n \ge 1$. Set $h_n = \sqrt{|\beta_n|}$ and $k_n := \theta_n \wedge (t_n + h_n)$.
- If the set $\{n \ge 1 : \beta_n = 0\}$ is not finite, then there exists a subsequence, renamed $(\beta_n)_{n\ge1}$, such that $\beta_n = 0$ for all $n \ge 1$. Set $h_n := n^{-1}$ and $k_n := \theta_n \wedge (t_n + h_n)$.

The final inequality still holds if we replace $t \wedge \theta_n$ with k_n. We then divide this inequality by h_n and send n to infinity by using (4.5), the dominated convergence theorem, and the right continuity of the filtration. The result is

$$\begin{aligned}0 \le \liminf_{n\to\infty} \frac{1}{h_n}\int_{t_n}^{t_n+h_n} &\left[b\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \mathcal{L}^{\nu_n(s)}\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right.\\ &\left. - \xi\left|a\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \sigma\left(s, X^{\nu_n}_{t_n,x_n}(s), \nu_n(s)\right) D\varphi\left(s, X^{\nu_n}_{t_n,x_n}(s)\right)\right|^2\right] ds.\end{aligned}$$

We continue by using the following result, whose proof is given after the proof of the supersolution property.

LEMMA 4.1. *Let $\psi : [0,T] \times \mathbb{R}^{d+1} \times U \to \mathbb{R}$ be locally Lipschitz in (t,z) uniformly in r. Then*

$$\frac{1}{h_n}\int_{t_n}^{t_n+h_n} \left[\psi\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) - \psi\left(t_0, z_0, \nu_n(s)\right)\right] ds \to 0 \qquad P - a.s.$$

along some subsequence.

In view of this lemma,

$$0 \le \liminf_{n\to\infty} \frac{1}{h_n} \int_{t_n}^{t_n+h_n} \Big[b\left(t_0, z_0, \nu_n(s)\right) - \mathcal{L}^{\nu_n(s)} \varphi\left(t_0, x_0\right) - \xi \left| a\left(t_0, z_0, \nu_n(s)\right) - \sigma\left(t_0, x_0, \nu_n(s)\right) D\varphi\left(t_0, x_0\right) \right|^2 \Big] ds.$$

Then, since $h_n^{-1} \int_{t_n}^{t_n+h_n} ds = 1$,

$$\frac{1}{h_n} \int_{t_n}^{t_n+h_n} \Big[b\left(t_0, z_0, \nu_n(s)\right) - \mathcal{L}^{\nu_n(s)} \varphi\left(t_0, x_0\right) - \xi \left| a\left(t_0, z_0, \nu_n(s)\right) - \sigma\left(t_0, x_0, \nu_n(s)\right) D\varphi\left(t_0, x_0\right) \right|^2 \Big] ds \in \bar{\text{co}}\mathcal{V}(t_0, z_0), \tag{4.6}$$

where $\bar{\text{co}}\mathcal{V}(t_0, z_0)$ is the closed convex hull of the set $\mathcal{V}(t_0, z_0)$ defined by

$$\mathcal{V}(t_0, z_0) := \left\{ b(t_0, z_0, \nu) - \mathcal{L}^{\nu}\varphi(t_0, x_0) - \xi \left| a(t_0, z_0, \nu) - \sigma(t_0, x_0, \nu) D\varphi(t_0, x_0) \right|^2 \;:\; \nu \in U \right\}.$$

Therefore, it follows from (4.6) that

$$\begin{aligned} 0 &\le \sup_{\phi \in \bar{\text{co}}\mathcal{V}} \phi \\ &= \sup_{\nu \in U} \left\{ \xi \left| -a(t_0, z_0, \nu) + \sigma(t_0, x_0, \nu) D\varphi(t_0, x_0) \right|^2 - \mathcal{L}^{\nu}\varphi(t_0, x_0) + b(t_0, z_0, \nu) \right\} \end{aligned} \tag{4.7}$$

for all $\xi \in \mathbb{R}$.

Step 3. For a large positive integer n, set $\xi = -n$. Since U is compact, the supremum in (4.7) is attained at some $\hat{\nu}_n \in U$, and

$$-n \left| a(t_0, z_0, \hat{\nu}_n) - \sigma(t_0, x_0, \hat{\nu}_n) D\varphi(t_0, x_0) \right|^2 - \mathcal{L}^{\hat{\nu}_n}\varphi(t_0, x_0) + b(t_0, z_0, \hat{\nu}_n) \ge 0.$$

By passing to a subsequence, we may assume that there exists $\hat{\nu} \in U$ such that $\hat{\nu}_n \to \nu_0$. Now let n to infinity in the last inequality to prove that

$$\left| a(t_0, z_0, \hat{\nu}_n) - \sigma(t_0, x_0, \hat{\nu}_n) D\varphi(t_0, x_0) \right|^2 \;\to\; 0 \tag{4.8}$$

and

$$-\mathcal{L}^{\nu_0}\varphi(t_0, x_0) + b(t_0, z_0, \nu_0) \ge 0. \tag{4.9}$$

In view of (4.8), we conclude that

$$\nu_0 = \psi\left(t_0, z_0, D\varphi(t_0, x_0)\right). \tag{4.10}$$

Since $\nu_0 \in U$, it follows from Remark 4.2 that

$$\inf_{\zeta \in \tilde{U}_1} \left(\delta_U(\zeta) - \zeta^* \nu_0 \right) \ge 0. \tag{4.11}$$

The supersolution property now follows from (4.9), (4.10), and (4.11). $\square$

Proof of Lemma 4.1. Since $\psi(t, z, r)$ is locally Lipschitz in (t, z) uniformly in r,

$$\begin{aligned} &\frac{1}{h_n} \int_{t_n}^{t_n+h_n} \left[\psi\left(s, Z_{t_n,z_n}^{\nu_n}(s), \nu_n(s)\right) - \psi\left(t_0, z_0, \nu_n(s)\right) \right] ds \\ &\qquad \le K \, \frac{1}{h_n} \int_{t_n}^{t_n+h_n} \left(|s - t_0| + \left| Z_{t_n,z_n}^{\nu_n}(s) - z_0 \right| \right) ds \\ &\qquad \le K \, \left(h_n + |t_n - t_0| + \sup_{t_n \le s \le t_n+h_n} \left| Z_{t_n,z_n}^{\nu_n}(s) - z_0 \right| \right) \end{aligned}$$

for some constant K. Thus, to complete the proof of this lemma, it suffices to show

$$\sup_{t_n \le s \le t_n+h_n} \left| Z^{\nu_n}_{t_n,z_n}(s) - z_0 \right| \longrightarrow 0 \qquad P-\text{a.s.}$$

along a subsequence. Set

$$\gamma(t,x,y,r) \ := \ \begin{pmatrix} \mu(t,x,r) \\ b(t,x,y,r) \end{pmatrix} \text{ and } \alpha(t,x,y,r) \ := \ \begin{pmatrix} \sigma^*(t,x,r) \\ a^*(t,x,y,r) \end{pmatrix}.$$

Functions α and γ inherit the pointwise bounds from μ, b, σ, and a. We directly calculate that, for $t_n \le s \le t_n + h_n$,

$$Z^{\nu_n}_{t_n,z_n}(s) - z_0 \le |z_n - z_0| + \|\gamma\|_\infty h_n + \left| \int_{t_n}^s \alpha\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) dW(s) \right|,$$

and, therefore,

$$\begin{aligned} \sup_{t_n \le s \le t_n+h_n} \left| Z^{\nu_n}_{t_n,z_n}(s) - z_0 \right| &\le |z_n - z_0| + \|\gamma\|_\infty h_n \\ &\quad + \sup_{t_n \le s \le t_n+h_n} \left| \int_{t_n}^s \alpha\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) dW(s) \right|. \end{aligned}$$

The first two terms on the right-hand side converge to zero. We estimate the third term by Doob's maximal inequality for submartingales.

The result is

$$\begin{aligned} E\left[\left(\sup_{t_n \le s \le t_n+h_n} \left| \int_{t_n}^s \alpha\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right) dW(s) \right| \right)^2 \right] & \\ \le 4\, E\left[\int_{t_n}^{t_n+h_n} \alpha\left(s, Z^{\nu_n}_{t_n,z_n}(s), \nu_n(s)\right)^2 ds \right] & \\ \le 4\, \|\alpha\|^2_\infty h_n. & \end{aligned}$$

This proves that

$$\sup_{t_n \le s \le t_n+h_n} \left| Z^{\nu_n}_{t_n,z_n}(s) - z_0 \right| \to 0 \quad \text{ in } \ L^2(P),$$

and, therefore, it also converges P-a.s. along some subsequence. □

4.2. Subsolution property. We start with a technical lemma which will be used both in the proof of the subsolution property and also in the next section on the characterization of the terminal data. We first introduce some notation. Given a smooth function $\varphi(t,x)$, we define the open subset of $[0,T] \times \mathbb{R}^d$:

$$\begin{aligned} \mathcal{M}_0(\varphi) := \Big\{ \ (t,x) \ : \ & \inf_{\zeta \in \tilde{U}_1} \left(\delta_U(\zeta) - \zeta^* \nu_0(t,x)\right) > 0 \text{ and} \\ & -\mathcal{L}^{\nu_0(t,x)} \varphi(t,x) + b\left(t,x,\varphi(t,x),\nu_0(t,x)\right) > 0 \ \Big\}, \\ = \{ \ (t,x) \ : \ & \nu_0(t,x) \in \operatorname{int}(U) \text{ and } - \mathcal{L}^{\nu_0(t,x)} \varphi(t,x) \\ & +b\left(t,x,\varphi(t,x),\nu_0(t,x)\right) > 0 \ \}, \end{aligned}$$

$$\text{where} \qquad \nu_0(t,x) \ = \ \psi(t,x,\varphi(t,x),D\varphi(t,x)) \ .$$

LEMMA 4.2. *Let φ be a smooth test function, and let $B = B_R(x_0)$ be the open ball around x_0 with radius $R > 0$. Suppose that there are $t_1 < t_2 \le T$ such that*

$$\mathrm{cl}(\mathcal{M}) \subset \mathcal{M}_0(\varphi), \text{ where } \mathcal{M} := (t_1, t_2) \times B.$$

Then

$$\sup_{\partial_p \mathcal{M}} (v - \varphi) = \max_{\mathrm{cl}(\mathcal{M})} (v^* - \varphi),$$

where $\partial_p \mathcal{M}$ is the parabolic boundary of $\mathcal{M}$; i.e., $\partial_p \mathcal{M} = ([t_1, t_2] \times \partial B) \cup \big(\{t_2\} \times \bar{B}\big)$.

Proof. We shall denote $\overline{\mathcal{M}} := \mathrm{cl}(\mathcal{M})$. Suppose, to the contrary, that

$$\max_{\overline{\mathcal{M}}} (v^* - \varphi) - \sup_{\partial_p \mathcal{M}} (v - \varphi) := 2\beta > 0,$$

and let us work toward a contradiction of (DP2).

Choose $(t_0, x_0) \in \mathcal{M}$ so that $(v - \varphi)(t_0, x_0) \ge -\beta + \max_{\overline{\mathcal{M}}} (v^* - \varphi)$, and

$$(v - \varphi)(t_0, x_0) \ge \beta + \sup_{\partial_p \mathcal{M}} (v - \varphi). \tag{4.12}$$

Step 1. In view of Remark 4.5, $\inf_{\zeta \in \tilde{U}_1} (\delta_U(\zeta) - \zeta^* \nu_0) > 0$ is equivalent to $\nu_0 \in \mathrm{int}(U)$. Set

$$\mathcal{N} := \big\{(t, x, y) \ : \ \hat{\nu}(t, x, y) \in \mathrm{int}(U) \ \text{ and } \ -\mathcal{L}^{\hat{\nu}(t,x,y)} \varphi(t, x) + b\,(t, x, y, \hat{\nu}(t, x, y)) > 0\big\},$$

where $\hat{\nu}(t, x, y) = \psi(t, x, y, D\varphi(t, x))$, and, for $\eta \ge 0$,

$$\mathcal{M}_\eta := \{\ (t, x) \ : \ \ (t, x, \varphi(t, x) - \eta) \in \mathcal{N}\ \}.$$

Note that this definition of $\mathcal{M}_0 := \mathcal{M}_0(\varphi)$ agrees with the previous definition. Moreover, in view of our hypothesis, for all sufficiently small η, $\overline{\mathcal{M}} \subset \mathcal{M}_\eta$. Fix $\eta \le \beta$ satisfying this inclusion.

Step 2. Let η be as in the previous step. Let (X_η, Y_η) be the solution of the state equation with initial data $X_\eta(t_0) = x_0$, $Y_\eta(t_0) = \varphi(t_0, x_0) - \eta$ and the control ν given in the feedback form

$$\nu(t, x) = \psi(t, x, \varphi(t, x) - \eta, D\varphi(t, x)).$$

Set

$$\nu(t) := \nu(t, X_\eta(t))$$

so that

$$(X_\eta, Y_\eta) = Z^\nu_{t_0, x_0, v(t_0, x_0) - \eta} = \big(X^\nu_{t_0, x_0}, Y^\nu_{t_0, x_0, v(t_0, x_0) - \eta}\big).$$

Set

$$\hat{Y}_\eta(t) := \varphi(t, X_\eta(t)) - \eta + (v - \varphi)(t_0, x_0),$$

and observe that $Y_\eta(0) = \hat{Y}_\eta(0) = v(t_0, x_0) - \eta$. In the next step, we will compare the processes Y_η and $\hat{Y}_\eta$.

Step 3. By Itô's rule,

$$d\hat{Y}_\eta(t) = \mathcal{L}^{\nu(t)}\varphi(t, X_\eta(t))dt + D\varphi(t, X_\eta(t)) \cdot \sigma^*(t, X_\eta(t), \nu(t))dW(t).$$

In view of (2.1) and the definition of $\nu(t)$,

$$D\varphi(t, X_\eta(t)) \cdot \sigma^*(t, X_\eta(t), \nu(t)) = a^*(t, X_\eta(t), \hat{Y}_\eta(t), \nu(t)).$$

Hence

$$d\hat{Y}_\eta(t) = \hat{b}(t)dt + a^*(t, X_\eta(t), \hat{Y}_\eta(t), \nu(t))dW(t),$$

where

$$\hat{b}(t) := \mathcal{L}^{\nu(t)}\varphi(t, X_\eta(t)).$$

Recall that Y_η solves the same SDE with a different drift term:

$$dY_\eta(t) = b(t)dt + a^*(t, X_\eta(t), Y_\eta(t), \nu(t))dW(t),$$

where $b(t) := b(t, X_\eta(t), Y_\eta(t), \nu(t))$.

Let θ be the stopping time

$$\theta := \inf\ \{\ s > 0\ :\ \ (t_0 + s, X_\eta(t_0 + s)) \notin \mathcal{M}\ \}.$$

Since $\mathcal{M}$ is an open set containing (t_0, x_0), the stopping time θ is positive a.s.

Now, from the definition of η, we have $\mathcal{M} \subset \mathcal{M}_\eta$. It follows that, for $t \in [t_0, t_0+\theta)$, $(t, X_\eta(t)) \in \mathcal{M}_\eta$ a.s.; i.e., $(t, X_\eta(t), \hat{Y}_\eta(t)) \in \mathcal{N}$ a.s. by definition of $\mathcal{M}_\eta$. Hence

$$b(t) > \mathcal{L}^{\nu(t)}\varphi(t, X_\eta(t))\ =\ \hat{b}(t), \qquad t \in [t_0, t_0 + \theta), \quad P - \text{a.s.}$$

Since $Y_\eta(0) = \hat{Y}_\eta(0) = v(t_0, x_0) - \eta$, it follows from stochastic comparison that

$$\hat{Y}_\eta(t) \le Y_\eta(t), \qquad t \in [t_0, t_0 + \theta), \quad P - \text{a.s.}$$

Step 4. We now proceed to contradict (DP2). First, observe that, by continuity of the process X_η, $(t_0 + \theta, X_\eta(t_0 + \theta)) \in \partial_p\mathcal{M}$. Also, from inequality (4.12), we have $v \le \varphi - \beta + (v - \varphi)(t_0, x_0)$ on $\partial_p\mathcal{M}$. Therefore,

$$\begin{aligned} Y_\eta(t_0 + \theta) - v(t_0 + \theta, X_\eta(t_0 + \theta)) &\ge \beta + Y_\eta(t_0 + \theta) - \varphi(t_0 + \theta, X_\eta(t_0 + \theta)) \\ &\qquad + (v - \varphi)(t_0, x_0) \\ &= (\beta - \eta) + Y_\eta(t_0 + \theta) - \hat{Y}_\eta(t_0 + \theta) \\ &\ge \beta - \eta\ \ge\ 0 \end{aligned}$$

from step 3. By (4.12) and the definition of (X_η, Y_η), we have $Y_\eta = Y^\nu_{t_0,x_0,v(t_0,x_0)-\eta}$ and $X_\eta = X^\nu_{t_0,x_0}$. Then the previous inequality contradicts (DP2). ∎

Proof of the subsolution property. Fix $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$, and let φ be a $C^2([0, T] \times \mathbb{R}^d)$ function satisfying

$$(v^* - \varphi)(t_0, x_0) = (\text{strict}) \max_{(t,x)\in[0,T)\times\mathbb{R}^d} (v^* - \varphi).$$

Set $z_0 := (x_0, \varphi(t_0, x_0))$. Let $\mathcal{M}_0 := \mathcal{M}_0(\varphi)$ be as in the previous lemma. Since (t_0, x_0) is a *strict* maximizer of $(v^* - \varphi)$ and since $\mathcal{M}_0$ is an open set, by the previous lemma we conclude that $(x_0, y_0) \notin \mathcal{M}_0$. Then, by the definition of $\mathcal{M}_0$,

$$\min\left\{ \inf_{\zeta\in\tilde{U}_1} (\delta_U(\zeta) - \zeta^*\hat{\nu}(t_0, z_0))\,,\ -\mathcal{L}^{\hat{\nu}(t_0,z_0)}\varphi(t_0, x_0) + b\,(t_0, z_0, \hat{\nu}(t_0, z_0)) \right\} \le 0,$$

and therefore v^* is a viscosity subsolution. ∎

5. Terminal condition. To characterize the value function as the unique solution of the dynamic programming equation, we need to specify the terminal data. The definition of the value function implies that

$$v(T,x) = g(x), \qquad x \in \mathbb{R}. \tag{5.1}$$

However, it is known that

$$\underline{G}(x) := \liminf_{t\uparrow T,\ x'\to x} \quad v(t,x')$$

may be strictly larger than $g(x)$ (see, for instance, [4] and Lemma 5.1 below).

In this section, we will characterize $\underline{G}$ as the viscosity supersolution of a first order PDE. We will also study

$$\overline{G}(x) := \limsup_{t\uparrow T,\ x'\to x} \quad v(t,x')$$

and prove that $\overline{G}$ is a viscosity subsolution of the same equation. More precisely, we have the following theorem.

THEOREM 5.1. *Let the assumptions of Theorem* 4.1 *hold, and assume that* $\underline{G}$ *and* $\overline{G}$ *are finite for every* $x \in \mathbb{R}^d$. *Suppose, further, that* $(g_*)^* \geq g$. *Then* $\overline{G}$ *and* $\underline{G}$, *respectively, are viscosity super- and subsolutions of the following equations on* $\mathbb{R}^d$*:*

$$\min\{\underline{G}(x) - g_*(x);\ H\left(T, x, \underline{G}(x), D\underline{G}(x)\right)\} \geq 0,$$
$$\min\{\overline{G}(x) - g^*(x);\ H\left(T, x, \overline{G}(x), D\overline{G}(x)\right)\} \leq 0.$$

In most cases, since a subsolution is not greater than a supersolution, this characterization implies that $\overline{G} \leq \underline{G}$ and therefore that $\overline{G} = \underline{G}$. In the next section, we provide examples for which this holds, and we will also compute $G := \overline{G} = \underline{G}$ explicitly in those examples.

The rest of this section is devoted to the proof of Theorem 5.1.

Remark 5.1. In the definition of $\overline{G}$, we may replace v by v^*:

$$\overline{G}(x) = \limsup_{t\uparrow T,\ x'\to x} \quad v^*(t,x').$$

Similarly,

$$\underline{G}(x) := \liminf_{t\uparrow T,\ x'\to x} \quad v_*(t,x').$$

We start with the following lemma.

LEMMA 5.1. *Suppose that* $\underline{G}(x)$ *and* $\overline{G}(x)$ *are finite for every* $x \in \mathbb{R}^d$. *Then*

$$\underline{G}(x) \ \geq \ g_*(x) \ \textit{for all} \ \ x \in \mathbb{R}^d.$$

Proof. Take a sequence $(x_n, t_n) \to (x, T)$ with $t_n < T$. Set $y_n := v(t_n, x_n) + (1/n)$. For each n, there exists a control $\nu_n \in \mathcal{U}$ satisfying

$$Y^{\nu_n}_{t_n,x_n,y_n}(T) \geq g\left(X^{\nu_n}_{t_n,x_n}(T)\right) \qquad \mathrm{P-a.s.}$$

Since a and b are bounded,

$$E\left[\ Y^{\nu_n}_{t_n,x_n,y_n}(T)\ \right] \leq y_n + \|b\|_\infty (T - t_n) \ = \ v(t_n, x_n) + \frac{1}{n} + \|b\|_\infty (T - t_n).$$

We continue by using the following claim, whose proof will be provided later:

$$\left\{ Y^{\nu_n}_{t_n,x_n,y_n}(T)\ ,\ \ n \geq 0 \right\} \text{ is uniformly integrable.} \tag{5.2}$$

Then

$$\begin{aligned}
\liminf_{n\to\infty} v(t_n,x_n) &\geq \liminf_{n\to\infty} E\left[\ Y^{\nu_n}_{t_n,x_n,y_n}(T)\ \right] \\
&= E\left[\ \liminf_{n\to\infty} Y^{\nu_n}_{t_n,x_n,y_n}(T)\ \right] \\
&\geq E\left[\ \liminf_{n\to\infty} g\left(X^{\nu_n}_{t_n,x_n}(T)\right)\ \right].
\end{aligned}$$

Since U is compact and (t_n,x_n) converges to (T,x), $X^{\nu_n}_{t_n,x_n}(T)$ approaches x as n tends to infinity. The required result then follows from the definition of the lower semicontinuous envelope g_* of g.

It remains to prove claim (5.2). Since b is bounded,

$$\begin{aligned}
\left|Y^{\nu_n}_{t_n,x_n,y_n}(T)\right| &\leq |y_n| + (T-t_n)\|b\|_\infty + \left|\int_{t_n}^T a\left(u, Z^{\nu_n}_{t_n,x_n,y_n}(u), \nu_n(u)\right)^* dW(u)\right| \\
&\leq T\|b\|_\infty + |v(t_n,x_n)| + \left|\int_{t_n}^T a\left(u, Z^{\nu_n}_{t_n,x_n,y_n}(u), \nu_n(u)\right)^* dW(u)\right|.
\end{aligned}$$

Now observe that $\limsup v(t_n,x_n) \leq \limsup v^*(t_n,x_n) \leq \overline{G}(x)$ and $\liminf v(t_n,x_n) \geq \liminf v_*(t_n,x_n) \geq \underline{G}(x)$. This proves that the sequence $v(t_n,x_n)$ is bounded. In order to complete the proof, it suffices to show that the sequence

$$\left\{ U_n := \int_{t_n}^T a\left(u, Z^{\nu_n}_{t_n,x_n,y_n}(u), \nu_n(u)\right)^* dW(u),\ n \geq 0 \right\}$$

is uniformly integrable. Since a is bounded,

$$\sup_{n\geq 0} E\left[U_n^2\right] \leq \sup_{n\geq 0} (T-t_n)\|a^*a\|_\infty\ \leq\ T\|a^*a\|_\infty.$$

Hence $\{U_n,\, n \geq 0\}$ is bounded in L^2, and, therefore, it is uniformly integrable. ☐

Next, we will show that $\underline{G}$ is a viscosity supersolution of $H \geq 0$, where H is as in (4.3).

LEMMA 5.2. *Suppose that $\underline{G}(x)$ is finite for every $x \in \mathbb{R}^d$. Then $\underline{G}$ is a viscosity supersolution of*

$$H(T,x,\underline{G}(x),D\underline{G}(x)) \geq 0.$$

Proof. By definition, $\underline{G}$ is lower semicontinuous. Let f be a $C^2(\mathbb{R}^d)$ function satisfying

$$0\ =\ (\underline{G}-f)(x_0) = \min_{x\in\mathbb{R}^d}(\underline{G}-f)$$

at some $x_0 \in \mathbb{R}^d$. Observe that $\underline{G} \geq f$ on $\mathbb{R}^d$.

Step 1. In view of Remark 5.1, there exists a sequence (s_n,ξ_n) converging to (T,x_0) such that $s_n < T$ and

$$\lim_{n\to\infty}\ v_*(s_n,\xi_n) = \underline{G}(x_0).$$

For a positive integer n, consider the auxiliary test function

$$\varphi_n(t,x) := f(x) - \frac{1}{2}|x-x_0|^2 + \frac{T-t}{(T-s_n)^2}.$$

Let $B := B_1(x_0)$ be the unit open ball in $\mathbb{R}^d$ centered at x_0. Choose $(t_n, x_n) \in [s_n, T] \times \bar{B}$, which minimizes the difference $v_* - \varphi_n$ on $[s_n, T] \times \bar{B}$.

Step 2. We claim that, for sufficiently large n, $t_n < T$, and x_n converges to x_0. Indeed, for sufficiently large n,

$$(v_* - \varphi_n)(s_n, \xi_n) \leq -\frac{1}{2(T-s_n)}.$$

On the other hand, for any $x \in \bar{B}$,

$$(v_* - \varphi_n)(T, x) = \underline{G}(x) - f(x) + \frac{1}{2}|x-x_0|^2 \geq \underline{G}(x) - f(x) \geq 0.$$

Comparing the two inequalities leads us to conclude that $t_n < T$ for large n. Suppose that, on a subsequence, x_n converges to x^*. Since $t_n \geq s_n$ and (t_n, x_n) minimizes the difference $(v_* - \varphi_n)$,

$$\begin{aligned}
&(\underline{G} - f)(x^*) - (\underline{G} - f)(x_0) \\
&\qquad \leq \liminf_{n\to\infty} (v_* - \varphi_n)(t_n, x_n) - (v_* - \varphi_n)(s_n, \xi_n) - \frac{1}{2}|x_n - x_0|^2 \\
&\qquad \leq \limsup_{n\to\infty} (v_* - \varphi_n)(t_n, x_n) - (v_* - \varphi_n)(s_n, \xi_n) - \frac{1}{2}|x_n - x_0|^2 \\
&\qquad \leq -\frac{1}{2}|x^* - x_0|^2.
\end{aligned}$$

Since x_0 minimizes the difference $\underline{G} - f$,

$$0 \leq (\underline{G} - f)(x^*) - (\underline{G} - f)(x_0) \leq -\frac{1}{2}|x^* - x_0|^2.$$

Hence $x^* = x_0$. The above argument also proves that

$$\begin{aligned}
0 &= \lim_{n\to\infty} (v_* - \varphi_n)(t_n, x_n) - (v_* - \varphi_n)(s_n, \xi_n) \\
&= -\underline{G}(x_0) + \lim_{n\to\infty} v_*(t_n, x_n) + \frac{(T-s_n) - (T-t_n)}{(T-s_n)^2} \\
&\geq -\underline{G}(x_0) + \limsup_{n\to\infty} v_*(t_n, x_n).
\end{aligned}$$

This proves that $\limsup_{n\to\infty} v_*(t_n, x_n) \leq \underline{G}(x_0)$. Since $\limsup v_*(t_n, x_n) \geq \liminf v_*(t_n, x_n) \geq \underline{G}(x_0)$, by definition of $\underline{G}$, this proves that

$$\lim_{n\to\infty} v_*(t_n, x_n) = \underline{G}(x_0). \tag{5.3}$$

This implies that, for all sufficiently large n, (t_n, x_n) is a local minimizer of the difference $(v_* - \varphi_n)$. In view of the general theory of viscosity solutions (see, for instance, Fleming and Soner [14]), the viscosity property of v_* holds at (t_n, x_n).

Step 3. We now use the viscosity property of v_* in $[0, T) \times \mathbb{R}^d$: for every n,

$$H(t_n, x_n, v_*(t_n, x_n), D\varphi_n(x_n, t_n)) \geq 0.$$

Note that $D\varphi_n(x_n,t_n) = Df(x_n,t_n) - (x_n - x_0)$, and recall that H is continuous; see Remark 4.4. Since (t_n,x_n) tends to (T,x_0), (5.3) implies that

$$H(T,x_0,\underline{G}(x_0),Df(x_0)) \geq 0. \qquad \square$$

These results imply that $\underline{G}$ is a viscosity supersolution of

$$\min \left\{\underline{G}(x) - g_*(x);\; H(T,x,\underline{G}(x),D\underline{G}(x))\right\} \geq 0, \tag{5.4}$$

proving the first part of Theorem 5.1. The following result concludes the proof of the theorem.

LEMMA 5.3. *Suppose that $\underline{G}(x)$ and $\overline{G}(x)$ are finite for every $x \in \mathbb{R}^d$ and that $(g_*)^* \geq g$. Then $\overline{G}$ is a viscosity subsolution on $\mathbb{R}^d$ of*

$$\min \left\{\overline{G}(x) - g^*(x);\; H(T,x,\overline{G}(x),D\overline{G}(x))\right\} \leq 0.$$

Proof. By definition, $\overline{G}$ is upper semicontinuous. Let $x_0 \in \mathbb{R}^d$ and $f \in C^2(\mathbb{R}^d)$ satisfy

$$0 \;=\; (\overline{G} - f)(x_0) = \max_{x \in \mathbb{R}^d}(\overline{G} - f).$$

We need to show that, if $\overline{G}(x_0) > g^*(x_0)$, then

$$H(T,x_0,\overline{G}(x_0),D\overline{G}(x_0)) \leq 0. \tag{5.5}$$

So we assume that

$$\overline{G}(x_0) > g^*(x_0). \tag{5.6}$$

For a positive integer n, set

$$s_n := T - \frac{1}{n^2},$$

and consider the auxiliary test function

$$\varphi_n(t,x) := f(x) + \frac{1}{2}|x - x_0|^2 + n(T-t), \qquad (t,x) \in [s_n,T] \times \mathbb{R}^d.$$

In order to obtain the required result, we shall first prove that the test function φ_n does not satisfy the condition of Lemma 4.2 on $[s_n,T] \times B_R(x_0)$ for some $R > 0$, and then we shall pass to the limit as $n \to \infty$.

Step 1. By definition, $\overline{G} \geq \underline{G}$. From Lemma 5.1, this provides $\overline{G} \geq g_*$ and then $\overline{G} \geq (g_*)^*$ by upper semicontinuity of $\overline{G}$. Hence, by assumption of the lemma,

$$\overline{G} \geq g. \tag{5.7}$$

This proves that $(v - \varphi_n)(T,x) = (g - f)(x) - |x - x_0|^2/2 \leq (\overline{G} - f)(x) \leq 0$ by definition of the test function f. Then, for all $R > 0$,

$$\sup_{B_R(x_0)} (v - \varphi_n)(T,.) \leq 0.$$

Now suppose that there exists a subsequence of (φ_n), still denoted (φ_n), such that

$$\lim_{n\to\infty} \sup_{B_R(x_0)} (v - \varphi_n)(T,.) = 0,$$

and let us work toward a contradiction. For each n, let $(x_n^k)_k$ be a maximizing sequence of $(v-\varphi_n)(T,.)$ on $B_R(x_0)$; i.e.,

$$\lim_{n\to\infty}\lim_{k\to\infty}(v-\varphi_n)(T,x_n^k)=0.$$

Then it follows from (5.7) that $(v-\varphi_n)(T,x_n^k)\le -|x_n^k-x_0|^2/2$, which provides

$$\lim_{n\to\infty}\lim_{k\to\infty}x_n^k=x_0.$$

Therefore,

$$\begin{aligned}0 \;&=\; \lim_{n\to\infty}\lim_{k\to\infty}(v-\varphi_n)(T,x_n^k) \;=\; \lim_{n\to\infty}\lim_{k\to\infty}g(x_n^k)-f(x_0)\\ &\le\; \limsup_{x\to x_0}g(x)-f(x_0) \;=\; (g^*-f)(x_0) \;<\; (G-f)(x_0)\end{aligned}$$

by (5.6). Since $(G-f)(x_0)=0$, this cannot happen since $(G-f)(x_0)=0$. The consequence of this is

$$\limsup_{n\to\infty}\sup_{B_R(x_0)}(v-\varphi_n)(T,.)<0 \qquad \text{for all } R>0. \tag{5.8}$$

Step 2. Let $(t_n,x_n)_n$ be a maximizing sequence of $(v^*-\varphi_n)$ on $[s_n,T]\times\partial B_R(x_0)$. Then, since $T-t_n\le T-s_n=n^{-2}$,

$$\limsup_{n\to\infty}\sup_{[s_n,T]\times\partial B_R(x_0)}(v^*-\varphi_n)\le\limsup_{n\to\infty}\left(v^*(t_n,x_n)-f(x_n)\right)-\frac{1}{2}R^2.$$

Since $t_n\longrightarrow T$ and, after passing to a subsequence, $x_n\longrightarrow x^*$ for some $x^*\in\partial B_R(x_0)$, we get

$$\limsup_{n\to\infty}\sup_{[s_n,T]\times\partial B_R(x_0)}(v^*-\varphi_n)\le(\overline{G}-f)(x^*)-\frac{1}{2}R^2 \;\le\; -\frac{1}{2}R^2.$$

This, together with (5.8), implies that, for all $R>0$, there exists $n(R)$ such that, for all $n>n(R)$,

$$\max\{\;(v-\varphi_n)\;:\;\;\partial_p\left((s_n,T)\times B_R(x_0)\right)\;\}<0\;=\;(v^*-\varphi_n)(T,x_0).$$

Hence it follows from Lemma 4.2 that

$$(s_n,T)\times B_R(x_0)\text{ is not a subset of }\mathcal{M}_0(\varphi_n)\quad\text{for all }\; n>n(R). \tag{5.9}$$

Step 3. Observe that, for all $\nu\in U$ and (t,x,y),

$$-\mathcal{L}^\nu\varphi_n(t,x)\;=\;n-\mathcal{L}^\nu f(x)-\mu(t,x,\nu)^*(x-x_0)-\frac{1}{2}\mathrm{Trace}[\sigma^*\sigma](t,x,\nu)>b(t,x,y,\nu),$$

provided that n is sufficiently large. Then, for large n,

$$\begin{aligned}&\mathcal{M}_0(\varphi_n)\cap\left((s_n,T)\times B_R(x_0)\right)\\ &\qquad=\{\;(t,x)\in(s_n,T)\times B_R(x_0)\;:\;\;H(t,x,\varphi_n(t,x),D\varphi_n(t,x))>0\;\}.\end{aligned}$$

In view of this, it follows from (5.9) that there is a sequence (t_n,x_n) converging to (T,x_0) such that

$$H(t_n,x_n,\varphi_n(t_n,x_n),D\varphi_n(t_n,x_n))\le 0.$$

We now let n tend to infinity to obtain (5.5). $\square$

6. Application: Superreplication problem in finance. Consider a financial market consisting of

- a nonrisky asset with price process $\tilde{X}^0$ normalized to unity,
- a risky asset $\tilde{X}$ defined by a positive price process with dynamics described by an SDE.

A trading strategy is an $\mathbb{F}$-adapted process $\nu = \{\nu(t),\, 0 \le t \le T\}$ valued in the closed interval $[-\ell, u]$ with $\ell, u \in [0,\infty)$ and $\ell + u > 0$. At each time $t \in [0,T]$, $\nu(t)$ represents the proportion of wealth invested in the risky asset $\tilde{X}$. The set of all trading strategies is denoted by $\mathcal{U}$.

Given an initial capital $\tilde{y} > 0$ and a trading strategy ν, the wealth process $\tilde{Y}$ is defined by

$$\tilde{Y}^{\nu}_{\tilde{y}}(0) \;=\; \tilde{y} \text{ and } d\tilde{Y}^{\nu}_{\tilde{y}}(t) \;=\; \tilde{Y}^{\nu}_{\tilde{y}}(t)\nu(t)\,\frac{d\tilde{X}(t)}{\tilde{X}(t)}.$$

We shall consider a "large investor" model in which the dynamics of the risky asset price process may be affected by trading strategies. Namely, given a trading strategy $\nu \in \mathcal{U}$,

$$\begin{aligned} \tilde{X}^{\nu}(0) = e^{X^{\nu}(0)} \;=\; e^{X(0)}, \quad \tilde{X}^{\nu}(t) \;=\; e^{X^{\nu}(t)}, \\ dX^{\nu}(t) = \mu\left(t, X^{\nu}(t), \nu(t)\right) dt + \sigma\left(t, X^{\nu}(t), \nu(t)\right) dW(t), \end{aligned}$$

where W is a one-dimensional Brownian motion. Define the log-wealth process:

$$Y^{\nu}_{y}(0) \;=\; y \;:=\; \ln\left(\tilde{y}\right) \text{ and } Y^{\nu}_{y}(t) \;=\; \ln\left(\tilde{Y}^{\nu}_{\tilde{y}}(t)\right).$$

Then a direct application of Itô's lemma provides

$$dY^{\nu}_{y}(t) = b\left(t, X^{\nu}(t), \nu(t)\right) dt + \nu(t)\sigma\left(t, X^{\nu}(t), \nu(t)\right) dW(t),$$

where

$$b(t,x,r) = r\left(\mu + \frac{1}{2}\sigma^2\right)(t,x,r) - \frac{1}{2}r^2\sigma^2(t,x,r).$$

Let f be a positive function defined on $[0,\infty)$. The superreplication problem is defined by

$$\tilde{v}(0, X(0)) := \inf\left\{\tilde{y} > 0 \;:\; \exists\, \nu \in \mathcal{U},\; \tilde{Y}^{\nu}_{\tilde{y}}(T) \ge f\left(X^{\nu}(T)\right) \quad P-\text{a.s.}\right\}.$$

Here $f\left(X^{\nu}(T)\right)$ is a contingent claim. The value function of the above superreplication problem is then the minimal initial capital which allows the seller of the contingent claim to face the promised payoff $f\left(X^{\nu}(T)\right)$ through some trading strategy $\nu \in \mathcal{U}$.

To see that the superreplication problem belongs to the general class of stochastic target problems studied in the previous sections, we introduce

$$v(0, X(0)) \;:=\; \ln \tilde{v}(0, X(0)) \text{ and } g \;:=\; \ln f.$$

Then

$$v(0, X(0)) := \inf\left\{y \in \mathbb{R} \;:\; \exists\, \nu \in \mathcal{U},\; Y^{\nu}_{y}(T) \ge g\left(X^{\nu}(T)\right) \quad P-\text{a.s.}\right\}.$$

Remark 6.1. Assume that function g is bounded. Then the value function v is bounded. Using the notation of previous sections, we also have that v_*, v^*, $\underline{G}$, and $\overline{G}$ are bounded functions.

Let us introduce the support function of the interval $[-\ell^{-1}, u^{-1}]$:

$$h(p) := u^{-1}p^+ \; + \; \ell^{-1}p^-,$$

with the convention $1/0 = +\infty$, and the usual notation $p^+ := p \vee 0$ and $p^- := (-p)^+$. Observe that h is a mapping from $\mathbb{R}$ into $\mathbb{R} \cup \{+\infty\}$. We also denote by $\overline{F}$ and $\underline{F}$ the functions

$$\overline{F} \; := \; e^{\overline{G}} \; = \; \limsup_{t \uparrow T, x' \to x} \tilde{v}(t, x') \text{ and } \underline{F} \; := \; e^{\underline{G}} \; = \; \liminf_{t \uparrow T, x' \to x} \tilde{v}(t, x').$$

Applying Theorems 4.1 and 5.1, we obtain the following characterization of the super-replication problem $\tilde{v}$ by a change of variable.

THEOREM 6.1. *Let μ and σ be bounded Lipschitz functions uniformly in the t variable, and $\sigma > 0$. Suppose further that g is bounded and satisfies $(g_*)^* \geq g$. Then*

(i) *$\tilde{v}$ is a discontinuous viscosity solution of*

$$\min \left\{ -\tilde{v}_t(t,x) - \frac{1}{2}\sigma^2\left(t, x, \tilde{v}_x(t,x)\right)\tilde{v}_{xx}(t,x) \; ; \; \tilde{v}(t,x) - h\left(\tilde{v}_x(t,x)\right) \right\} = 0$$
$$\text{on} \quad [0, T) \times \mathbb{R} \; .$$

(ii) *The terminal value functions $\underline{F}$ and $\overline{F}$ satisfy in the viscosity sense*

$$\begin{aligned} &\min\{\, \underline{F} - f_*;\; \underline{F} - h(\underline{F}_x) \,\} \geq 0, \\ &\min\{\, \overline{F} - f^*;\; \overline{F} - h(\overline{F}_x) \,\} \leq 0 \qquad \text{on} \quad \mathbb{R}. \end{aligned}$$

The rest of this section is devoted to the characterization of the terminal functions $\overline{F}$ and $\underline{F}$. It is known that the first order variational inequality appearing in part (ii) of the above theorem could fail to have a unique bounded discontinuous viscosity solution: under our condition $(f_*)^* \geq f$, all viscosity discontinuous bounded solutions have the same lower semicontinuous envelope; see Barles [3]. Therefore, we do not have much to say in the case where the payoff function f is not continuous.

We provide a characterization of the terminal condition of the superreplication problem in the case of Lipschitz payoff function f.

PROPOSITION 6.1. *Let the conditions of Theorem* 6.1 *hold. Assume, further, that the payoff function f is Lipschitz on $\mathbb{R}$. Then*

$$\overline{F}(x) \; = \; \underline{F}(x) = \hat{f}(x) \; := \; \sup_{y \in \mathbb{R}} f(x+y)e^{-\delta(y)},$$

where $\delta := \delta_U$ is the support function of the interval $U = [-\ell, u]$.

Proof. From Theorem 6.1, functions $\overline{F}$ and $\underline{F}$ are, respectively, upper and lower semicontinuous viscosity sub- and supersolutions of

$$\text{(VI)} \qquad \min\left\{u - f \; ; \; u - h(u_x)\right\} = 0 \quad \text{on} \quad \mathbb{R}.$$

In order to obtain the required result, we shall first prove that $\hat{f}$ is a (continuous) viscosity supersolution of (VI) (step 1). Then we will prove that $\underline{F} \geq \hat{f}$ (step 2). The proof is then concluded by means of a comparison theorem (Barles [2, Theorem 4.3, p. 93]); since f is Lipschitz, conditions (H1), (H4), and (H11) of this theorem are easily seen to hold. Since $\overline{F} \geq \underline{F}$ by definition, the above claims provide $\hat{f} \geq \overline{F} \geq \underline{F} \geq \hat{f}$.

Step 1. Let us prove that $\hat{f}$ is a continuous viscosity supersolution of (VI).

(i) $\hat{f}$ is a Lipschitz function. To see this, observe that, since δ is a sublinear function, it follows that $\hat{\hat{f}} = \hat{f}$, where $\hat{\hat{f}}$ is defined by the same formula as $\hat{f}$ with $\hat{f}$ substituted to f. Then, since $\hat{f}$ and δ are nonnegative,

$$\begin{aligned}\hat{f}(x+y) - \hat{f}(x) &\leq \hat{f}(x+y)(1-e^{-\delta(y)}) \quad \text{for all} \quad y \in \mathbb{R}\\ &\leq \hat{f}(x+y)\delta(y) \;\leq\; \|f\|_\infty \max(u,\ell)|y|.\end{aligned}$$

(ii) $\hat{f}$ is a supersolution of (VI). To see this, let $x_0 \in \mathbb{R}$ and $\varphi \in C^1(\mathbb{R})$ be such that $0 = (\hat{f} - \varphi)(x_0) = \min(\hat{f} - \varphi)$. Observe that $\hat{f} \geq \varphi$. Since $\hat{f} > 0$, we can assume without loss of generality that $\varphi > 0$. By definition, we have $\hat{f}(x_0) \geq f(x_0)$.

It remains to prove that $(\varphi'/\varphi)(x_0) \in [-\ell, u]$. Since $\hat{\hat{f}} = \hat{f}$, we have

$$\varphi(x_0) \;=\; \hat{f}(x_0) \geq \hat{f}(x_0+h)e^{-\delta(h)} \;\geq\; \varphi(x_0+h)e^{-\delta(h)}$$

for all $h \in \mathbb{R}$. Now let h be an arbitrary positive constant. Then

$$\frac{\varphi(x_0+h) - \varphi(x_0)}{h} \leq \varphi(x_0+h)\frac{1-e^{-uh}}{h},$$

and, by sending h to zero, we get $\varphi'(x_0) \leq u\varphi(x_0)$. Similarly, by considering an arbitrary constant $h < 0$, we see that $\varphi'(x_0) \geq -\ell\varphi(x_0)$.

Step 2. We now prove that $\underline{F} \geq \hat{f}$. From the supersolution property of $\underline{F}$, we have that $\underline{F} \geq f$, and, for all $y \in \mathbb{R}$, $\underline{F}$ satisfies in the viscosity sense

$$\delta(y)\underline{F} - y\underline{F}_x \geq 0.$$

By an easy change of variable, we see that $\underline{G} = \ln \underline{F}$ satisfies in the viscosity sense

$$\delta(y) - y\underline{G}_x \geq 0.$$

This proves that the function $x \longmapsto \delta(y)x - y\underline{G}(x)$ is nondecreasing (see, e.g., Cvitanić, Pham, and Touzi [9]), and therefore

$$\begin{aligned}\delta(y)(x+y) - y\underline{G}(x+y) &\geq \delta(y)x - y\underline{G}(x) \quad \text{for all} \quad y > 0,\\ \delta(y)(x+y) - y\underline{G}(x+y) &\leq \delta(y)x - y\underline{G}(x) \quad \text{for all} \quad y < 0.\end{aligned}$$

Recalling that $\underline{F} \geq f$, this provides

$$\underline{F}(x) \geq \sup_{y\in\mathbb{R}} \underline{F}(x+y)e^{-\delta(y)} \;\geq\; \sup_{y\in\mathbb{R}} f(x+y)e^{-\delta(y)} \;=\; \hat{f}(x). \qquad \square$$

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