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HOMOGENIZATION OF THE STOKES FLOW IN A CONNECTED POROUS MEDIUM

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Abstract. In this paper we prove the convergence of the homogenization process of the Stokes equations with Dirichlet boundary condition in a periodic porous medium. We consider here the case where the solid part of the porous medium is connected, and we generalize to this case the results obtained by Tartar (1980).

Introduction

We define a porous medium as the periodic repetition in a bounded domain Ω of an elementary ε-sized cell in which the solid part of the porous medium is also of size ε. A typical case of such a porous medium is a regular lattice of interconnected cylinders (see Fig. 1). Let Ω_ε be the fluid part contained in Ω. The flow of an incompressible viscous fluid in Ω_ε under the action of an exterior force f is ruled by the Stokes equations (S_ε) with Dirichlet boundary condition:

$$(S_\varepsilon): \quad \nabla p_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \qquad \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon$$

$(u_\varepsilon, p_\varepsilon)$ being the velocity and the pressure of the fluid.

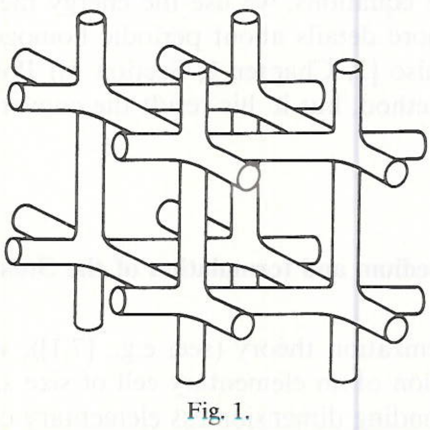

Fig. 1.

The goal of this paper is to prove the following result: let $\tilde{u}_\varepsilon$ and P_ε be extensions of u_ε and p_ε to the whole of Ω, defined in the following way:

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega - \Omega_\varepsilon; \end{cases} \qquad P_\varepsilon = \begin{cases} p_\varepsilon & \text{in } \Omega_\varepsilon, \\ \dfrac{1}{|Y^\varepsilon_{F_i}|} \displaystyle\int_{Y^\varepsilon_{F_i}} p_\varepsilon & \text{in } Y^\varepsilon_{S_i} \text{ for each } i; \end{cases}$$

where, for an ε-cell Y_i^ε, we denote by $Y^\varepsilon_{F_i}$ and $Y^\varepsilon_{S_i}$ the corresponding fluid and solid parts. Then

$$u_\varepsilon/\varepsilon^2 \to u \quad \text{in } \left[L^2(\Omega)\right]^N \text{ weakly}, \qquad P_\varepsilon \to p \quad \text{in } L^2_{\text{loc}}(\Omega)/\mathbb{R} \text{ strongly}$$

where (u, p) is the unique solution of Darcy's law (S):

$$\text{(S)}: \qquad u = \bar{A}(f - \nabla p) \quad \text{in } \Omega, \qquad \nabla \cdot u = 0 \quad \text{in } \Omega, \qquad u \cdot n = 0 \quad \text{on } \partial\Omega.$$

Here $\bar{A}$ is a constant, symmetric and positive definite matrix depending only on the elementary cell's geometry.

It is worth noticing that if Ω_ε is defined as the union of entire elementary cells, then the convergence of the sequence P_ε occurs in $L^2(\Omega)/\mathbb{R}$ (there is no more "loc"). This is the case, for example, when Ω is a cube.

The above result is the generalization of results obtained by Tartar, Lipton, and Avellaneda to another geometry. These authors considered the case of a porous medium the solid part of which is composed of disconnected grains, each included in a corresponding ε-sized cell. For that geometry, Tartar proved in [8] that there exists an extension P_ε of the pressure which allows to pass to the limit in (S_ε) obtaining (S). Lipton and Avellaneda recently noticed [4] that, actually, this extension P_ε is just obtained by taking the mean value of the pressure p_ε in the fluid part of each ε-sized cell as the value of P_ε in the solid part of the same cell. This remark illuminates the meaning of the pressure's extension defined by Tartar using a transposition process. However, Tartar's way of defining the extension seems to be necessary to prove that the sequence P_ε is bounded (and even relatively compact) in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$. Here we are concerned with the physically realistic case of a porous medium which has connected solid and fluid parts, and a diphasic boundary $\partial\Omega$. So we follow the scheme of [8], except for the pressure's extension which is new. In order to pass to the limit in the equations, we use the energy method introduced by Tartar [9] (see also [1, Chapter 1]). For more details about periodic homogenization of Stokes equations, see [7, Chapter 7]. In [6] (see also [2, Chapter 1, Section 5]) Polisevsky has already proved a similar result with a different method, but in his result the convergence of P_ε occurs in $L^{6/5}(\Omega)$ instead of $L^2(\Omega)$.

1. Modelization of the porous medium and formulation of the Stokes problem

As usual in periodic homogenization theory (see, e.g., [7,1]), we consider a porous medium obtained by the periodic repetition of an elementary cell of size ε, in a bounded domain of $\mathbb{R}^N$. We will first define the corresponding dimensionless elementary cell Y.

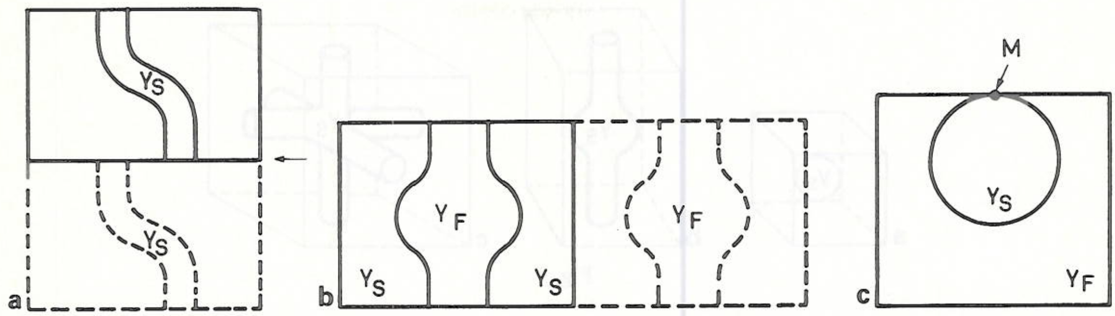

Fig. 2. Forbidden situations. (a) The boundary of E_S is not of class C^1 because Y_S is not Y-periodic. (b) No contact between the fluid parts of two adjacent cells implies E_F is not connected. (c) Although E_F has a smooth boundary, ∂Y_F is not locally Lipschitz at point M.

1.1. Definition of the elementary cell Y

Let $Y =]-1; +1[^N$ be the open unit cube of $\mathbb{R}^N$, $N \geqslant 2$. Let Y_S be a closed subset of $\overline{Y}$. We define Y_F, open set of $\mathbb{R}^N$, by $Y_F = Y - Y_S$, where Y_S represents the part of Y occupied by the solid and Y_F represents the part of Y occupied by the fluid.

The closed set Y_S is repeated by Y-periodicity and fills the entire space $\mathbb{R}^N$, in order to obtain a closed set of $\mathbb{R}^N$, noted E_S. Let the open set E_F be the complementary of E_S in $\mathbb{R}^N$, i.e.

$$E_S = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid \exists (k_1, \ldots, k_N) \in \mathbb{Z}^N \text{ such that } (x_1 - 2k_1, \ldots, x_N - 2k_N) \in Y_S\}, \qquad E_F = \mathbb{R}^N - E_S.$$

We assume the following hypotheses on Y_F and E_F:

(i) Y_F and Y_S have strictly positive measures in $\overline{Y}$;

(ii) E_F and the interior of E_S are open sets with boundary of class C^1, and are locally located on one side of their boundary. Moreover E_F is connected; (1.1)

(iii) Y_F is an open connected set with a locally Lipschitz boundary.

What is the concrete meaning of those hypotheses?

(i) means that the elementary cell Y contains fluid and solid together.

(ii) implies that Y_F has some properties:

- Y_F is "Y-periodic", because E_F has a boundary of class C^1; for example, the situation of Fig. 2(a) is forbidden.
- $\overline{Y}_F$ has an intersection with each face of the cube $\overline{Y}$ which has a strictly positive surface measure; if not, E_F could not be connected when E_F and E_S are locally located on one side of their boundary; for example, the situation of Fig. 2(b) is forbidden.

(iii) is a technical assumption necessary for the proof of Lemma 3.4; for example, the situation of Fig. 2(c) is forbidden.

In Fig. 3 we give three typical situations which agree with assumptions (1.1). Note that in Figs. 3(b) and 3(c), Y_F has a locally Lipschitz boundary which is not of class C^1. This motivates (iii) in (1.1).

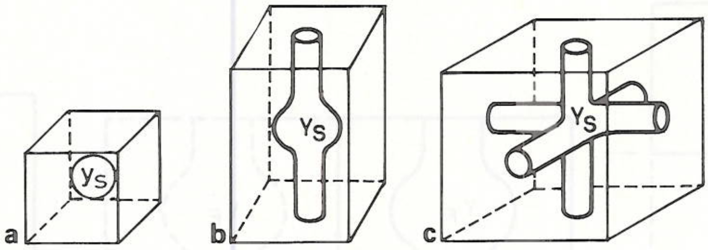

Fig. 3.

Let $(\Sigma_K)_{K\in\mathscr{K}}$ be the $2N$ faces of the cube $\overline{Y}$ with $\mathscr{K}=\{-N;\ -(N-1);\ldots;\ -1;\ +1;\ldots;\ N-1;\ N\}$ such that Σ_K and Σ_{-K} are the two faces of $\overline{Y}$ orthogonal to the Kth unit vector e_K. A first consequence of the hypotheses (1.1) is the existence of a family of functions $(\phi_K)_{K\in\mathscr{K}}$ such that

$$\begin{aligned}&\text{(i)} && \phi_K\in C^\infty(\overline{Y}) \text{ and } \phi_K\geqslant 0;\\ &\text{(ii)} && \phi_K\not\equiv 0 \quad \text{on } \Sigma_K, \quad \phi_K\equiv 0 \quad \text{on } Y_S \text{ and } \Sigma_{K'} \text{ for each } K'\neq K;\\ &\text{(iii)} && \text{let } \Sigma_K \text{ and } \Sigma_{-K} \text{ be two opposite faces of } \overline{Y}; \text{ then } \phi_K|_{\Sigma_K}\equiv\phi_{-K}|_{\Sigma_{-K}}.\end{aligned} \tag{1.2}$$

($\phi_K|_{\Sigma_K}$ denotes the restriction of ϕ_K to Σ_K). Examples are shown in Figs. 4 and 5.

Remark 1.1. In [8], Tartar considered the case (corresponding to Fig. 3(a)) $Y_S\subset\subset Y$ (Y_S is strictly included in Y). This assumption is not physically realistic in three dimensions because the solid part E_S is not a connected body.

1.2. Definition of the open set Ω_ε

Let Ω be a bounded and connected open set of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ of class C^1 ($N\geqslant 2$). Let $\varepsilon>0$. The set Ω is covered with a regular mesh of size 2ε, each cell being a cube Y_i^ε

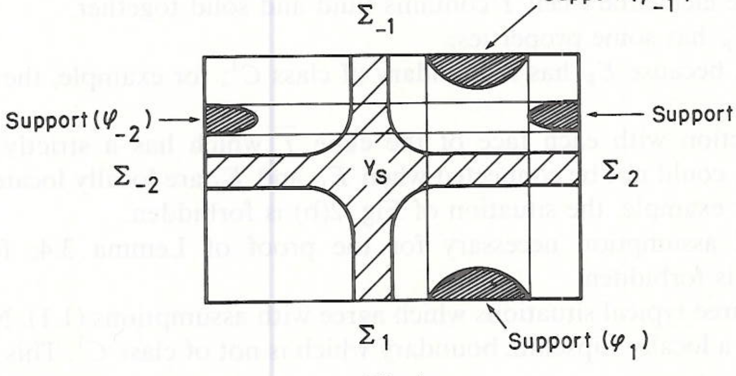

Fig. 4.

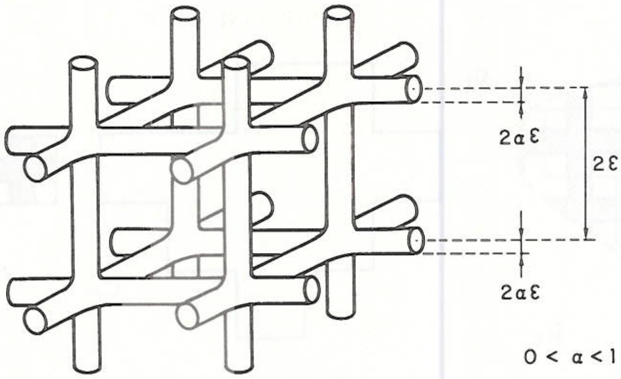

Fig. 5.

with $1 \leqslant i \leqslant N(\varepsilon)$. An elementary geometrical consideration gives

$$N(\varepsilon) = \frac{|\Omega|}{(2\varepsilon)^N}[1 + o(1)].$$

Let π_i^ε be the linear continuous invertible application, composed of a translation and an homothety of ratio $1/\varepsilon$, which maps Y_i^ε onto Y:

$$Y_i^\varepsilon \xrightarrow{\pi_i^\varepsilon} Y, \qquad x \to y = x/\varepsilon + \text{translation}. \tag{1.3}$$

Now we define

$$Y_{S_i}^\varepsilon = (\pi_i^\varepsilon)^{-1}(Y_S), \qquad Y_{F_i}^\varepsilon = (\pi_i^\varepsilon)^{-1}(Y_F), \qquad \Sigma_K^\varepsilon = (\pi_i^\varepsilon)^{-1}(\Sigma_K).$$

We construct Ω_ε by picking out from Ω the solid parts $Y_{S_i}^\varepsilon$: $\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} Y_{S_i}^\varepsilon$. Ω_ε denotes the part of Ω occupied by the fluid.

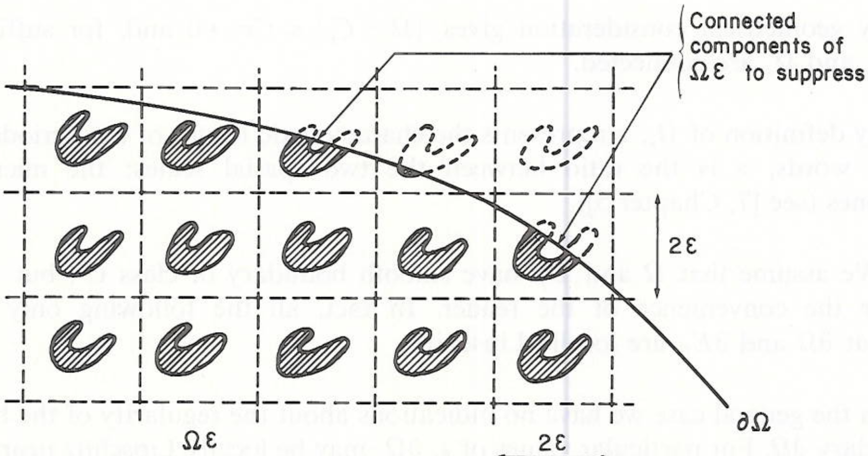

Fig. 6.

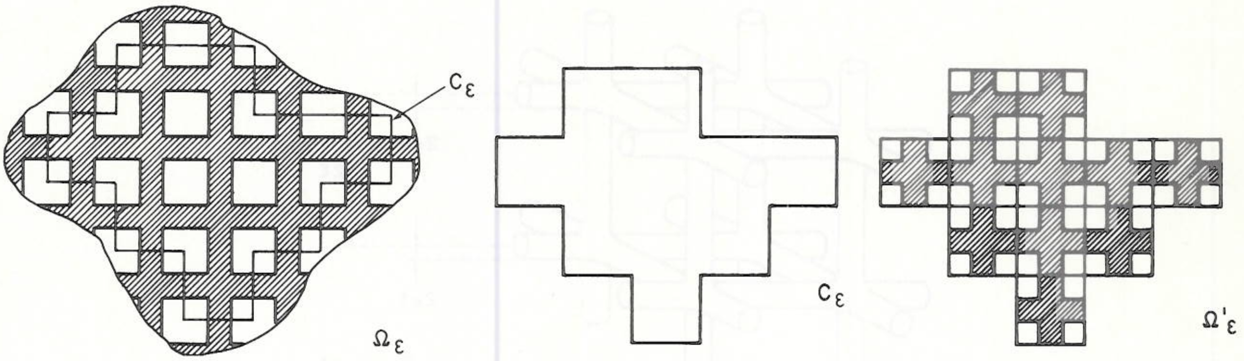

Fig. 7.

By construction Ω_ε is a bounded open set of $\mathbb{R}^N$. Unfortunately, Ω_ε is not necessarily connected. Indeed, near the boundary $\partial\Omega$ there may be connected components of Ω_ε which have a maximum size ε (see Fig. 6). In order to simplify the exposition and without loss of generality, we can suppress those ε-sized connected components, and from now on we assume Ω_ε to be connected (it will allow us to define a unique pressure in Ω_ε, up to a single additive constant in Ω_ε).

The set Ω represents the porous medium, and Ω_ε its fluid part. Remark that Ω_ε is supposed to be connected, but the solid part of the porous medium, which is represented by $\Omega - \Omega_\varepsilon$, may be connected or not. Moreover, one can see that the boundary $\partial\Omega$ of the porous medium may be either diphasic (i.e., $\partial\Omega \cap \overline{\Omega}_\varepsilon \neq \emptyset$ and $\partial\Omega \cap \overline{(\Omega - \Omega_\varepsilon)} \neq \emptyset$) or not. Obviously, the physically realistic case (a connected solid part, and a diphasic boundary) is taken into account in the present paper.

We now define C_ε and Ω'_ε which approach respectively Ω and Ω_ε in the following sense: let C_ε be the polygonal open set constituted by all the cells Y_i^ε entirely included in Ω, and let Ω'_ε be the fluid part of C_ε (see Fig. 7). More precisely,

$$\overline{C}_\varepsilon = \bigcup_{i \in I(\varepsilon)} \overline{Y}_i^\varepsilon \quad \text{with } I(\varepsilon) = \{ i \in [1;\ N(\varepsilon)] \mid Y_i^\varepsilon \subset \Omega \}, \qquad \Omega'_\varepsilon = C_\varepsilon \cap \Omega_\varepsilon. \tag{1.4}$$

An elementary geometrical consideration gives $|\Omega - C_\varepsilon| \leqslant C\varepsilon \to 0$ and, for sufficiently small values of ε, C_ε and Ω'_ε are connected.

Remark 1.2. By definition of Ω_ε, ε represents the characteristic length of the periodic elementary cell. In other words, ε is the ratio between the two spatial scales: the microscopic and macroscopic ones (see [7, Chapter 5]).

Remark 1.3. We assume that Ω and E_{F} have smooth boundary of class C^1, but this has been done only for the convenience of the reader. In fact, all the following only requires the assumption that $\partial\Omega$ and ∂E_{F} are locally Lipschitz.

Remark 1.4. In the general case we have no indications about the regularity of the boundary $\partial\Omega_\varepsilon$ near the boundary $\partial\Omega$. For particular values of ε, $\partial\Omega_\varepsilon$ may be locally Lipschitz near $\partial\Omega$, but it is, in general, not true for all the values of ε. That is why, in the sequel, the pressure's extension will be defined only in $L^2_{\mathrm{loc}}(\Omega)$ (instead of $L^2(\Omega)$). This is the price to pay in order to be able to handle the diphasic boundary $\partial\Omega$.

1.3. Formulation of the Stokes problem

The flow of an incompressible viscous fluid in the domain Ω_ε under the action of an exterior force f, and with a no-slip (Dirichlet) boundary condition, is described by the following Stokes equations, where u_ε is the fluid velocity, p_ε is the fluid pressure and f given in $[L^2(\Omega)]^N$:

$$(\mathrm{S}_\varepsilon): \qquad \nabla p_\varepsilon - \Delta u_\varepsilon = f \ \text{ in } \Omega_\varepsilon, \qquad \nabla \cdot u_\varepsilon = 0 \ \text{ in } \Omega_\varepsilon, \qquad u_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon. \tag{1.5–7}$$

(The viscosity and density of the fluid are supposed to be equal to 1.)

If Ω_ε is a bounded and connected open set with a locally Lipschitz boundary, a classical result asserts that there exists a unique solution $(u_\varepsilon, p_\varepsilon)$ of (S_ε) belonging to $[H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}]$. But in the present case the boundary $\partial\Omega_\varepsilon$ is not locally Lipschitz in the vicinity of $\partial\Omega$, hence the existence and uniqueness of the solution $(u_\varepsilon, p_\varepsilon)$ stand only in the space $V_\varepsilon \times L_\varepsilon$ with V_ε being the closure in $[H_0^1(\Omega_\varepsilon)]^N$ of the space ϑ_ε, where

$$\vartheta_\varepsilon = \left\{ \phi \in \left[\mathscr{D}(\Omega_\varepsilon)\right]^N \mid \nabla \cdot \phi = 0 \text{ in } \Omega_\varepsilon \right\},$$
$$L_\varepsilon = \left\{ q \in \mathscr{D}'(\Omega_\varepsilon) \mid \forall \omega \subset\subset \Omega,\ q \in L^2(\omega \cap \Omega_\varepsilon) \right\} / \mathbb{R}. \tag{1.8}$$

We have the following inclusions which may be strict:

$$V_\varepsilon \subset \left\{ \phi \in \left[H_0^1(\Omega_\varepsilon)\right]^N \mid \nabla \cdot \phi = 0 \text{ in } \Omega_\varepsilon \right\}, \qquad L_\varepsilon \supset L^2(\Omega_\varepsilon)/\mathbb{R}.$$

For more details about those existence and uniqueness results for Stokes equations, see, for example, [11, Chapter 1, Section 2].

In order to predict the "homogenized" limit of equations (S_ε), one can apply the asymptotic expansion method to the system (S_ε). Assume that

$$\begin{cases} u^\varepsilon(x) = \varepsilon^2\left[u^0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots\right], \\ p^\varepsilon(x) = p_0(x, x/\varepsilon) + \varepsilon p_1(x, x/\varepsilon) + \varepsilon^2 p_2(x, x/\varepsilon) + \cdots \end{cases} \tag{1.9}$$

where $u_i(x, y)$ and $p_i(x, y)$ are Y-periodic in the variable y. Then one can heuristically obtain (see [7, Chapter 7]) that $p_0(x, x/\varepsilon) = p(x)$, where $p \in H^1(\Omega)$ is the unique solution of

$$(\mathrm{S}): \qquad \nabla \cdot \left[\bar{A}(f - \nabla p)\right] = 0 \quad \text{in } \Omega, \qquad \left[\bar{A}(f - \nabla p)\right] \cdot n = 0 \quad \text{on } \partial\Omega, \tag{1.10}$$

and that $u_0(x, x/\varepsilon) = A(x/\varepsilon) \cdot [f(x) - \nabla p(x)]$. Here, $\bar{A} = (1/|Y|)\int_Y A(y)\,\mathrm{d}y$, a symmetric, positive definite matrix and $A(y)$ is the matrix composed of the column vectors $v_K(y)$ for $1 \leqslant K \leqslant N$, defined as the unique solutions $(v_K, q_K) \in [H^1(Y_\mathrm{F})]^N \times [L^2(Y_\mathrm{F})/\mathbb{R}]$ of the following systems (S_K):

$$(\mathrm{S}_K): \qquad \begin{array}{ll} \nabla q_K - \Delta v_K = e_K \ \text{ in } Y_\mathrm{F}, & \nabla \cdot v_K = 0 \ \text{ in } Y_\mathrm{F}, \\ v_K = 0 \ \text{ on } \partial Y_\mathrm{S}, & v_K \text{ and } q_K \text{ are } Y\text{-periodic} \end{array} \tag{1.11}$$

where e_K is the Kth unit vector of $\mathbb{R}^N$.

Remark 1.5. (1.10) is a Darcy's law for the pressure p in a medium of permeability $\bar{A}$. (For more details about those results, see [7, Chapter 7].)

2. Convergence of the homogenization process

Notation. Let $\tilde{\ }$ denote the extension by zero operator in $\Omega - \Omega_\varepsilon$. More precisely, for each $\phi \in H_0^1(\Omega_\varepsilon)$, we define $\tilde{\phi} \in H_0^1(\Omega)$ by

$$\tilde{\phi} = \begin{cases} \phi & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega - \Omega_\varepsilon. \end{cases}$$

2.1. Statement of the main results

The purpose of this paper is to prove the following theorem.

Theorem 2.1. *Consider the unique solution* $(u_\varepsilon, p_\varepsilon)$ *of* (S_ε). *Under the assumptions* (1.1) *on the elementary cell, there exists* $P_\varepsilon \in L^2_{\text{loc}}(\Omega)$, *which extends the pressure* p_ε *to the whole of* Ω *(i.e.* $P_\varepsilon \equiv p_\varepsilon$ *in* Ω_ε*), such that*

$$\tilde{u}_\varepsilon/\varepsilon^2 \to u \quad \text{in } \left[L^2(\Omega)\right]^N \text{ weakly}, \qquad P_\varepsilon \to p \quad \text{in } L^2_{\text{loc}}(\Omega)/\mathbb{R} \text{ strongly}$$

where $u = \bar{A}(f - \nabla p)$ *and* p *is the unique solution of* (S) *(see* (1.10)).

Remark 2.2. In the statement of Theorem 2.1 the convergence of the pressure's extension occurs in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$, instead of $L^2(\Omega)/\mathbb{R}$ as one could expect. Nevertheless, if Ω is such that $\Omega = C_\varepsilon$ for a sequence of ε values which tends to zero (e.g., this is the case if Ω is a cube), then the pressure's convergence really occurs in $L^2(\Omega)/\mathbb{R}$ (without the "loc") for this sequence. This "local" result is imposed only because there are "cutted" cubes Y_i^ε in the vicinity of $\partial\Omega$ (see also Remark 3.2).

The main difficulty in the proof of Theorem 2.1 is to extend the pressure to the whole of Ω. For this purpose we need the following crucial theorem.

Theorem 2.3. *Assume that the hypotheses* (1.1) *on the elementary cell Y hold true. Then there exists a linear continuous operator* R_ε *such that*

(i) $R_\varepsilon \in \mathcal{L}\{[H_0^1(C_\varepsilon)]^N; [H_0^1(\Omega'_\varepsilon)]^N\}$,

(ii) $u \in [H_0^1(\Omega'_\varepsilon)]^N$ *implies* $R_\varepsilon \tilde{u} = u$ *in* Ω'_ε,

(iii) $\nabla \cdot u = 0$ *in* C_ε *implies* $\nabla \cdot (R_\varepsilon u) = 0$ *in* Ω'_ε,

(iv) *there exists a constant C, which does not depend on* ε, *such that, for each* $u \in [H_0^1(C_\varepsilon)]^N$, *we have*

$$\| R_\varepsilon u \|_{L^2(\Omega'_\varepsilon)} + \varepsilon \| \nabla (R_\varepsilon u) \|_{L^2(\Omega'_\varepsilon)} \leqslant C\Big[\| u \|_{L^2(C_\varepsilon)} + \varepsilon \| \nabla u \|_{L^2(C_\varepsilon)} \Big]$$

(see (1.4) *for the definitions of* C_ε *and* Ω'_ε*).*

Remark 2.4. In order to construct an extension P_ε of the pressure p_ε, we will proceed as follows. In fact, we construct an extension of ∇p_ε in $[H^{-1}(\Omega)]^N$ which is defined as the "dual" operator of R_ε. Because R_ε is some kind of "restriction" operator from the set of the divergence-free vectors of $[H_0^1(C_\varepsilon)]^N$ into the set of the divergence-free vectors of $[H_0^1(\Omega'_\varepsilon)]^N$, the extension of

∇p_ε is also the gradient of a function P_ε. And, thanks to the properties of R_ε, P_ε is just the bounded pressure's extension we were looking for.

Remark 2.5. Theorems 2.1 and 2.3 have already been proved by Tartar in [8], in the particular case where Y_S is strictly included in Y (see Remark 1.1 and Fig. 3(a)). In this case, the pressure's extension converges strongly in $L^2(\Omega)/\mathbb{R}$ (without the "loc") if we get rid of the solid parts $Y^\varepsilon_{S_i}$ which cut the boundary $\partial\Omega$.

The originality of the present paper comes from the proof of Theorem 2.3 which follows the lines of Tartar's proof but needs additional ideas, due to the more complex geometry. Another original aspect is the "local" convergence of the pressure's extension in Theorem 2.1 (this is a consequence of the diphasic boundary $\partial\Omega$). Polisevsky has also proved theorems similar to Theorems 2.1 and 2.3 (see [6] or [2, Chapter 1, Section 5]), but his pressure's extension converges in $L^{6/5}(\Omega)$ instead of $L^2(\Omega)$.

Whereas R_ε is explicitly constructed (see the proof of Theorem 2.3 in Section 3), the extension P_ε of the pressure p_ε is not explicitly constructed but is derived from R_ε through a theoretical "duality" argument. An interesting problem is then to find the explicit values of P_ε in the "solid part" $\Omega - \Omega_\varepsilon$. Lipton and Avelladena [4] work out this problem, and we reproduce their important result.

Theorem 2.6. *Let R_ε be the operator of $\mathscr{L}\{[H^1_0(C_\varepsilon)]^N; [H^1_0(\Omega'_\varepsilon)]^N\}$ which is explicitly constructed in the proof of Theorem 2.3. Then the extension P_ε of the pressure p_ε, which is derived from R_ε in Theorem 2.1, satisfies the following*:

(i) *$P_\varepsilon = p_\varepsilon$ in the fluid part Ω_ε*

(ii) *in each cell Y^ε_i included in C_ε, P_ε is a constant in the solid part $Y^\varepsilon_{S_i}$, which is explicitly given by*

$$P_\varepsilon = \frac{1}{|Y^\varepsilon_{F_i}|}\int_{Y^\varepsilon_{F_i}} p_\varepsilon \quad \text{in } Y^\varepsilon_{S_i}.$$

The proofs of Theorems 2.3 and 2.6 can be found in Section 3.

2.2. Some technical lemmas

Lemma 2.7 (Poincaré's inequality in Ω_ε). *There exists a constant C which depends only on Y_F, and not on Ω or ε, such that, for each $u \in H^1_0(\Omega_\varepsilon)$, one has $\| u \|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon \| \nabla u \|_{L^2(\Omega_\varepsilon)}$.*

Proof. See [8]. □

Lemma 2.8. *Let ω be a bounded, connected, open set of $\mathbb{R}^N$, with a locally Lipschitz boundary. Let p be a distribution in ω. If $\nabla p \in [H^{-1}(\omega)]^N$, then $p \in L^2(\omega)/\mathbb{R}$ and one has $\| p \|_{L^2(\omega)/\mathbb{R}} \leqslant C \| \nabla p \|_{H^{-1}(\omega)}$ where the constant C depends only on ω (and not on p).*

Proof. See [5]. □

Lemma 2.9. *Let ω be a bounded, connected, open set of $\mathbb{R}^N$, with a locally Lipschitz boundary. Let $f \in [H^{-1}(\omega)]^N$ such that, for each $u \in [H_0^1(\omega)]^N$ with $\nabla \cdot u = 0$ in ω, one has $\langle f, u \rangle_{H^{-1}, H_0^1(\omega)} = 0$; then there exists $p \in L^2(\omega)/\mathbb{R}$ such that $f = \nabla p$ in ω.*

Proof. See [11, Chapter I, Remark 1.9]. □

Lemma 2.10. *Let ω be a bounded, connected, open set of $\mathbb{R}^N$, with a locally Lipschitz boundary. For each $f \in L^2(\omega)$ with $\int_\omega f = 0$, there exists $u \in [H_0^1(\omega)]^N$ such that $\nabla \cdot u = f$ in ω. Moreover, one can choose u in such a way that the application $f \to u$ is linear and continuous, with $\| u \|_{H_0^1(\omega)} \leqslant C \| f \|_{L^2(\omega)}$ where the constant C depends only on ω.*

Proof. See [11, Chapter I, Lemma 2.4]. □

Remark 2.11. Lemmas 2.8 through 2.10 are strongly connected. For example, Temam proved Lemmas 2.9 and 2.10 in [11] with the help of Lemma 2.8 proved by Necas in [5]. If ω has a boundary of class C^1, Tartar gives a self-contained proof of Lemmas 2.8–2.10 in [10, pp. 26-31] (see also [3, Chapter 1, Section 2] which reproduces the proof of Tartar).

2.3. Proof of convergence

This section is devoted to the proof of Theorem 2.1 under the assumption that Theorem 2.3 holds true. The proof is divided into two parts:

(2.3.1) extending the pressure,

(2.3.2) passing to the limit in the equations.

Part (2.3.1) follows [8] with slight modifications due to the "local" character of convergence of the pressure's extension. Part (2.3.2) reproduces [8] and is given here in order for the present paper to be self-contained.

2.3.1. Extending the pressure

Multiplying the following equation by u_ε

$$\nabla p_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon \tag{1.5}$$

and integrating by parts on Ω_ε, we obtain

$$\| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon)}^2 = \int_{\Omega_\varepsilon} f \cdot u_\varepsilon . \tag{2.1}$$

Using Lemma 2.7 we find that

$$\| \nabla u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon \| f \|_{L^2(\Omega)} \tag{2.2}$$

implying

$$\| u_\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon^2 \| f \|_{L^2(\Omega)} . \tag{2.3}$$

Thus $\tilde{u}_\varepsilon / \varepsilon^2$ is a bounded sequence in $[L^2(\Omega)]^N$, and therefore we can extract a subsequence such that there exists $u \in [L^2(\Omega)]^N$ with

$$\tilde{u}_\varepsilon / \varepsilon^2 \to u \quad \text{in } \left[L^2(\Omega) \right]^N \text{ weakly.} \tag{2.4}$$

While the velocity u_ε can be naturally continued by zero in $\Omega - \Omega_\varepsilon$, it is not obvious to construct an extension to Ω of the pressure p_ε (which is defined only on Ω_ε). With the help of Theorem 2.3 this will be achieved.

Let F_ε be defined in $[H^{-1}(C_\varepsilon)]^N$ by the following formula (see (1.4) for the definitions of C_ε and Ω'_ε):

$$\text{for each } v \in \left[H_0^1(C_\varepsilon)\right]^N \quad \langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(C_\varepsilon)} = \langle \nabla p_\varepsilon, R_\varepsilon v\rangle_{H^{-1},H_0^1(\Omega'_\varepsilon)} \tag{2.5}$$

where R_ε is given by Theorem 2.3 (we also denote by ∇p_ε the restriction of ∇p_ε to Ω'_ε).

Because R_ε is linear, F_ε is a linear functional on $[H_0^1(C_\varepsilon)]^N$. In order to estimate its norm, we write

$$\langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(C_\varepsilon)} = \langle f + \Delta u_\varepsilon, R_\varepsilon v\rangle_{H^{-1},H_0^1(\Omega'_\varepsilon)}.$$

Integrating by parts, we obtain

$$\langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(C_\varepsilon)} = \int_{\Omega'_\varepsilon} f \cdot R_\varepsilon v - \int_{\Omega'_\varepsilon} \nabla u_\varepsilon \cdot \nabla(R_\varepsilon v).$$

With the help of property (iv) of Theorem 2.3 and of inequality (2.2) we can majorate the functional and we obtain

$$|\langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(C_\varepsilon)}| \leqslant C \| f \|_{L^2(\Omega)} \left[\| v \|_{L^2(C_\varepsilon)} + \varepsilon \| \nabla v \|_{L^2(C_\varepsilon)} \right]. \tag{2.6}$$

Thus, if $\varepsilon < 1$, $\| F_\varepsilon \|_{H^{-1}(C_\varepsilon)} \leqslant C \| f \|_{L^2(\Omega)}$.
Moreover property (iii) of Theorem 2.3 implies that, for each $v \in [H_0^1(C_\varepsilon)]^N$ with $\nabla \cdot v = 0$ in C_ε, we have $\langle F_\varepsilon, v\rangle_{H^{-1},H_0^1(C_\varepsilon)} = 0$. Applying Lemma 2.9 we deduce the existence of $P_\varepsilon \in L^2(C_\varepsilon)/\mathbb{R}$ such that $F_\varepsilon = \nabla P_\varepsilon$ in C_ε. On the other hand, Lemma 2.8 provides an estimate of P_ε depending on the norm of its gradient:

$$\| P_\varepsilon \|_{L^2(C_\varepsilon)/\mathbb{R}} \leqslant C \| F_\varepsilon \|_{H^{-1}(C_\varepsilon)} \leqslant C \| f \|_{L^2(\Omega)}. \tag{2.7}$$

Thus we have defined P_ε on C_ε, and we complete this definition on $\Omega - C_\varepsilon$ with

$$P_\varepsilon \equiv p_\varepsilon \quad \text{on } (\Omega - C_\varepsilon) \cap \Omega_\varepsilon, \qquad P_\varepsilon \equiv 0 \quad \text{on } (\Omega - C_\varepsilon) - \Omega_\varepsilon. \tag{2.8}$$

Now we prove that P_ε is an extension of p_ε with the help of property (ii) of Theorem 2.3: for each $v \in [H_0^1(\Omega'_\varepsilon)]^N$ we have $R_\varepsilon \tilde{v} = v$ in Ω'_ε, implying

$$\langle \nabla P_\varepsilon, \tilde{v}\rangle_{H^{-1},H_0^1(C_\varepsilon)} = \langle \nabla p_\varepsilon, v\rangle_{H^{-1},H_0^1(\Omega'_\varepsilon)}$$

which is equivalent to

$$\int_{C_\varepsilon} P_\varepsilon \nabla \cdot \tilde{v} = \int_{\Omega'_\varepsilon} P_\varepsilon \nabla \cdot v = \int_{\Omega'_\varepsilon} p_\varepsilon \nabla \cdot v.$$

Thanks to Lemma 2.10 we can assert that, for each $f \in L^2(\Omega'_\varepsilon)$ with $\int_{\Omega'_\varepsilon} f = 0$, there exists $v \in [H_0^1(\Omega'_\varepsilon)]^N$ such that $\nabla \cdot v = f$ in Ω'_ε. Then $\int_{\Omega'_\varepsilon} f(P_\varepsilon - p_\varepsilon) = 0$ for each $f \in L^2(\Omega'_\varepsilon)$ with $\int_{\Omega'_\varepsilon} f = 0$ implying $(P_\varepsilon - p_\varepsilon) \equiv 0$ in $L^2(\Omega'_\varepsilon)/\mathbb{R}$. Moreover, by definition (2.8) we have $P_\varepsilon \equiv p_\varepsilon$ on $(\Omega - C_\varepsilon) \cap \Omega_\varepsilon$, so we can conclude $P_\varepsilon \equiv p_\varepsilon$ in Ω'_ε (up to an additive constant). Thus P_ε is strictly an extension of the pressure p_ε.

Now we prove the existence of $p \in L^2(\Omega)/\mathbb{R}$ such that $P_\varepsilon \to p$ in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$ strongly. First recall the estimate (2.7): $\| P_\varepsilon \|_{L^2(C_\varepsilon)/\mathbb{R}} \leqslant C \| f \|_{L^2(\Omega)}$. For each open set ω, strictly included in Ω, choose sufficiently small values of ε such that $\omega \subset C_\varepsilon$. Thus one can deduce from (2.7) that $\| P_\varepsilon \|_{L^2(\omega)/\mathbb{R}} \leqslant C \| f \|_{L^2(\Omega)}$. This means that P_ε is bounded in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$. So we can extract a subsequence, still denoted by P_ε, and there exists $p \in L^2_{\text{loc}}(\Omega)/\mathbb{R}$ such that $P_\varepsilon \to p$ in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$ weakly. Because of the weak convergence of P_ε in $L^2(\omega)/\mathbb{R}$, we have, for any open set ω strictly included in Ω,

$$\| p \|_{L^2(\omega)/\mathbb{R}} \leqslant \liminf_{\varepsilon \to 0} \| P_\varepsilon \|_{L^2(\omega)/\mathbb{R}}.$$

By combining this estimate with inequality (2.7), we obtain $\| p \|_{L^2(\omega)/\mathbb{R}} \leqslant C \| f \|_{L^2(\Omega)}$ for each $\omega \subset\subset \Omega$ and C does not depend on ω, implying $p \in L^2(\Omega)/\mathbb{R}$ (there is no more "loc").

To show that the convergence of P_ε to p is, in fact, strong in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$, we consider a sequence w_ε such that $w_\varepsilon \to w$ in $[H^1_0(\omega)]^N$ weakly, where ω is a smooth open set, strictly included in Ω. Consider the following inequality:

$$|\langle \nabla P_\varepsilon, w_\varepsilon \rangle_{H^{-1}, H^1_0(\omega)} - \langle \nabla p, w \rangle_{H^{-1}, H^1_0(\omega)}| \leqslant |\langle \nabla P_\varepsilon, w_\varepsilon - w \rangle| + |\langle \nabla P_\varepsilon - \nabla p, w \rangle|. \tag{2.9}$$

Integrating by parts on ω the second right-hand side member of (2.9), we obtain

$$\langle \nabla P_\varepsilon - \nabla p, w \rangle = -\int_\omega (P_\varepsilon - p) \nabla \cdot w \xrightarrow[\varepsilon \to 0]{} 0$$

because $P_\varepsilon \to p$ in $L^2(\omega)/\mathbb{R}$ weakly. Using inequality (2.6) for the first right-hand side member of (2.9), we obtain

$$|\langle \nabla P_\varepsilon, w_\varepsilon - w \rangle| \leqslant C \| f \|_{L^2(\Omega)} \big[\| w_\varepsilon - w \|_{L^2(\omega)} + \varepsilon \| \nabla w_\varepsilon - \nabla w \|_{L^2(\omega)} \big]. \tag{2.10}$$

By the definition of w_ε, we have that (∇w_ε) is bounded in $[L^2(\omega)]^{N^2}$. By virtue of Rellich's Theorem, $w_\varepsilon \to w$ in $[L^2(\omega)]^N$ strongly. Recalling inequality (2.9), we have

$$|\langle \nabla P_\varepsilon, w_\varepsilon \rangle - \langle \nabla p, w \rangle| \xrightarrow[\varepsilon \to 0]{} 0$$

for each sequence w_ε which converges weakly in $[H^1_0(\omega)]^N$. This is just the definition of the strong convergence of ∇P_ε in $[H^{-1}(\omega)]^N$. Because ω has a smooth boundary, we can apply Lemma 2.8, and we obtain $P_\varepsilon \to p$ in $L^2(\omega)/\mathbb{R}$ strongly and this is true for each smooth open set ω strictly included in Ω. Thus we can conclude $P_\varepsilon \to p$ in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$ strongly.

2.3.2. Passing to the limit in the equations

We apply the energy method introduced by Tartar in [9] (see also [1]). First we multiply the equation $\nabla \cdot u_\varepsilon = 0$ by $\phi \in H^1(\Omega)$, and we integrate by parts on Ω. We obtain

$$\int_\Omega (\nabla \cdot \tilde{u}_\varepsilon) \phi = 0 \iff -\int_\Omega \tilde{u}_\varepsilon \cdot \nabla \phi + \int_{\partial\Omega} \phi \tilde{u}_\varepsilon \cdot n = 0$$

and $\int_{\partial\Omega} \phi \tilde{u}_\varepsilon \cdot n = 0$ because $\tilde{u}_\varepsilon \in [H^1_0(\Omega)]^N$. We pass to the limit $\varepsilon \to 0$, and we get

$$\int_\Omega u \cdot \nabla \phi = 0 \iff -\int_\Omega \phi \nabla \cdot u + \int_{\partial\Omega} \phi u \cdot n = 0$$

and this is true for each $\phi \in H^1(\Omega)$. Thus,

$$\begin{cases} \nabla \cdot u = 0 & \text{in } \Omega \ (\text{in } H^{-1}(\Omega)), \\ u \cdot n = 0 & \text{on } \partial\Omega \ (\text{in } H^{-1/2}(\partial\Omega)), \end{cases} \tag{2.11}$$

On the other hand, using definition (1.11), we define the following functions: let $(v_K^\varepsilon, q_K^\varepsilon) \in [H^1(\Omega)]^N \times [L^2(\Omega_\varepsilon)]$ such that

$$\begin{cases} v_K^\varepsilon(x) = v_K(x/\varepsilon), \\ q_K^\varepsilon(x) = q_K(x/\varepsilon) \end{cases} \quad (\text{extended in } \Omega \text{ by } \varepsilon Y\text{-periodicity}). \tag{2.12}$$

$(v_K^\varepsilon, q_K^\varepsilon)$ satisfy the following system:

$$\varepsilon \nabla q_K^\varepsilon - \varepsilon^2 \Delta v_K^\varepsilon = e_K \quad \text{in } \Omega_\varepsilon, \qquad \nabla \cdot v_K^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \qquad v_K^\varepsilon = 0 \quad \text{in } \Omega - \Omega_\varepsilon \tag{2.13–15}$$

with the following estimates:

$$\| q_K^\varepsilon \|_{L^2(\Omega_\varepsilon)} \leqslant C, \qquad \| v_K^\varepsilon \|_{L^2(\Omega)} \leqslant C; \qquad \| \nabla v_K^\varepsilon \|_{L^2(\Omega)} \leqslant C/\varepsilon \tag{2.16; 17}$$

where C does not depend on ε. Moreover, classically we know that

$$q_K^\varepsilon \to 0 \quad \text{in } \left[L^2(\Omega)/\mathbb{R}\right] \text{ weakly,}$$

$$v_K^\varepsilon \to \frac{1}{|Y|} \int_Y v_K(y)\, \mathrm{d}y = \bar{A} e_K \quad \text{in } \left[L^2(\Omega)\right]^N \text{ weakly.}$$

Let $\phi \in \mathscr{D}(\Omega)$. We multiply the equations (2.13) by $\phi \tilde{u}_\varepsilon$, and (1.5) by ϕv_K^ε, and we integrate by parts on Ω. We obtain

$$\int_\Omega \nabla v_K^\varepsilon \cdot \nabla(\phi \tilde{u}_\varepsilon) = \frac{1}{\varepsilon} \int_\Omega q_K^\varepsilon \tilde{u}_\varepsilon \cdot \nabla\phi + \frac{1}{\varepsilon^2} \int_\Omega e_K \cdot \tilde{u}_\varepsilon \phi, \tag{2.18}$$

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla(\phi v_K^\varepsilon) = \int_\Omega f \cdot v_K^\varepsilon \phi + \int_\Omega P_\varepsilon v_K^\varepsilon \cdot \nabla\phi. \tag{2.19}$$

For sufficiently small values of ε, we have Support$(\phi) \subset C_\varepsilon$. Consequently, the *local* but *strong* convergence of P_ε in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$ is sufficient to pass to the limit in the following term:

$$\int_\Omega P_\varepsilon v_K^\varepsilon \cdot \nabla\phi \xrightarrow[\varepsilon \to 0]{} \int_\Omega p\left(\bar{A} e_K\right) \cdot \nabla\phi.$$

Therefore we have

$$\int_\Omega \nabla v_K^\varepsilon \cdot \nabla(\phi \tilde{u}_\varepsilon) \xrightarrow[\varepsilon \to 0]{} \int_\Omega \phi u \cdot e_K,$$

$$\int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla(\phi v_K^\varepsilon) \xrightarrow[\varepsilon \to 0]{} \int_\Omega \phi f \cdot \left(\bar{A} e_K\right) - \int_\Omega \phi\left(\bar{A} e_K\right) \cdot \nabla p.$$

On the other hand, using the estimates of u_ε and v_K^ε we obtain

$$\left| \int_\Omega \nabla \tilde{u}_\varepsilon \cdot \nabla(\phi v_K^\varepsilon) - \int_\Omega \nabla v_K^\varepsilon \cdot \nabla(\phi \tilde{u}_\varepsilon) \right| \leqslant C\varepsilon \to 0.$$

Hence,

$$\int_\Omega \phi u \cdot e_K = \int_\Omega \phi f \cdot (\bar{A} e_K) - \int_\Omega \phi \nabla p \cdot (\bar{A} e_K)$$

and this is true for each $\phi \in \mathscr{D}(\Omega)$. Thus, $u = \bar{A}(f - \nabla p)$ in $\mathscr{D}'(\Omega)$. Recalling (2.11) we obtain (S):

$$\text{(S):} \qquad \nabla \cdot [\bar{A}(f - \nabla p)] = 0 \quad \text{in } \Omega, \qquad [\bar{A}(f - \nabla p)] \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1.10}$$

A classical result asserts that (S) has a unique solution $p \in H^1(\Omega)/\mathbb{R}$; therefore, not only a subsequence, but the whole sequence P_ε converges to p in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$. The same result holds for the velocity: the whole sequence $\tilde{u}_\varepsilon$ converges to u in $[L^2(\Omega)]^N$ weakly. And thus Theorem 2.1 has been proved.

3. Construction of the operator R_ε

The third section of this paper is devoted to the proofs of Theorems 2.3 and 2.6. First we establish two important lemmas in order to deal, just afterwards, with the proof of Theorem 2.3. Lastly we reproduce the proof of Theorem 2.6 following [4].

The idea of the construction of an operator R_ε, which satisfies the properties of Theorem 2.3 in order to extend the pressure, is due to Tartar [8]. Unfortunately, his construction of R_ε applies only to the case where Y_S is strictly included in Y and is explicitly local in each cube Y_i^ε. We propose a generalization of this result when Y_S is no longer strictly included in Y, and, in this case, the construction of R_ε is partly global (because of the introduction of an operator Q_ε which projects $H_0^1(C_\varepsilon)$ on $H_0^1(\Omega_\varepsilon')$), and partly local (because R_ε is still constructed in each cube Y_i^ε). In fact, $Q_\varepsilon u$ defines the boundary values on ∂Y_i^ε of the function $R_\varepsilon u$ in each cell Y_i^ε.

Remark 3.1. Using several trace properties of the functions belonging to $W^{1,6}(Y)$, Polisevsky has constructed in [6] (see also [2, Chapter 1, Section 5]) an operator R_ε in the case where Y_S is not strictly included in Y. But R_ε operates from $[W_0^{1,6}(\Omega)]^3$, instead of $[H_0^1(\Omega)]^3$, into $[H_0^1(\Omega_\varepsilon)]^3$.

Remark 3.2. Why do we choose to construct $R_\varepsilon \in \mathscr{L}\{[H_0^1(C_\varepsilon)]^N; [H_0^1(\Omega_\varepsilon')]^N\}$ and not, "more naturally", $R_\varepsilon \in \mathscr{L}\{[H_0^1(\Omega)]^N; [H_0^1(\Omega_\varepsilon)]^N\}$? This difference between Ω and C_ε, or Ω_ε and Ω_ε' is the reason why the pressure's extension P_ε converges "locally" in $L^2_{\text{loc}}(\Omega)/\mathbb{R}$. (Recall that this local convergence is sufficient to pass to the limit in the equations). The answer is that the construction of $R_\varepsilon u$ in each cell Y_i^ε is reduced, with the help of the translation–homothety π_i^ε, to the construction of Ru in the unit cell Y (see Lemma 3.4). Now, if the cell Y_i^ε is "cut" by the boundary $\partial\Omega$, the construction of $R_\varepsilon u$ in $Y_i^\varepsilon \cap \Omega$ is reduced to the construction of Ru in a part of Y which can have a very small size compared with Y (and we cannot "control" this size). Unfortunately, the various constants C which appear in the estimates of Lemmas 2.10 and 3.4, depend strongly on the size of the considered open set. And because we cannot estimate those constants for any small subset of Y, we cannot obtain estimate (iv) of Theorem 2.3 if we define $R_\varepsilon u$ in Ω instead of C_ε.

Lemma 3.3. *There exists a linear continuous operator Q_ε such that*
(i) $Q_\varepsilon \in \mathscr{L}\{[H_0^1(C_\varepsilon)]^N; [H_0^1(\Omega'_\varepsilon)]^N\}$,
(ii) $u \in [H_0^1(\Omega'_\varepsilon)]^N$ *implies* $Q_\varepsilon \tilde{u} = u$ *in* Ω'_ε,
(iii) *for each* $u \in [H_0^1(C_\varepsilon)]^N$ *we have*

$$\| Q_\varepsilon u \|_{L^2(\Omega'_\varepsilon)} + \varepsilon \| \nabla (Q_\varepsilon u) \|_{L^2(\Omega'_\varepsilon)} \leqslant C \Big[\| u \|_{L^2(C_\varepsilon)} + \varepsilon \| \nabla u \|_{L^2(C_\varepsilon)} \Big]$$

where the constant C depends only on Y_F, *and not on* ε.

Proof. There are many ways to construct such an operator Q_ε. We choose the following one: let $u \in [H_0^1(C_\varepsilon)]^N$. We consider the following problem:

$$\text{find } v_\varepsilon \in \big[H_0^1(\Omega'_\varepsilon)\big]^N \text{ such that } \quad -\Delta v_\varepsilon = -\Delta u \text{ in } \Omega'_\varepsilon. \tag{3.1}$$

Thanks to the Lax–Milgram Theorem, we know that this problem has a unique solution v_ε belonging to $H_0^1(\Omega'_\varepsilon)$. Then we define Q_ε by

$$Q_\varepsilon u = v_\varepsilon \tag{3.2}$$

Properties (i) and (ii) of Q_ε can be easily checked, and in order to obtain estimate (iii), we multiply equation (3.1) by v_ε and we integrate by parts on Ω'_ε:

$$\int_{\Omega'_\varepsilon} |\nabla v_\varepsilon|^2 = \int_{\Omega'_\varepsilon} \nabla v_\varepsilon \cdot \nabla u \quad \Rightarrow \quad \| \nabla v_\varepsilon \|_{L^2(\Omega'_\varepsilon)} \leqslant \| \nabla u \|_{L^2(C_\varepsilon)}.$$

Using Lemma 2.7 (Poincaré's inequality), we obtain estimate (iii) with $Q_\varepsilon u = v_\varepsilon$:

$$\| v_\varepsilon \|_{L^2(\Omega'_\varepsilon)} + \varepsilon \| \nabla v_\varepsilon \|_{L^2(\Omega'_\varepsilon)} \leqslant C \Big[\| u \|_{L^2(C_\varepsilon)} + \varepsilon \| \nabla u \|_{L^2(C_\varepsilon)} \Big]. \qquad \square$$

Lemma 3.4. *Let Q be a linear operator belonging to* $\mathscr{L}\{[H^1(Y)]^N; [H^1(Y)]^N\}$ *satisfying*

$$\text{for each } u \in \big[H^1(Y)\big]^N \quad Qu \equiv 0 \text{ in } Y_S. \tag{3.3}$$

We consider the following problem:

$$\begin{cases} \text{find } (v, q) \in \big[H^1(Y_F)\big]^N \times \big[L^2(Y_F)/\mathbb{R}\big] \text{ such that} & \\ \nabla q - \Delta v = -\Delta u & \text{in } Y_F, \\ \nabla \cdot v = \nabla \cdot u + \dfrac{1}{|Y_F|} \displaystyle\int_{Y_S} \nabla \cdot u & \text{in } Y_F, \\ v = Qu + \dfrac{\phi_K}{\displaystyle\int_{\Sigma_K} \phi_K} \left[\displaystyle\int_{\Sigma_K} (u - Qu) \cdot e_K \right] e_K & \text{on } \Sigma_K \cap \overline{Y}_F, \\ v = 0 & \text{on } \partial Y_S. \end{cases} \tag{3.4}$$

Then we have the following results:
(i) (3.4) *has a unique solution*,

(ii) *the application R defined by* $Ru = v$ *belongs to* $\mathscr{L}\{[H^1(Y)]^N; [H^1(Y_F)]^N\}$,
(iii) *for each* $u \in [H^1(Y)]^N$ *we have*

$\| Ru \|_{H^1(Y_F)} \leqslant C\left[\| u \|_{H^1(Y)} + \| Qu \|_{H^1(Y)} \right]$ *where C depends only on* Y_F.

N.B.: See (1.2) for the definitions of ϕ_K and Σ_K.

Proof. A necessary condition for the existence of a solution of (3.4) is the following compatibility condition:

$$\int_{Y_F} \nabla \cdot v = \int_{\partial Y_F} v \cdot n. \tag{3.5}$$

If in this formula we substitute the functions by their assigned values given in (3.4), we obtain

$$\int_{Y_F} \nabla \cdot v = \int_{Y_F} \left[\nabla \cdot u + \frac{1}{|Y_F|} \int_{Y_S} \nabla \cdot u \right] = \int_{Y} \nabla \cdot u = \int_{\partial Y} u \cdot n$$

and

$$\int_{\partial Y_F} v \cdot n = \sum_{\substack{K=-N \\ K \neq 0}}^{+N} \int_{\Sigma_K} \left[Qu + \frac{\phi_K}{\int_{\Sigma_K} \phi_K} \left(\int_{\Sigma_K} (u - Qu) \cdot e_K \right) e_K \right] \cdot e_K$$

$$= \sum_{\substack{K=-N \\ K \neq 0}}^{+N} \left[\int_{\Sigma_K} Qu \cdot e_K + \int_{\Sigma_K} (u - Qu) \cdot e_K \right] = \int_{\partial Y} u \cdot n.$$

So (3.5) is satisfied.

In order to show that system (3.4) has a unique solution, we transform (3.4) into a divergence-free system with homogeneous boundary conditions. Let u_1 be defined in $[H^1(Y_F)]^N$ by

$$u_1 = Qu + \sum_{\substack{K=-N \\ K \neq 0}}^{+N} \frac{\phi_K}{\int_{\Sigma_K} \phi_K} \left[\int_{\Sigma_K} (u - Qu) \cdot e_K \right] e_K. \tag{3.6}$$

It is easy to see that if v is a solution of (3.4), we have $(v - u_1) \in [H_0^1(Y_F)]^N$. Moreover, because Y_F is connected with a locally Lipschitz boundary (see assumption (1.1)) we can apply Lemma 2.10 (about the lift of divergence): there exists $u_2 \in [H_0^1(Y_F)]^N$ such that

$$\nabla \cdot u_2 = \left(\nabla \cdot u - \nabla \cdot u_1 + \frac{1}{|Y_F|} \int_{Y_S} \nabla \cdot u \right) \quad \text{in } Y_F$$

and

$$\| u_2 \|_{H_0^1(Y_F)} \leqslant C\left[\| u \|_{H^1(Y)} + \| Qu \|_{H^1(Y)} \right]. \tag{3.7}$$

Consequently, finding a solution (v, q) of (3.4) is equivalent to finding a solution $(u_3; q) = (v - u_1 - u_2; q) \in [H_0^1(Y_F)]^N \times [L^2(Y_F)/\mathbb{R}]$ of the following system:

$$\nabla q - \Delta u_3 = -\Delta(u - u_1 - u_2) \quad \text{in } Y_F, \qquad \nabla \cdot u_3 = 0 \quad \text{in } Y_F. \tag{3.8}$$

Because Y_F is connected with a locally Lipschitz boundary, classical results (see, for example, [11, Chapter 1, Section 2]) state that there exists a unique solution of (3.8), which satisfies

$$\| \nabla u_3 \|_{L^2(Y_F)} \leqslant \| \nabla (u - u_1 - u_2) \|_{L^2(Y_F)}. \tag{3.9}$$

Then (3.4) has also a unique solution (v, q).

This allows us to define an application R by the following formula:

$$\text{for each } u \in [H^1(Y)]^N \quad Ru = v \text{ in } Y_F.$$

It is easy to see that $R \in \mathscr{L}\{[H^1(Y)]^N; [H^1(Y_F)]^N\}$, and recalling estimates (3.9), (3.7) and (3.6) we obtain

$$\| Ru \|_{H^1(Y_F)} \leqslant C\left[\| u \|_{H^1(Y)} + \| Qu \|_{H^1(Y)} \right]$$

where C depends only on Y_F, because the functions ϕ_K depend only on Y_F too. Note that C does not depend on Q, provided (3.3) holds true. And thus Lemma 3.4 has been proved. □

Proof of Theorem 2.3. Let $(Y_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ be the cubes which cover Ω. Let C_ε be the polygonal open set such that (see (1.4)) $\overline{C}_\varepsilon = \bigcup_{i \in I(\varepsilon)} \overline{Y}_i^\varepsilon$ with $I(\varepsilon) = \{ i \in [1;\ N(\varepsilon)] \mid Y_i^\varepsilon \subset \Omega \}$. Let $u \in [H_0^1(C_\varepsilon)]^N$. Recall that π_i^ε is the translation–homothety which maps Y_i^ε on Y (see (1.3)).

We define $R_\varepsilon u$ by its values in each cell $Y_i^\varepsilon \subset \Omega$, which are denoted by $R_\varepsilon u\,|_{Y_i^\varepsilon}$:

$$R_\varepsilon u\,|_{Y_i^\varepsilon} = \left[R\left(u \circ (\pi_i^\varepsilon)^{-1}\right)\right] \circ \pi_i^\varepsilon \quad \text{in } Y_i^\varepsilon \tag{3.10}$$

where R is defined by Lemma 3.4 in which Q is defined by

$$\left[Q\left(u \circ (\pi_i^\varepsilon)^{-1}\right)\right] \circ \pi_i^\varepsilon = \begin{cases} Q_\varepsilon u & \text{in } Y_{F_i}^\varepsilon, \\ 0 & \text{in } Y_{S_i}^\varepsilon. \end{cases} \tag{3.11}$$

Definition (3.10) is meaningful, provided that the operator Q defined by (3.11) satisfies the assumptions of Lemma 3.4. And it is clear that property (i) of Q_ε (defined by Lemma 3.3) implies condition (3.3) of Lemma 3.4.

By its very construction, $R_\varepsilon u\,|_{Y_i^\varepsilon}$ satisfies

$$\begin{cases} \text{there exists } q_\varepsilon \in L^2\left(Y_{F_i}^\varepsilon\right)/\mathbb{R} \text{ such that} & \\ \nabla q_\varepsilon - \Delta\left(R_\varepsilon u\,|_{Y_i^\varepsilon}\right) = -\Delta u & \text{in } Y_{F_i}^\varepsilon, \\ \nabla \cdot \left(R_\varepsilon u\,|_{Y_i^\varepsilon}\right) = \nabla \cdot u + \dfrac{1}{\left|Y_{F_i}^\varepsilon\right|} \displaystyle\int_{Y_{S_i}^\varepsilon} \nabla \cdot u & \text{in } Y_{F_i}^\varepsilon, \\ R_\varepsilon u\,|_{Y_i^\varepsilon} = Q_\varepsilon u + \dfrac{\phi_K \circ \pi_i^\varepsilon}{\displaystyle\int_{\Sigma_K^\varepsilon} \phi_K \circ \pi_i^\varepsilon} \left[\displaystyle\int_{\Sigma_K^\varepsilon} (u - Q_\varepsilon u) \cdot e_K \right] e_K & \text{on } \Sigma_K^\varepsilon, \\ R_\varepsilon u\,|_{Y_i^\varepsilon} = 0 & \text{on } \partial Y_{S_i}^\varepsilon. \end{cases} \tag{3.12}$$

Thus $R_\varepsilon u$ is defined on $\Omega_\varepsilon' = \bigcup_{i \in I(\varepsilon)} Y_{F_i}^\varepsilon$. It remains to prove properties (i) to (iv) of R_ε.

(i) Do we have $R_\varepsilon \in \mathscr{L}\{[H_0^1(C_\varepsilon)]^N; [H_0^1(\Omega_\varepsilon')]^N\}$? By construction, R_ε is linear because R is linear. Moreover, it is easy to see in (3.12) that $R_\varepsilon u = 0$ on $\partial\Omega_\varepsilon' = \partial C_\varepsilon \cup \{\bigcup_i \partial Y_{S_i}^\varepsilon\}$.

But it is crucial to check that $R_\varepsilon u \in [H_0^1(\Omega'_\varepsilon)]^N$. Because $R_\varepsilon u \in [H^1(Y^\varepsilon_{F_i})]^N$ for each $i \in I(\varepsilon)$, it remains to show that $R_\varepsilon u$ is continuous through the faces of Y_i^ε.

$$(3.12) \quad \Rightarrow \quad R_\varepsilon u = Q_\varepsilon u + \frac{\phi_K \circ \pi_i^\varepsilon}{\int_{\Sigma_K^\varepsilon} \phi_K \circ \pi_i^\varepsilon} \left[\int_{\Sigma_K^\varepsilon} (u - Q_\varepsilon u) \cdot e_K \right] e_K \quad \text{on } \Sigma_K^\varepsilon .$$

By construction (see (1.2)) the functions ϕ_K are equal on opposite faces of the same cell Y, and, by definition, $Q_\varepsilon u \in [H_0^1(\Omega'_\varepsilon)]^N$; then $R_\varepsilon u$ is continuous through Σ_K^ε and $R_\varepsilon \in \mathscr{L}\{[H_0^1(C_\varepsilon)]^N; [H_0^1(\Omega'_\varepsilon)]^N\}$.

(ii) If $u \in [H_0^1(\Omega'_\varepsilon)]^N$, then Lemma 3.3 implies that $Q_\varepsilon \tilde{u} = u$ in Ω'_ε. Then, from (3.11), we have $Q(u \circ \pi_i^\varepsilon) = u \circ \pi_i^\varepsilon$ in Y. In this case, the system (3.4) has an obvious solution which is u, and the uniqueness of the solution implies that $R(u \circ \pi_i^\varepsilon) = u \circ \pi_i^\varepsilon$ in Y_F. Thus, $R_\varepsilon \tilde{u} = u$ in Ω'_ε.

(iii) If $\nabla \cdot u = 0$ in C_ε, then from (3.12) we deduce that $\nabla \cdot (R_\varepsilon u) = 0$ in Ω'_ε.

(iv) Estimate of the norm of $R_\varepsilon u$: let $\psi \in H^1(Y)$. The norm of $\psi \circ \pi_i^\varepsilon$ in $H^1(Y_i^\varepsilon)$ in terms of the norm of ψ in $H^1(Y)$ is given by

$$\int_Y \psi^2(y)\, \mathrm{d}y = \int_{Y_i^\varepsilon} [\psi \circ \pi_i^\varepsilon(x)]^2 \cdot |\mathrm{Jac}\, \pi_i^\varepsilon|\, \mathrm{d}x = \frac{1}{\varepsilon^N} \int_{Y_i^\varepsilon} (\psi \circ \pi_i^\varepsilon)^2\, \mathrm{d}x$$

and

$$\int_Y |\nabla \psi|^2\, \mathrm{d}y = \int_{Y_i^\varepsilon} \frac{1}{|\pi_i^{\varepsilon\prime}(x)|} |\nabla(\psi \circ \pi_i^\varepsilon)|^2 \cdot |\mathrm{Jac}\, \pi_i^\varepsilon|\, \mathrm{d}x = \frac{\varepsilon^2}{\varepsilon^N} \int_{Y_i^\varepsilon} |\nabla(\psi \circ \pi_i^\varepsilon)|^2\, \mathrm{d}x.$$

Consequently, the estimate of Lemma 3.4 becomes, after the change of variables π_i^ε,

$$\begin{aligned} &\| R_\varepsilon u \|^2_{L^2(Y^\varepsilon_{F_i})} + \varepsilon^2 \| \nabla (R_\varepsilon u) \|^2_{L^2(Y^\varepsilon_{F_i})} \\ &\leqslant C \Big[\| u \|^2_{L^2(Y_i^\varepsilon)} + \| Q_\varepsilon u \|^2_{L^2(Y^\varepsilon_{F_i})} + \varepsilon^2 \Big(\| \nabla u \|^2_{L^2(Y_i^\varepsilon)} + \| \nabla (Q_\varepsilon u) \|^2_{L^2(Y^\varepsilon_{F_i})} \Big) \Big]. \end{aligned} \tag{3.13}$$

After summation of the estimates (3.13) for each $i \in I(\varepsilon)$, we obtain

$$\begin{aligned} &\| R_\varepsilon u \|^2_{L^2(\Omega'_\varepsilon)} + \varepsilon^2 \| \nabla (R_\varepsilon u) \|^2_{L^2(\Omega'_\varepsilon)} \\ &\leqslant C \Big[\| u \|^2_{L^2(C_\varepsilon)} + \| Q_\varepsilon u \|^2_{L^2(\Omega'_\varepsilon)} + \varepsilon^2 \Big(\| \nabla u \|^2_{L^2(C_\varepsilon)} + \| \nabla (Q_\varepsilon u) \|^2_{L^2(\Omega'_\varepsilon)} \Big) \Big], \end{aligned}$$

and with the help of estimate (iii) of Lemma 3.3, we get

$$\| R_\varepsilon u \|^2_{L^2(\Omega'_\varepsilon)} + \varepsilon^2 \| \nabla (R_\varepsilon u) \|^2_{L^2(\Omega'_\varepsilon)} \leqslant C \Big[\| u \|^2_{L^2(C_\varepsilon)} + \varepsilon^2 \| \nabla u \|^2_{L^2(C_\varepsilon)} \Big]$$

where C depends only on Y_F (and not on ε). This inequality is just property (iv) we were looking for. Thus, Theorem 2.3 has been proved. □

Proof of Theorem 2.6. Let us first show that the pressure's extension P_ε is a constant in the solid part $Y^\varepsilon_{S_i}$ of each cell included in C_ε.

Let w be a function of $[C^\infty(Y^\varepsilon_{S_i})]^N$ with compact support contained in $Y^\varepsilon_{S_i}$ (i.e. $w \in [\mathscr{D}(Y^\varepsilon_{S_i})]^N$). By its very construction in Lemma 3.3, the operator Q_ε is such that $Q_\varepsilon w = 0$ in Ω'_ε. Furthermore, system (3.12), which is satisfied by $R_\varepsilon w$ in $Y^\varepsilon_{\mathrm{F}_i}$, reduces to

$$\begin{cases} \text{there exists } q_\varepsilon \in L^2(Y^\varepsilon_{\mathrm{F}_i})/\mathbb{R} \text{ such that} & \\ \nabla q_\varepsilon - \Delta(R_\varepsilon w) = -\Delta w = 0 & \text{in } Y^\varepsilon_{\mathrm{F}_i}, \\ \nabla \cdot (R_\varepsilon w) = \dfrac{1}{|Y^\varepsilon_{\mathrm{F}_i}|} \displaystyle\int_{Y^\varepsilon_{\mathrm{S}_i}} \nabla \cdot w = 0 & \text{in } Y^\varepsilon_{\mathrm{F}_i}, \\ R_\varepsilon w = 0 & \text{on } \partial Y^\varepsilon_{\mathrm{F}_i} \end{cases} \tag{3.14}$$

because w has a compact support contained in $Y^\varepsilon_{\mathrm{S}i}$. Then, (3.14) implies that $R_\varepsilon w = 0$ in $Y^\varepsilon_{\mathrm{F}_i}$, which means that

$$R_\varepsilon w = 0 \quad \text{in } \Omega'_\varepsilon. \tag{3.15}$$

Recall definition (2.5) of P_ε: $\langle \nabla P_\varepsilon, w \rangle_{H^{-1}, H^1_0(C_\varepsilon)} = \langle \nabla p_\varepsilon, R_\varepsilon w \rangle_{H^{-1}, H^1_0(\Omega'_\varepsilon)}$. Then, for each $w \in [\mathscr{D}(Y^\varepsilon_{\mathrm{S}_i})]^N$:

$$\langle \nabla P_\varepsilon, w \rangle = 0 \iff \int_{Y^\varepsilon_{\mathrm{S}_i}} P_\varepsilon \nabla \cdot w = 0 \tag{3.16}$$

Using Lemma 2.10 this implies that P_ε is a constant in each solid part $Y^\varepsilon_{\mathrm{S}_i}$.

Let us now compute explicitly the value of this constant in each $Y^\varepsilon_{\mathrm{S}_i}$. Let v be a function of $[C^\infty(Y^\varepsilon_i)]^N$ with compact support contained in the cell Y^ε_i (i.e., $v \in [\mathscr{D}(Y^\varepsilon_i)]^N$). Remark that v is different from the previous function w because the compact support of v is contained in Y^ε_i, while the support of w is in $Y^\varepsilon_{\mathrm{S}_i}$ which is strictly included in Y^ε_i. Recalling the definition (2.5) of P_ε, we obtain

$$\langle \nabla P_\varepsilon, v \rangle_{H^{-1}, H^1_0(C_\varepsilon)} = \langle \nabla p_\varepsilon, R_\varepsilon v \rangle_{H^{-1}, H^1_0(\Omega'_\varepsilon)} \iff \int_{Y^\varepsilon_i} P_\varepsilon \nabla \cdot v = \int_{Y^\varepsilon_{\mathrm{F}_i}} p_\varepsilon \nabla \cdot (R_\varepsilon v). \tag{3.17}$$

By the very construction of R_ε and P_ε, we have

$$P_\varepsilon \equiv \begin{cases} p_\varepsilon & \text{in } Y^\varepsilon_{\mathrm{F}_i}, \\ \text{constant} & \text{in } Y^\varepsilon_{\mathrm{S}_i} \end{cases}$$

and (see (3.12))

$$\nabla \cdot (R_\varepsilon v) = \nabla \cdot v + \frac{1}{|Y^\varepsilon_{\mathrm{F}_i}|} \int_{Y^\varepsilon_{\mathrm{S}_i}} \nabla \cdot v \quad \text{in } Y^\varepsilon_{\mathrm{F}_i}.$$

We substitute in (3.17)

$$\int_{Y^\varepsilon_{\mathrm{F}_i}} p_\varepsilon \nabla \cdot v + \int_{Y^\varepsilon_{\mathrm{S}_i}} (P_\varepsilon|_{Y^\varepsilon_{\mathrm{S}_i}}) \nabla \cdot v = \int_{Y^\varepsilon_{\mathrm{F}_i}} p_\varepsilon \left[\nabla \cdot v + \frac{1}{|Y^\varepsilon_{\mathrm{F}_i}|} \int_{Y^\varepsilon_{\mathrm{S}_i}} \nabla \cdot v \right]$$

$$\Rightarrow \left(P_\varepsilon|_{Y^\varepsilon_{\mathrm{S}_i}}\right) \int_{Y^\varepsilon_{\mathrm{S}_i}} \nabla \cdot v = \left(\frac{1}{|Y^\varepsilon_{\mathrm{F}_i}|} \int_{Y^\varepsilon_{\mathrm{S}_i}} \nabla \cdot v \right) \int_{Y^\varepsilon_{\mathrm{F}_i}} p_\varepsilon. \tag{3.18}$$

and this is true for each $v \in [\mathscr{D}(Y^\varepsilon_i)]^N$. Thus,

$$P_\varepsilon|_{Y^\varepsilon_{\mathrm{S}_i}} = \frac{1}{|Y^\varepsilon_{\mathrm{F}_i}|} \int_{Y^\varepsilon_{\mathrm{F}_i}} p_\varepsilon \tag{3.19}$$

This proves the fact that the pressure's extension P_ε is constant in the solid part of each cell included in C_ε, and (3.19) gives the value of the constant. □

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