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Homogenization of the Neumann problem with nonisolated holes *

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Abstract

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We consider the homogenization of second-order elliptic equations with a Neumann boundary condition in open sets periodically perforated with holes of the size of the period. When the holes are isolated, Cioranescu and Saint Jean Paulin (1979) proved the convergence of the homogenization process. One of their main tool was the construction of an extension of the solution, which is uniformly bounded. In the present paper, we give a new proof of the convergence, which avoids the use of such an extension. The main advantage of our approach is that it generalizes the result of Cioranescu and Saint Jean Paulin to the general case of periodic holes which may be not isolated (including, for example in three dimensions, the case of a domain perforated by interconnected cylinders).

0. Introduction

This paper is devoted to the homogenization of second-order elliptic equations in a domain periodically perforated by infinitely many small holes (having the same size as the period), with a Neumann boundary condition. This type of problems arises from several fields of physics or mechanics. Let us mention a few of them: the convection–diffusion of a chemical in a porous medium [12,10], the elasticity (resp. viscoplasticity) problem for a perforated material [9] (resp. [11]), or the Navier–Stokes equations for a gas condensating on rods [8]. For all those problems, the heuristic derivation of the homogenized problem is by now well known and understood, thanks to the celebrated two-scale method (see e.g. [5,14]). Here we focus on the mathematical problem of proving the convergence of the homogenization process. The first result in this direction is due to Cioranescu and Saint Jean Paulin [7]. Following the lines of Tartar [15], they rigorously proved the convergence in the case of isolated and periodically distributed holes

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which do not meet the boundary of the domain. Their main tools were the so-called energy method of Tartar and the construction of an extension operator. Here, we generalize their result to the case of periodically distributed holes which are either isolated or connected, and which may meet the exterior boundary. Our main tools are, again, the energy method, and a new compactness lemma in perforated domains, which avoid the use of any extension operator.

Now, we turn to a more precise presentation of our results. Let Ω be a bounded set in $\mathbb{R}^N$ $(N \geqslant 2)$. The set Ω is covered by identical small cells εY, where $Y = (-1, +1)^N$ is the unit cell, and ε is the period which will tend to zero. Let Y^* be a subset of the unit cell Y (we call it the *material part*). The domain Ω_ε is defined as the intersection of Ω with the union of the small material parts εY^*. We assume that the material part Y^*, and the union of all the material parts which cover $\mathbb{R}^N$, are connected, and that the volume fraction of the material $\theta = |Y^*|/|Y|$ is strictly positive (see hypotheses (H1), (H2) and (H3) in Section 1). Those assumptions are not too restrictive, and the holes are allowed to be isolated (i.e. $Y - Y^*$ is strictly included in Y), or to be connected (i.e. $Y - Y^*$ meets the boundary ∂Y; this case only occurs when the dimension is greater or equal to 3). We consider the following scalar equation in the domain Ω_ε

$$\begin{cases} -\nabla \cdot \left(A\left(\dfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) + u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}} = \left[A\left(\dfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \right] \cdot n = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{0.1}$$

where the matrix $A(y)$ is Y-periodic, uniformly bounded, and coercive, and the right-hand side f belongs to $L^2(\Omega)$. It is well known that this problem has a unique solution in $H^1(\Omega_\varepsilon)$.

Using the two-scale method, it is easy to see that the corresponding homogenized problem is

$$\begin{cases} -\nabla \cdot \left[\bar{A}\, \nabla u \right] + \theta u = \theta f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_{\bar{A}}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where the matrix $\bar{A}$ is a constant which can be computed through the so-called cell problem (see (1.5) and (1.6) in Section 1), and θ is the material volume fraction. Our main result (Theorem 1.4) is the following.

Theorem 0.1. *The sequence of solutions u_ε of* (0.1) *converges to the solution u of the homogenized problem* (0.2) *in the following sense*

$$\text{for any open set } \omega \text{ with } \bar{\omega} \subset \Omega, \quad \lim_{\varepsilon \to 0} \| u_\varepsilon - u \|_{L^2(\Omega_\varepsilon \cap \omega)} = 0.$$

In the above result, the introduction of the set ω means that the convergence is local inside Ω (this local result is forced by a possible "wild" boundary $\partial\Omega_\varepsilon$ in the vicinity of $\partial\Omega$). The proof of this theorem relies upon a compactness lemma which states that "the embedding of $H^1(\Omega_\varepsilon)$ in $L^2(\Omega_\varepsilon)$ is compact, uniformly in ε" (Lemma 2.3). This avoids the use of any extension of the sequence u_ε in the holes $\Omega - \Omega_\varepsilon$ (this was the technical part of the proof in [7]).

In the Appendix, written in collaboration with A.K. Nandakumar, we adapt the above result to a slightly different problem in the same geometric situation. Instead of having a Neumann condition, both on the holes boundary, and on the exterior boundary, we consider there a system with a Neumann condition on the holes boundary, and a Dirichlet one on the exterior boundary; namely

$$\begin{cases} -\nabla\cdot\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right)=f & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}}=\left[A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right]\cdot n=0 & \text{on } \partial\Omega_\varepsilon-\partial\Omega, \\ u_\varepsilon=0 & \text{on } \partial\Omega\cap\partial\Omega_\varepsilon. \end{cases} \tag{0.3}$$

Again, there is a unique solution of this problem in $H^1(\Omega_\varepsilon)$, and the homogenized system is

$$\begin{cases} -\nabla\cdot(\bar{A}\,\nabla u)=\theta f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

where the matrix $\bar{A}$ is the same as above. Then, we prove the following result (see Theorem A.1).

Theorem 0.2. (G. Allaire, F. Murat, A.K. Nandakumar). *The sequence of the solutions u_ε of* (0.3) *converges to the solution u of the homogenized problem* (0.4) *in the following sense*

$$\lim_{\varepsilon\to 0}\|u_\varepsilon-u\|_{L^2(\Omega_\varepsilon)}=0.$$

Observe that, in this case, the result is no longer local, but valid up to the exterior boundary $\partial\Omega$. This is due to the Dirichlet boundary condition on $\partial\Omega$.

After this work had been completed, we learned that Acerbi et al. [1] obtained the same result as ours (i.e. Theorem 0.1), but with a completely different method; indeed, they construct a bounded extension operator from $H^1(\Omega_\varepsilon)$ into $H^1(\Omega)$, as in [7], but with no restrictions on the geometry of the holes (which may be isolated or connected). Theorem 0.1 can also be proved by using the two-scale convergence method (see [3] and [4]). Anyway, we believed that our main tool (the compactness Lemma 2.3), which is interesting by itself, provides the simplest proof of Theorem 0.1.

1. Setting of the problem

As usual in the periodic homogenization theory, we first define a so-called unit cell, which, upon rescalling to size ε, becomes the period of a periodic medium. The unit cube $Y=(-1,+1)^N$ is perforated by a hole, and the part of Y occupied by the material is called Y^*. The volume fraction of the material is denoted by $\theta=|Y^*|/|Y|$. We make the following hypotheses on the material part Y^*:

(H1) Y^* is a connected open set of $\mathbb{R}^N$, has a Lipschitz boundary ∂Y^*, and is locally located on one side of its boundary;

(H2) the union E^* of all material parts, defined as the periodic open set obtained by covering $\mathbb{R}^N$ with the material part Y^*, is connected, has a Lipschitz boundary, and lies locally on one side of its boundary;

(H3) the material volume fraction θ is strictly positive.

Hypotheses (H1) and (H2) imply that the material is in one piece, while hypothesis (H3) means that there is actually some material. However, they do not restrict the topology of the holes. In particular, the holes may be isolated, or connected in one piece, or any intermediate situation.

Remark 1.1. In hypothesis (H2) we have skipped a little technical difficulty in the definition of E^*. Because the material part Y^* is an open set, it does not contain its boundary ∂Y^*. Thus the physically realistic material part of two contiguous cells Y_1 and Y_2 is the union of the two open sets Y_1^* and Y_2^* *plus* the material interface $\partial Y_1^* \cap \partial Y_2^*$. Consequently the union E^* of all material parts is rigorously defined as the interior of the union of the closures of all the open sets Y^*.

Now, let Ω be a bounded open set in $\mathbb{R}^N$, with Lipschitz boundary $\partial\Omega$, Ω being locally located on one side of its boundary. Let ε be a sequence of strictly positive real numbers which tends to zero. The set Ω is periodically covered by cells Y_i^ε, similar to the unit cell Y rescaled to size ε. More precisely, we define

$$Y_i^\varepsilon = \left\{ x \in \mathbb{R}^N \mid \left(\frac{x}{\varepsilon} - 2i\right) \in Y \right\}, \qquad Y_i^{*\varepsilon} = \left\{ x \in \mathbb{R}^N \mid \left(\frac{x}{\varepsilon} - 2i\right) \in Y^* \right\}, \tag{1.1}$$

where i is an element of $\mathbb{Z}^N$.

We also define the open set εE^* as the material part E^* rescaled to size ε. Up to material interfaces, εE^* is equal to the union of the $Y_i^{*\varepsilon}$. Then, the material part Ω_ε is defined by

$$\Omega_\varepsilon = \Omega \cap \varepsilon E^*. \tag{1.2}$$

Denoting by $\mathbf{1}_{\Omega_\varepsilon}$ the characteristic function of the set Ω_ε, a well-known result states that the sequence $\mathbf{1}_{\Omega_\varepsilon}$ converges to θ in the weak star topology of $L^\infty(\Omega)$.

Remark 1.2. Although we have assumed (H2), the set Ω_ε may be not connected. Indeed there may be some connected components of Ω_ε in the neighborhood of $\partial\Omega$, which have a size smaller than ε. In the same vein, because of (H2) the boundary $\partial\Omega_\varepsilon$, is smooth "in the interior of Ω", but "in the neighborhood of $\partial\Omega$" nothing can be said about its regularity, because, under our assumptions, the holes may meet the boundary $\partial\Omega$ (contrary to reference [7]). The definition of Ω_ε is similar to that of a porous medium in [2], where the homogenization of Stokes flows was studied.

In the material domain Ω_ε, we consider the Neumann problem for the second-order elliptic equation

$$\begin{cases} -\nabla \cdot \left(A\left(\dfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) + u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\[2ex] \dfrac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}} = \left[A\left(\dfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \right] \cdot n = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.3}$$

As in [7], we make the following assumptions:

(A1) $f \in L^2(\Omega)$;

(A2) The coefficients a_{ij} of the matrix A are periodic of period Y, and belong to $L^\infty(\mathbb{R}^N)$;

(A3) there is a strictly positive number α such that

$$^t\xi A(y)\xi \geqslant \alpha |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N \text{ and } y \in Y.$$

Under these assumptions it is well-known that (1.3) admits a unique solution in $H^1(\Omega_\varepsilon)$ (the zero-order term $+u_\varepsilon$ is here to enforce existence and uniqueness).

Remark 1.3. The boundary condition in (1.3) is of Neumann type, both on the boundary of the holes $\partial\Omega_\varepsilon - \partial\Omega$ and on the "exterior boundary" $\partial\Omega_\varepsilon \cap \partial\Omega$. In the appendix, written in collaboration with A.K. Nandakumar, we consider a problem analogous to (1.3), where the Neumann boundary condition on $\partial\Omega_\varepsilon \cap \partial\Omega$ is replaced by a Dirichlet boundary condition; this allows us to remove the zero-order term $+u_\varepsilon$ in the equation.

Using the celebrated two-scale method (see, e.g., [5] or [14]), it is easy to see heuristically that the limit problem of (1.3), when ε goes to zero, is

$$\begin{cases} -\nabla \cdot \left[\tilde{A}\,\nabla u\right] + \theta u = \theta f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_{\tilde{A}}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The constant matrix $\tilde{A}$ is given by

$$^te_j\,{}^t\tilde{A}e_i = \frac{1}{|Y|}\int_{Y^*} {}^t\nabla w_j\,{}^tA\,\nabla w_i, \tag{1.5}$$

where the functions $(w_i)_{1 \leqslant i \leqslant N}$ are the solutions of the so-called cell problem

$$\begin{cases} -\nabla \cdot \left[{}^tA(y)\,\nabla w_i\right] = 0 & \text{in } Y^*, \\ \dfrac{\partial w_i}{\partial \nu_{{}^tA}} = 0 & \text{on } \partial Y^* - \partial Y, \\ (w_i - y_i) & Y\text{-periodic}. \end{cases} \tag{1.6}$$

From (1.5) it is easy to deduce that there exists a strictly positive number β such that

$$^t\xi\tilde{A}\xi \geqslant \beta|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N.$$

Thus, system (1.4) admits a unique solution in $H^1(\Omega)$.

The goal of the present paper is to rigorously prove the convergence of the sequence of the solutions of (1.3) to the solution of the homogenized problem (1.4), i.e., to prove the following theorem.

Theorem 1.4. *Let u_ε (resp. u) be the unique solution of* (1.3) (*resp.* (1.4)). *Under the hypotheses* (H1), (H2) *and* (H3) *on the geometry of the unit cell, the sequence u_ε tends to u in the following sense*:

$$\text{for any open set } \omega \text{ with } \overline{\omega} \subset \Omega, \quad \lim_{\varepsilon \to 0} \| u_\varepsilon - u \|_{L^2(\Omega_\varepsilon \cap \omega)} = 0. \tag{1.7}$$

Remark 1.5. Let us recall the result obtained by Cioranescu and Saint Jean Paulin [7]. Under a certain further hypothesis on the holes (namely, the holes are isolated in each cell, and no holes meet the boundary $\partial\Omega$), they built an extension operator P_ε from $H^1(\Omega_\varepsilon)$ in $H^1(\Omega)$, such that the sequence $P_\varepsilon u_\varepsilon$ converges weakly to u in $H^1(\Omega)$. In their context, the convergence (1.7) appears as a consequence of the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ (Rellich's theorem). Note however that the present result (1.7) is local (i.e. holds only in the interior of Ω) because some holes may meet the boundary $\partial\Omega$.

The main interest of Theorem 1.4 is obviously that it holds true under less restrictive assumptions than in [7]. For example in three dimensions, the holes may be connected like a mesh of cylinders.

2. Proof of convergence

The proof of Theorem 1.4 is based on the so-called energy method introduced by Tartar (see [15], partially written in [13]) and on Lemma 2.3 which, loosely speaking, states that the embedding of $H^1(\Omega_\varepsilon)$ in $L^2(\Omega_\varepsilon)$ is compact, uniformly in ε. In [7] the energy method was also the main tool; thus the originality of our approach lies in Lemma 2.3 which, more or less, replaces the extension operator and Rellich's theorem used in [7].

Definition 2.1. We denote by $\tilde{}$ the extension operator by zero in the holes $\Omega - \Omega_\varepsilon$. Thus, for any function v_ε of $L^2(\Omega_\varepsilon)$, $\tilde{v}_\varepsilon$ is defined by

$$\tilde{v}_\varepsilon = \begin{cases} v_\varepsilon & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega - \Omega_\varepsilon. \end{cases}$$

Lemma 2.2. *Assume that hypothesis* (H1), (H2) *and* (H3) *hold. We then have*

(1) *There exists a positive constant C, which depends only on Y^*, such that, for any function $v \in H^1(Y^*)$, we have*

$$\left\| v - \frac{1}{|Y^*|} \int_{Y^*} v \right\|_{L^2(Y^*)} \leq C \| \nabla v \|_{L^2(Y^*)}. \tag{2.1}$$

(2) *Let Y and Y' be two contiguous cells* (*i.e. two cells which share a common side*). *Let us denote by Z^* the material part of the two cells, namely $Z^* = Y^* \cup Y'^* \cup (\partial Y^* \cap \partial Y'^*)$. There exists a positive constant C, which depends only on Y^*, such that, for any function $v \in H^1(Z^*)$, we have*

$$\left| \frac{1}{|Y^*|} \int_{Y^*} v - \frac{1}{|Y'^*|} \int_{Y'^*} v \right| \leq C \| \nabla v \|_{L^2(Z^*)}. \tag{2.2}$$

Proof. Inequality (2.1) is nothing but the Poincaré–Wirtinger inequality in Y^*, which is easily proved by contradiction since Y^* is connected (hypothesis (H1)). Similarly, inequality (2.2) is easily proved by contradiction since hypotheses (H1) and (H2) obviously implies that Z^* is connected. □

Lemma 2.3. *Let u_ε be a sequence with uniformly bounded norm in $H^1(\Omega_\varepsilon)$, i.e.*

$$\| u_\varepsilon \|_{H^1(\Omega_\varepsilon)} \leqslant C, \tag{2.3}$$

where the constant C does not depend on ε. The sequence $\tilde{u}_\varepsilon$ being bounded in $L^2(\Omega)$, there exists a function u in $L^2(\Omega)$ such that, up to a subsequence, we have

$$\tilde{u}_\varepsilon \rightharpoonup \theta u \text{ weakly in } L^2(\Omega). \tag{2.4}$$

Then, this subsequence u_ε is "compact" in the following sense

For any sequence v_ε in $L^2(\Omega_\varepsilon)$ such that $\tilde{v}_\varepsilon \rightharpoonup \theta v$ weakly in $L^2(\Omega)$, and for any function $\phi \in D(\Omega)$, we have:

$$\int_{\Omega_\varepsilon} \phi u_\varepsilon v_\varepsilon \to \int_\Omega \theta \phi u v. \tag{2.5}$$

Furthermore, the limit u actually belongs to $H^1(\Omega)$.

Remark 2.4. Although the sequence u_ε is "compact" in the sense of (2.5), we emphasize that $\tilde{u}_\varepsilon$ is definitely not compact in $L^2(\Omega)$. Nevertheless, it is easy to deduce from (2.5) that, for any open set ω satisfying $\overline{\omega} \subset \Omega$, we have

$$\| u_\varepsilon - u \|_{L^2(\Omega_\varepsilon \cap \omega)} \to 0.$$

Note also that the compactness (2.5) could be easily deduced from the existence of a bounded extension operator, if any. Indeed, if we assume that there exists an extension operator P_ε such that, further to (2.3), $P_\varepsilon u_\varepsilon$ is bounded in $H^1(\Omega)$, it is easily seen that (2.4) and the equality $\tilde{u}_\varepsilon = 1_{\Omega_\varepsilon} P_\varepsilon u_\varepsilon$ in Ω imply

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega),$$

and (2.5) holds true.

Proof of Lemma 2.3. Let ω be a convex subset of Ω such that $\overline{\omega} \subset \Omega$. The domain Ω is covered by cells Y_i^ε, but is usually not exactly equal to an union of entire cells (some cells meet the boundary $\partial\Omega$). For that reason, we introduce the set C_ε which is the largest union of entire cells included in Ω, namely $C_\varepsilon = \bigcup_{i \in I_\varepsilon} Y_i^\varepsilon$, with $I_\varepsilon = \{i \mid Y_i^\varepsilon \subset \Omega\}$. For sufficiently small values of ε, we have $\omega \subset C_\varepsilon \subset \Omega$. In C_ε we define a piecewise constant function $\overline{u}_\varepsilon$ by

$$\overline{u}_\varepsilon = \frac{1}{|Y_i^{*\varepsilon}|} \int_{Y_i^{*\varepsilon}} u_\varepsilon \quad \text{in the cell } Y_i^\varepsilon \text{ for } i \in I_\varepsilon. \tag{2.6}$$

Let us prove that the sequence $\overline{u}_\varepsilon$ is relatively compact in $L^2(\omega)$ by application of the Kolmogorov criterion. For any vector e_k of the canonical basis of $\mathbb{R}^N$, let $h \in \mathbb{R}^+$ be sufficiently small, such that, for any point $x \in \omega$, $x + he_k$ belongs to Ω. Let Y_i^ε and $Y_{i'}^\varepsilon$ be two contiguous cells such that $i' - i = 2e_k$. By rescaling inequality (2.2) we obtain for $x \in Y_i^\varepsilon$

$$\varepsilon^N |\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(x + 2\varepsilon e_k)|^2 \leqslant C\varepsilon^2 \| \nabla u_\varepsilon \|^2_{L^2(Y_i^{*\varepsilon} \cup Y_{i'}^{*\varepsilon})}. \tag{2.7}$$

If $0 \leqslant h \leqslant 2\varepsilon$, denoting by c_i^ε its center, the cell $Y_i^\varepsilon = \{x \in \Omega \,|\, (x - c_i^\varepsilon) \in (-\varepsilon, -\varepsilon)^N\}$ is made of two parts $A_i^\varepsilon = \{x \in Y_i^\varepsilon \,|\, -\varepsilon \leqslant (x - c_i^\varepsilon) \cdot e_k \leqslant \varepsilon - h\}$ and $B_i^\varepsilon = \{x \in Y_i^\varepsilon \,|\, \varepsilon - h \leqslant (x - c_i^\varepsilon) \cdot e_k \leqslant +\varepsilon\}$, such that

$$x \in A_i^\varepsilon \Rightarrow (x + he_k) \in Y_i^\varepsilon \quad \text{and} \quad x \in B_i^\varepsilon \Rightarrow (x + he_k) \in Y_{i'}^\varepsilon.$$

Since $\bar{u}_\varepsilon$ is constant in each cell, we deduce that

$$\begin{cases} \varepsilon^N |\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(x + he_k)|^2 = 0 & \text{for } x \in A_i^\varepsilon, \\ \varepsilon^N |\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(x + he_k)|^2 \leqslant C\varepsilon^2 \| \nabla u_\varepsilon \|^2_{L^2(Y_i^{*\varepsilon} \cup Y_{i'}^{*\varepsilon})} & \text{for } x \in B_i^\varepsilon. \end{cases} \tag{2.8}$$

Integrating (2.8) over Y_i^ε and noticing that $|B_i^\varepsilon| = (2\varepsilon)^{N-1}h$, then summing on i, leads to

$$\varepsilon^N \| \bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(x + he_k) \|^2_{L^2(\omega)} \leqslant 2C\varepsilon^{N+1} h \| \nabla u_\varepsilon \|^2_{L^2(\Omega_\varepsilon)}.$$

Thus

$$\| \bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(x + he_k) \|_{L^2(\omega)} \leqslant Ch^{1/2}\varepsilon^{1/2} \quad \text{for } h \leqslant 2\varepsilon. \tag{2.9}$$

If $h > 2\varepsilon$, then there exists an integer $n \geqslant 1$ and a positive real $h' < 2\varepsilon$ such that $h = 2n\varepsilon + h'$. Since ω is convex, and since $\bar{u}_\varepsilon(x)$ is a constant in each cell Y_i^ε, it is easy to relate $\bar{u}_\varepsilon(x)$ to $\bar{u}_\varepsilon(x + he_k)$ by using a path made of segments of the type $(x + 2j\varepsilon e_k,\ x + 2(j+1)\varepsilon e_k)$, for $0 \leqslant j \leqslant n - 1$, and an end segment $(x + 2n\varepsilon e_k,\ x + (2n\varepsilon + h')e_k)$. For each segment $(x + 2j\varepsilon e_k,\ x + 2(j+1)\varepsilon e_k)$, integrating (2.7) over Y_i^ε, then summing on i, leads to

$$\varepsilon^N \| \bar{u}_\varepsilon(x + 2j\varepsilon e_k) - \bar{u}_\varepsilon(x + 2j\varepsilon e_k + 2\varepsilon e_k) \|^2_{L^2(\omega)} \leqslant C\varepsilon^{N+2} \| \nabla u_\varepsilon \|^2_{L^2(\Omega_\varepsilon)},$$

which implies

$$\| \bar{u}_\varepsilon(x + 2j\varepsilon e_k) - \bar{u}_\varepsilon(x + 2j\varepsilon e_k + 2\varepsilon e_k) \|_{L^2(\omega)} \leqslant C\varepsilon.$$

Thus, summing over all segments (including the end segment for which formula (2.9) holds) gives

$$\| \bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(x + he_k) \|_{L^2(\omega)} \leqslant C(n\varepsilon + h'^{1/2}\varepsilon^{1/2}) \leqslant Ch \quad \text{for } h > 2\varepsilon. \tag{2.10}$$

Since $\bar{u}_\varepsilon$ is easily seen to be bounded in $L^2(\omega)$, inequalities (2.9) and (2.10) are nothing but the Kolmogorov criterion for the relative compactness of the sequence $\bar{u}_\varepsilon$ in $L^2(\omega)$. Therefore, there exists $\bar{u}$ such that, extracting a subsequence, we have

$$\bar{u}_\varepsilon \to \bar{u} \quad \text{strongly in } L^2(\omega).$$

Passing to the limit in (2.10), we obtain for any value of h

$$\| \bar{u}(x) - \bar{u}(x + he_k) \|_{L^2(\omega)} \leqslant C|h|, \tag{2.11}$$

where the constant C depends neither on h nor on ω. Inequality (2.11) implies that $\bar{u}$ belongs to $H^1(\Omega)$ (see if necessary Proposition IX.3 in [6]).

For any smooth function ϕ, with compact support in Ω, and for any sequence v_ε in $L^2(\Omega_\varepsilon)$ such that $\tilde{v}_\varepsilon$ weakly converges to θv in $L^2(\Omega)$, we now study the limit of

$$\int_{\Omega_\varepsilon} \phi u_\varepsilon v_\varepsilon = \int_\Omega \phi \bar{u}_\varepsilon \tilde{v}_\varepsilon + \int_{\Omega_\varepsilon} \phi (u_\varepsilon - \bar{u}_\varepsilon) v_\varepsilon. \tag{2.12}$$

Because $\bar{u}_\varepsilon$ is relatively compact in $L^2_{\mathrm{loc}}(\Omega)$, we pass to the limit (for a subsequence) in the first term of the r.h.s. of (2.12)

$$\int_\Omega \phi \bar{u}_\varepsilon \tilde{v}_\varepsilon \to \int_\Omega \phi \theta \bar{u} v. \tag{2.13}$$

For ε small enough, the support of ϕ is included in C_ε, and the second term of the r.h.s. of (2.12) is bounded by

$$\left| \int_{\Omega_\varepsilon} \phi (u_\varepsilon - \bar{u}_\varepsilon) v_\varepsilon \right| \leqslant C \| \phi \|_{L^\infty(\Omega)} \| u_\varepsilon - \bar{u}_\varepsilon \|_{L^2(\Omega_\varepsilon \cap C_\varepsilon)}.$$

Rescaling the Poincaré–Wirtinger inequality (2.1), and summing over all the cells of C_ε leads to

$$\| u_\varepsilon - \bar{u}_\varepsilon \|_{L^2(\Omega_\varepsilon \cap C_\varepsilon)} \leqslant C\varepsilon \| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon \cap C_\varepsilon)}. \tag{2.14}$$

Thus, we deduce from (2.12) that

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \phi u_\varepsilon v_\varepsilon = \int_\Omega \theta \phi \bar{u} v.$$

Finally it remains to prove that $\bar{u} = u$, where u is defined by (2.4). This is obvious because (2.14) implies

$$\lim_{\varepsilon \to 0} \| \tilde{u}_\varepsilon - \mathbf{1}_{\Omega_\varepsilon} \bar{u}_\varepsilon \|_{L^2(\omega)} = 0,$$

while the strong convergence of $\bar{u}_\varepsilon$ implies that $\mathbf{1}_{\Omega_\varepsilon} \bar{u}_\varepsilon$ converges weakly to $\theta \bar{u}$ in $L^2(\omega)$. □

Proof of Theorem 1.4. In order to prove the convergence of the homogenization process, we use the energy method, introduced by Tartar [15]. We follow along the lines of [7], with some modifications since here we are not using any extension operator.

First step: a priori estimates for the sequence u_ε

Multiplying equation (1.3) by u_ε, and integrating by parts, we obtain

$$\int_{\Omega_\varepsilon} {}^t\nabla u_\varepsilon A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon + \int_{\Omega_\varepsilon} (u_\varepsilon)^2 = \int_{\Omega_\varepsilon} f u_\varepsilon. \tag{2.15}$$

From (2.15) we easily deduce that

$$\| u_\varepsilon \|_{H^1(\Omega_\varepsilon)} \leqslant C. \tag{2.16}$$

Defining a function $\xi_\varepsilon = A(x/\varepsilon) \nabla u_\varepsilon$ in Ω_ε, (2.16) and assumption (A2) yield

$$\| \xi_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C. \tag{2.17}$$

In view of (2.16) and (2.17), there exist two functions $u \in L^2(\Omega)$ and $\xi \in [L^2(\Omega)]^N$, such that, up to a subsequence, we have

$$\begin{cases} \tilde{u}_\varepsilon \rightharpoonup \theta u & \text{weakly in } L^2(\Omega), \\ \tilde{\xi}_\varepsilon \rightharpoonup \theta \xi & \text{weakly in } \left[L^2(\Omega)\right]^N. \end{cases} \tag{2.18}$$

Since ξ_ε belongs to $[L^2(\Omega_\varepsilon)]^N$, and $\nabla \cdot \xi_\varepsilon$ belongs also to $L^2(\Omega_\varepsilon)$, there is no problem to define the trace $\xi_\varepsilon \cdot n$ as an element of $H^{-1/2}(\partial\Omega_\varepsilon)$. Furthermore, because of the Neumann boundary condition satisfied by u_ε, the normal component $\xi_\varepsilon \cdot n$ is continuous through the boundary $\partial\Omega_\varepsilon$, and thus $\nabla \cdot \tilde{\xi}_\varepsilon$ is a well-defined function of $L^2(\Omega)$ which satisfies

$$\begin{cases} -\nabla \cdot \tilde{\xi}_\varepsilon + \tilde{u}_\varepsilon = 1_{\Omega_\varepsilon} f & \text{in } \Omega, \\ \tilde{\xi}_\varepsilon \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

Passing to the limit in (2.19), and dividing by θ gives

$$\begin{cases} -\nabla \cdot \xi + u = f & \text{in } \Omega, \\ \xi \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Second step: definition of the test functions

Rescaling the solutions of the cell problem (1.6), we define in the union εE^* of all material parts (see hypothesis (H2) and (1.2))

$$w_i^\varepsilon(x) = \varepsilon w_i\left(\frac{x}{\varepsilon}\right), \qquad \eta_i^\varepsilon = {}^{t}A\left(\frac{x}{\varepsilon}\right) \nabla w_i^\varepsilon. \tag{2.21}$$

The functions w_i^ε satisfy

$$\begin{cases} -\nabla \cdot \left[{}^{t}A\left(\dfrac{x}{\varepsilon}\right) \nabla w_i^\varepsilon\right] = 0 & \text{in } \varepsilon E^* \\ \dfrac{\partial w_i^\varepsilon}{\partial \nu_{{}^{t}A_\varepsilon}} = 0 & \text{on } \partial(\varepsilon E^*), \end{cases} \tag{2.22}$$

and we have the estimates

$$\|w_i^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C \quad \text{and} \quad \|\eta_i^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C. \tag{2.23}$$

Since $w_i^\varepsilon(x) = x_i + \varepsilon\chi_i(x/\varepsilon)$ in εE^*, where χ_i is Y-periodic, we have

$$\begin{cases} \tilde{w}_i^\varepsilon \rightharpoonup \theta x_i & \text{weakly in } L^2(\Omega), \\ \tilde{\eta}_i^\varepsilon \rightharpoonup \dfrac{1}{|Y|} \displaystyle\int_{Y^*} {}^{t}A(y)\, \nabla w_i & \text{weakly in } \left[L^2(\Omega)\right]^N. \end{cases} \tag{2.24}$$

Furthermore, multiplying equation (1.6) by $\chi_j(y) = w_j(y) - y_j$ and integrating by parts yields

$$\int_{Y^*} \left[{}^{t}A(y)\, \nabla w_i\right] \cdot \nabla \chi_j = 0.$$

Thus

$$\frac{1}{|Y|}\int_{Y^*} {}^{t}A(y)\,\nabla w_i = \sum_{j=1}^{N} e_j \frac{1}{|Y|}\int_{Y^*}\left[{}^{t}A(y)\,\nabla w_i\right]\cdot\left[e_j + \nabla\chi_j\right] = \sum_{j=1}^{N} e_j\left[{}^{t}\tilde{A}e_i\right]\cdot e_j = {}^{t}\tilde{A}e_i .$$

Consequently (2.24) implies that

$$\tilde{\eta}_i^{\varepsilon} \rightharpoonup \theta\frac{{}^{t}\tilde{A}e_i}{\theta} \quad \text{weakly in } \left[L^2(\Omega)\right]^N . \tag{2.25}$$

Third step: passing to the limit in the equations

For any function $\phi \in D(\Omega)$, we multiply (2.22) by ϕu_ε, and (1.3) by ϕw_i^ε. Integrating by parts, and subtracting one from the other, lead to

$$\int_{\Omega_\varepsilon} \phi\,{}^{t}\nabla w_i^\varepsilon A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon + \int_{\Omega_\varepsilon} w_i^\varepsilon\left[A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right]\cdot\nabla\phi + \int_{\Omega_\varepsilon}\phi w_i^\varepsilon u_\varepsilon$$
$$-\int_{\Omega_\varepsilon}\phi\,{}^{t}\nabla u_\varepsilon\,{}^{t}A\left(\frac{x}{\varepsilon}\right)\nabla w_i^\varepsilon - \int_{\Omega_\varepsilon} u_\varepsilon\left[{}^{t}A\left(\frac{x}{\varepsilon}\right)\Omega w_i^\varepsilon\right]\cdot\nabla\phi = \int_{\Omega_\varepsilon}\phi f w_i^\varepsilon . \tag{2.26}$$

The first and the fourth terms of (2.26) cancel out. For the remaining ones, we apply Lemma 2.3 to obtain

$$\int_{\Omega_\varepsilon} w_i^\varepsilon\left[A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right]\cdot\nabla\phi \to \int_\Omega \theta x_i \xi\cdot\nabla\phi, \qquad \int_{\Omega_\varepsilon}\phi w_i^\varepsilon u_\varepsilon \to \int_\Omega \theta\phi x_i u,$$
$$\int_{\Omega_\varepsilon} u_\varepsilon\left[{}^{t}A\left(\frac{x}{\varepsilon}\right)\nabla w_i^\varepsilon\right]\cdot\nabla\phi \to \int_\Omega \theta u\left(\frac{{}^{t}\tilde{A}e_i}{\theta}\right)\cdot\nabla\phi, \qquad \int_{\Omega_\varepsilon}\phi f w_i^\varepsilon \to \int_\Omega \theta\phi f x_i . \tag{2.27}$$

Thus (2.26) yields

$$\int_\Omega x_i\,\nabla\phi\cdot\xi + \int_\Omega \phi x_i u - \int_\Omega u\,\nabla\phi\cdot\left(\frac{{}^{t}\tilde{A}e_i}{\theta}\right) = \int_\Omega \phi f x_i .$$

Integrating by parts, and recalling (2.20) gives

$$-\int_\Omega \phi e_i\cdot\xi + \int_\Omega \phi\,\nabla u\cdot\left(\frac{{}^{t}\tilde{A}e_i}{\theta}\right) = 0,$$

hence

$$\xi = \frac{\tilde{A}\,\nabla u}{\theta} . \tag{2.28}$$

Together with (2.20), (2.28) is the homogenized problem (1.4) which has a unique solution $u \in H^1(\Omega)$. Thus the entire sequence u_ε converges. This proves Theorem 1.4. □

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Appendix. Homogenization with a Neumann boundary condition on the holes and a Dirichlet condition on the exterior boundary [1]

In the same geometric situation as in Section 1, we consider in the appendix the homogenization of a system slightly different from (1.3), namely

$$\begin{cases} -\nabla\cdot\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right)=f & \text{in } \Omega_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial\nu_{A_\varepsilon}}=\left[A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right]\cdot n=0 & \text{on } \partial\Omega_\varepsilon-\partial\Omega \\ u_\varepsilon=0 & \text{on } \partial\Omega\cap\partial\Omega_\varepsilon. \end{cases} \tag{A.1}$$

System (A.1) is similar to (1.3), except that the boundary condition on the exterior boundary (and on the exterior boundary only) is different: Dirichlet here, while it was Neumann in Section 1. Passing from (1.3) to (A.1) we have dropped the linear term $+u_\varepsilon$, which was there only to ensure existence and uniqueness in (1.3). Anyway, whether this zero-order term is present or not does not matter for the homogenization process.

The same assumptions A1, A2, and A3, as in Section 1, are made on the matrix A; consequently it is well-known that (A.1) has a unique solution in $H^1(\Omega_\varepsilon)$. With the help of the two-scale method, it is easy to heuristically obtain the limit problem of (A.1)

$$\begin{cases} -\nabla\cdot(\tilde{A}\,\nabla u)=\theta f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{A.2}$$

where the matrix $\tilde{A}$ is still defined by (1.5) (the cell problem is the same as it was in Section 1).

In this appendix we prove the rigorous convergence of the sequence of solutions of (A.1) to the solution of (A.2) when ε goes to zero.

Theorem A.1. *Let u_ε (resp. u) be the unique solution of* (A.1) *(resp.* (A.2)*). Under the hypotheses* (H1), (H2) *and* (H3) *on the geometry of the unit cell, u_ε tends to u in the following sense*

$$\lim_{\varepsilon\to 0}\|u_\varepsilon-u\|_{L^2(\Omega_\varepsilon)}=0. \tag{A.3}$$

Remark A.2. Theorem A.1 has already been proved by Cioranescu and Saint Jean Paulin in [7] when the holes are isolated in each cell. As already mentioned in the introduction of this paper, Theorem A.1 generalizes their result to the case of connected holes. Furthermore, even in the case of isolated holes, their result is improved here because we do not "remove" the holes which meet the exterior boundary $\partial\Omega$.

Remark that the convergence is not local in the interior of Ω, as it was the case in Theorem 1.4. This is due to the Dirichlet boundary condition which allows us to get a result up to the exterior boundary.

Before proving Theorem A.1, we modify Lemma 2.3 to take into account the Dirichlet boundary condition on $\partial\Omega$.

[1] Written jointly with A.K. Nandakumar.

Lemma A.3. *Let u_ε be a sequence such that*

$$\begin{cases} u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \cap \partial\Omega \\ \| u_\varepsilon \|_{H^1(\Omega_\varepsilon)} \leqslant C, \end{cases} \tag{A.4}$$

where the constant C does not depend on ε.

The sequence $\tilde{u}_\varepsilon$ is bounded in $L^2(\Omega)$, and thus, extracting a subsequence, we can define a function u in $L^2(\Omega)$ such that

$$\tilde{u}_\varepsilon \rightharpoonup \theta u \quad \text{weakly in } L^2(\Omega). \tag{A.5}$$

Then the sequence u_ε is relatively "compact" in the following sense

For any sequence v_ε in $L^2(\Omega_\varepsilon)$, such that $\tilde{v}_\varepsilon \rightharpoonup \theta v$ weakly in $L^2(\Omega)$, we have

$$\int_{\Omega_\varepsilon} u_\varepsilon v_\varepsilon \to \int_\Omega \theta u v. \tag{A.6}$$

Furthermore, the limit u actually belongs to $H_0^1(\Omega)$.

Proof of Lemma A.3. We proceed as in Lemma 2.3, but, instead of defining the function $\bar{u}_\varepsilon$ in Ω only, we define it in the whole of $\mathbb{R}^N$. Before that, we need to extend a function defined only in Ω_ε to the union of all material parts εE^* (see hypothesis (H2) and (1.2)). For any function $v_\varepsilon \in H^1(\Omega_\varepsilon)$ we define its extension $Q_\varepsilon v_\varepsilon$ in εE^* by

$$Q_\varepsilon v_\varepsilon = \begin{cases} v_\varepsilon & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \varepsilon E^* - \Omega_\varepsilon. \end{cases} \tag{A.7}$$

The key point is now to remark that, if v_ε satisfies a Dirichlet boundary condition on the exterior boundary $\partial\Omega \cap \partial\Omega_\varepsilon$, then the extension $Q_\varepsilon v_\varepsilon$ actually belongs to $H^1(\varepsilon E^*)$.

Applying this result to a sequence u_ε satisfying (A.4), we define a piecewise constant function $\bar{u}_\varepsilon$ by

$$\bar{u}_\varepsilon = \frac{1}{|Y_i^{*\varepsilon}|} \int_{Y_i^{*\varepsilon}} Q_\varepsilon u_\varepsilon \quad \text{in the cell } Y_i^\varepsilon \text{ for } i \in \mathbb{Z}^N. \tag{A.8}$$

Then, as in Lemma 2.3, we prove that the sequence $\bar{u}_\varepsilon$ is relatively compact in $L^2(\omega)$ for any convex subset ω of $\mathbb{R}^N$. In particular, $\bar{u}_\varepsilon$ is relatively compact in $L^2(\Omega)$. Furthermore, the limit $\bar{u}$ of a subsequence of $\bar{u}_\varepsilon$ is known to belong to $H^1_{\text{loc}}(\mathbb{R}^N)$. In order to prove that $\bar{u}$ is actually equal to zero in $\mathbb{R}^N - \Omega$, i.e. belongs to $H_0^1(\Omega)$, we simply note that in $\mathbb{R}^N - \Omega$, at a distance of $\partial\Omega$ greater than ε, the function $\bar{u}_\varepsilon$ is equal to zero.

The end of Lemma A.3 is as Lemma 2.3, except that we do not need to localize inside Ω by a function ϕ. □

Now, we give a Poincaré inequality in Ω_ε.

Lemma A.4. *There exists a constant C, which does not depend on ε, such that, for any $v_\varepsilon \in H^1(\Omega_\varepsilon)$ satisfying $v_\varepsilon = 0$ on $\partial\Omega_\varepsilon \cap \partial\Omega$, we have*

$$\| v_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C \| \nabla v_\varepsilon \|_{L^2(\Omega_\varepsilon)}. \tag{A.9}$$

Proof. For any function $v_\varepsilon \in H^1(\Omega_\varepsilon)$, let $\bar{v}_\varepsilon$ be the function defined by (A.8).

$$\| v_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C \| v_\varepsilon - \bar{v}_\varepsilon \|_{L^2(\Omega_\varepsilon)} + \| \bar{v}_\varepsilon \|_{L^2(\Omega_\varepsilon)}. \tag{A.10}$$

The first term in the right-hand side of (A.10) is bounded with the help of the Poincaré–Wirtinger-type inequality (2.14), i.e.

$$\| v_\varepsilon - \bar{v}_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon \| \nabla v_\varepsilon \|_{L^2(\Omega_\varepsilon)}.$$

For the second term in the right-hand side of (A.10), we use inequality (2.10), i.e.

$$\| \bar{v}_\varepsilon(x) - \bar{v}_\varepsilon(x+h) \|^2_{L^2(\Omega)} \leqslant C \,|\, h \,|^2 \, \| \nabla v_\varepsilon \|^2_{L^2(\Omega_\varepsilon)}. \tag{A.11}$$

Because of the Dirichlet condition on the exterior boundary $\partial\Omega_\varepsilon \cap \partial\Omega$, the function $\bar{v}_\varepsilon$ is equal to zero outside a neighborhood of Ω. Thus there exist a $h \in \mathbb{R}^N$ such that $\bar{v}_\varepsilon(x+h) = 0$, and (A.11) yields

$$\| \bar{v}_\varepsilon \|_{L^2(\Omega)} \leqslant C \| \nabla v_\varepsilon \|_{L^2(\Omega_\varepsilon)}. \qquad \square$$

Proof of Theorem A.1. The only difference with the proof of Theorem 1.4 comes from the first step, establishing a priori estimates for the sequences u_ε.

Multiplying equation (A.1) by u_ε, and integrating by parts, we obtain

$$\int_{\Omega_\varepsilon} {}^{\mathrm{t}}\nabla u_\varepsilon A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon = \int_{\Omega_\varepsilon} f u_\varepsilon. \tag{A.11}$$

Using the Poincaré inequality of Lemma A.4, we deduce from (A.11)

$$\| u_\varepsilon \|_{H^1(\Omega_\varepsilon)} \leqslant C. \tag{A.12}$$

At this point, we proceed as in the proof of Theorem 1.4, except that we know from Lemma A.3. that the limit u of u_ε belongs to $H^1_0(\Omega)$. Thus, we replace the last result (2.20) of the first step by

$$\begin{cases} -\nabla \cdot \xi = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.13}$$

and we repeat the second and third step. $\square$

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