OPTIMAL DESIGN OF STRUCTURES (MAP 562)

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CHAPTER VII (continued)

TOPOLOGY OPTIMIZATION

BY THE HOMOGENIZATION METHOD

Brief review of the preceding course

1. Topology optimization versus geometric optimization
2. Homogenization method in the periodic case (two-scale asymptotic expansions)
3. An explicit class of composite materials: sequential laminates.

What remains to be done:

☞ To characterize the set G_θ of all composites materials

☞ Towards this goal, prove bounds on A^*.

☞ Application to shape optimization

☞ To build numerical algorithms for topology optimization

7.3.4 Variational characterization of homogenized tensors

From now on, we assume that the microscopic tensor $A(y)$ is symmetric. Then A^* is symmetric too.

Furthermore, A^* is characterized by the variational principle

$$A^*\xi \cdot \xi = \min_{w \in H^1_\#(Y)/\mathbb{R}} \int_Y A(y)\,(\xi + \nabla w) \cdot (\xi + \nabla w)\, dy$$

Indeed, if w_ξ is the minimizer, then it satisfies the Euler optimality condition

$$\begin{cases} -\operatorname{div}\Big(A(y)\,(\xi + \nabla w_\xi(y))\Big) = 0 & \text{in } Y \\ y \to w_\xi(y) & Y\text{-periodic.} \end{cases}$$

By linearity, we have $w_\xi = \displaystyle\sum_{i=1}^N \xi_i w_i$ and thus

$$\int_Y A(y)\,(\xi + \nabla w_\xi) \cdot (\xi + \nabla w_\xi)\, dy = \sum_{i,j=1}^N \xi_i \xi_j A^*_{ij} = A^*\xi \cdot \xi.$$

Arithmetic and harmonic mean bounds

Taking $w = 0$ in the variational principle, we deduce the arithmetic mean bound

$$A^* \xi \cdot \xi \leq \left(\int_Y A(y)\, dy \right) \xi \cdot \xi$$

Enlarging the minimization space, we obtain the harmonic mean bound

$$\left(\int_Y A^{-1}(y)\, dy \right)^{-1} \xi \cdot \xi \leq A^* \xi \cdot \xi$$

These bounds can be improved for two-phase composites !

Indeed, since $\int_Y \nabla w \, dy = 0$, we enlarge the minimization space by replacing ∇w with any vector field $\zeta(y)$ with zero-average on Y

$$A^*\xi \cdot \xi \geq \min_{\zeta \in L^2_\#(Y)^N, \ \int_Y \zeta \, dy = 0} \int_Y A(y) \left(\xi + \zeta(y)\right) \cdot \left(\xi + \zeta(y)\right) dy$$

The Euler equation for the minimizer $\zeta_\xi(y)$ of this convex problem is

$$A(y) \left(\xi + \zeta_\xi(y)\right) = \lambda$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier for the constraint $\int_Y \zeta \, dy = 0$. We deduce

$$\xi = \left(\int_Y A(y)^{-1} \, dy \right) \lambda$$

and thus

$$\int_Y A(y) \Big(\xi + \zeta_\xi(y) \Big) \cdot \Big(\xi + \zeta_\xi(y) \Big) dy = \left(\int_Y A(y)^{-1} \, dy \right)^{-1} \xi \cdot \xi.$$

7.3.5 Characterization of G_θ

We consider two isotropic phases $A = \alpha\,\mathrm{Id}$ and $B = \beta\,\mathrm{Id}$ with $0 < \alpha < \beta$.

Theorem 7.17. The set G_θ of all homogenized tensors obtained by mixing α and β in proportions θ and $(1-\theta)$ is the set of all symmetric matrices A^* with eigenvalues $\lambda_1, ..., \lambda_N$ such that

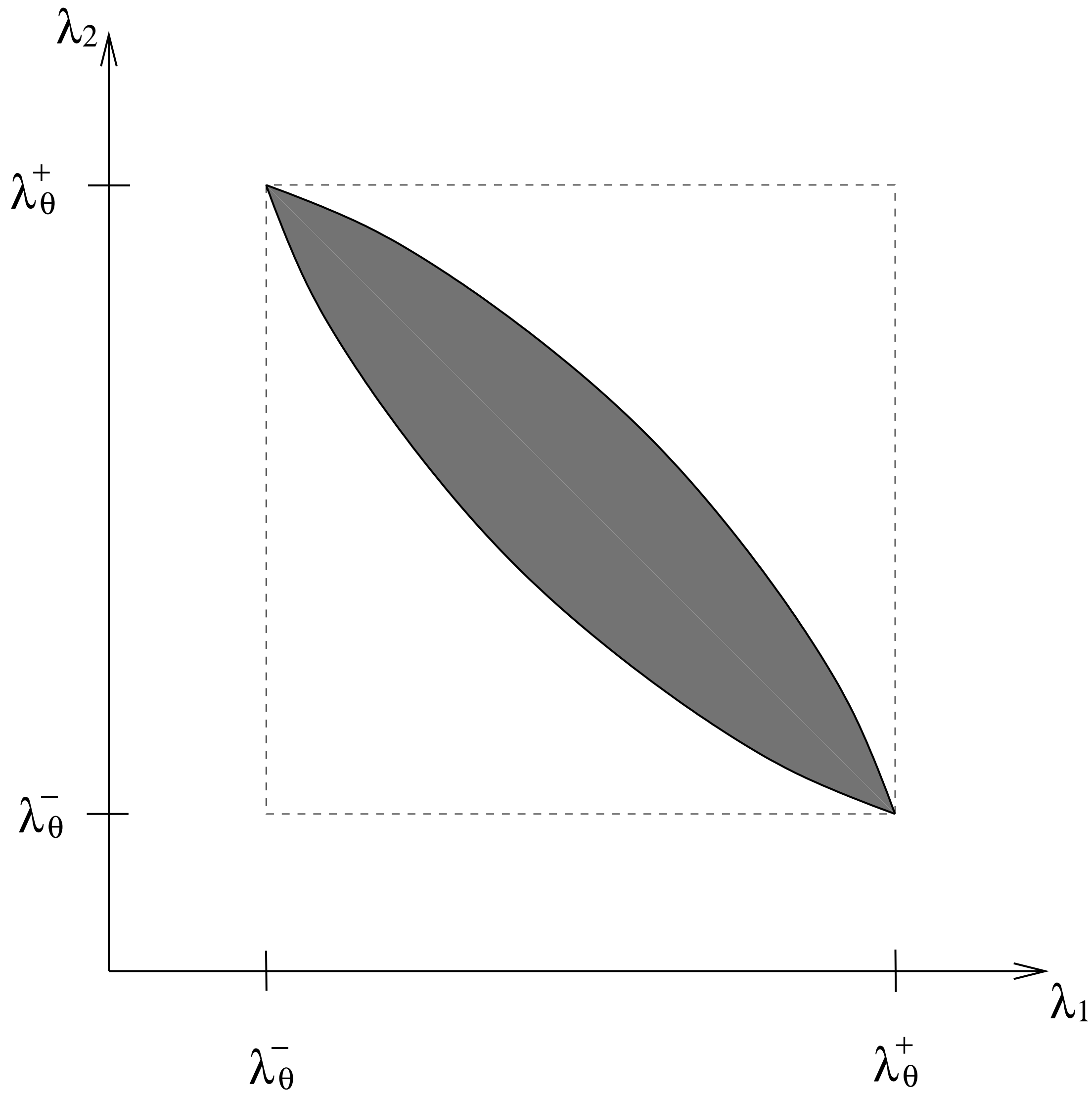
$$\left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1} = \lambda_\theta^- \leq \lambda_i \leq \lambda_\theta^+ = \theta\alpha + (1-\theta)\beta \qquad 1 \leq i \leq N$$

$$\sum_{i=1}^{N} \frac{1}{\lambda_i - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{N-1}{\lambda_\theta^+ - \alpha}$$

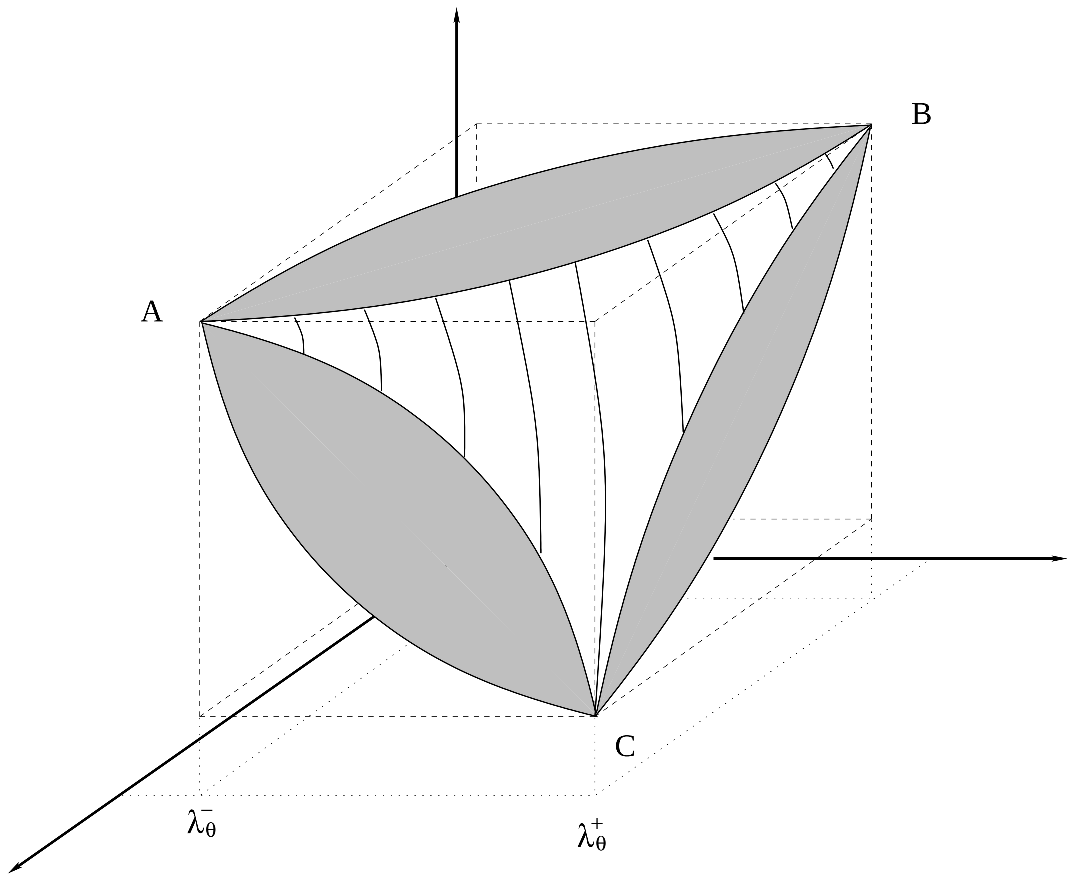
$$\sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+},$$

Furthermore, these so-called Hashin and Shtrikman bounds are optimal and attained by rank-N sequential laminates.

Set G_θ in dimension $N = 2$



Set G_θ in dimension $N = 3$



Proof. We first show that all matrices satisfying these inequalities (Hashin-Shtrikman bounds) belong to G_θ.

Let us start by showing that the upper bound is attained by sequential laminates. Take a matrix A^* such that

$$\sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} = \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+}.$$

Define a rank-N sequentiel laminate A_L^* of matrix β and inclusion α, with lamination directions being the (orthogonal) eigenvectors of A^*

$$\theta \left(A_L^* - \beta \,\mathrm{Id} \right)^{-1} = \frac{1}{\alpha - \beta} \mathrm{Id} + (1-\theta) \sum_{i=1}^{N} m_i \frac{e_i \otimes e_i}{\beta} \quad \text{with} \quad m_i \geq 0, \sum_{i=1}^{N} m_i = 1.$$

We have $A^* = A_L^*$ if we can choose the m_i's such that

$$\frac{\theta}{\lambda_i - \beta} = \frac{1}{\alpha - \beta} + \frac{m_i(1-\theta)}{\beta} \;\Leftrightarrow\; m_i = \frac{\beta\left(\lambda_\theta^+ - \lambda_i\right)}{(1-\theta)(\beta-\alpha)(\beta-\lambda_i)}$$

We check that $0 < m_i < 1$ is equivalent to $\lambda_\theta^- < \lambda_i < \lambda_\theta^+$ and that

$$\sum_{i=1}^{N} m_i = 1 \;\Leftrightarrow\; \sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} = \frac{1}{\beta - \lambda_\theta^-} + \frac{N-1}{\beta - \lambda_\theta^+},$$

thus any matrix on the upper bound is a rank-N sequential laminate with matrix β and inclusion α.

The same proof works for the lower bound upon exchanging the role of α (now the matrix) and β (now the inclusions).

Then, the next easy computation shows that the matrices “inside” G_θ are attained by simple lamination of two matrices, one on the upper bound, the other on the lower bound.

Computation for the interior of G_θ

Recall the lamination formula:

$$\tau (A^* - B)^{-1} = (A - B)^{-1} + \frac{(1-\tau)}{Be_1 \cdot e_1} e_1 \otimes e_1$$

Particular case: $A, B \in G_\theta$ diagonal in the same basis $(e_1, ..., e_N)$.

$$A = \text{diag}(a_1, ..., a_N) \qquad B = \text{diag}(b_1, ..., b_N)$$

Then, for any $\tau \in [0,1]$, $A^* \in G_\theta$ and

$$a_1^* = \left(\frac{\tau}{a_1} + \frac{1-\tau}{b_1} \right)^{-1} \qquad a_i^* = \tau a_i + (1-\tau) b_i \quad 2 \le i \le N.$$

Branches of hyperbolas which connect the upper and lower bounds of G_θ.

It remains to prove that the lower and upper Hashin-Shtrikman bounds hold true.

To establish the lower bound we introduce the so-called Hashin and Shtrikman variational principle.

Main idea: use Fourier analysis and Plancherel theorem, but, in a first step, eliminate the cubic terms.

By definition of A^*, for $\xi \in \mathbb{R}^N$, we have

$$A^*\xi \cdot \xi = \min_{w(y) \in H^1_\#(Y)} \int_Y \Big(\chi(y)\alpha + (1-\chi(y))\beta\Big)(\xi + \nabla w) \cdot (\xi + \nabla w) dy$$

Substracting a reference material α

$$\int_Y (\chi\alpha + (1-\chi)\beta) |\xi + \nabla w|^2 dy =$$

$$\int_Y (1-\chi)(\beta - \alpha)|\xi + \nabla w|^2 dy + \int_Y \alpha|\xi + \nabla w|^2 dy.$$

We use convex duality (or Legendre transform): for any symmetric positive definite matrix M

$$M\zeta\cdot\zeta = \max_{\eta\in\mathbb{R}^N}\left(2\zeta\cdot\eta - M^{-1}\eta\cdot\eta\right) \qquad \forall\,\zeta\in\mathbb{R}^N.$$

Since $\beta-\alpha>0$, we apply the above formula at each point in Y, and we get

$$\int_Y (1-\chi)(\beta-\alpha)|\xi+\nabla w|^2 dy =$$

$$\max_{\eta(y)\in L^2_\#(Y)^N}\int_Y (1-\chi)\Big(2(\xi+\nabla w)\cdot\eta-(\beta-\alpha)^{-1}|\eta|^2\Big)dy,$$

which becomes an inequality if we restrict the minimization to constant η in Y

$$\int_Y (1-\chi)(\beta-\alpha)|\xi+\nabla w|^2 dy \geq$$

$$\geq \max_{\eta}\int_Y (1-\chi)\Big(2(\xi+\nabla w)\cdot\eta-(\beta-\alpha)^{-1}|\eta|^2\Big)dy$$

$$\geq (1-\theta)\Big(2\xi\cdot\eta-(\beta-\alpha)^{-1}|\eta|^2\Big)-2\int_Y \chi\nabla w\cdot\eta\,dy.$$

On the other hand, because of periodicity, $\int_Y \nabla w dy = 0$ which implies

$$\int_Y \alpha|\xi + \nabla w|^2 dy = \alpha|\xi|^2 + \int_Y \alpha|\nabla w|^2 dy.$$

Overall, we obtain, for any $\eta \in \mathbb{R}^N$,

$$A^*\xi \cdot \xi \geq \alpha|\xi|^2 + (1-\theta)\Big(2\xi \cdot \eta - (\beta-\alpha)^{-1}|\eta|^2\Big) - g(\chi,\eta),$$

where $g(\chi,\eta)$ is a so-called non-local term, defined by

$$g(\chi,\eta) = - \min_{w(y)\in H^1_\#(Y)} \int_Y \left(\alpha|\nabla w|^2 - 2\chi\nabla w \cdot \eta\right) dy.$$

We can now use Fourier analysis to compute $g(\chi,\eta)$.

By periodicity, χ and the test function w can be written as Fourier series

$$\chi(y) = \sum_{k \in \mathbb{Z}^N} \hat{\chi}(k) e^{2i\pi k \cdot y}, \qquad w(y) = \sum_{k \in \mathbb{Z}^N} \hat{w}(k) e^{2i\pi k \cdot y}.$$

Since χ and w are real-valued, their Fourier coefficients satisfy

$$\overline{\hat{\chi}(k)} = \hat{\chi}(-k) \quad \text{and} \quad \overline{\hat{w}(k)} = \hat{w}(-k).$$

The gradient of w is

$$\nabla w(y) = \sum_{k \in \mathbb{Z}^N} 2i\pi e^{2i\pi k \cdot y} \hat{w}(k) k.$$

Plancherel formula yields

$$\int_Y \left(\alpha |\nabla w|^2 - 2\chi \nabla w \cdot \eta \right) dy$$

$$= \sum_{k \in \mathbb{Z}^N} \left(4\pi^2 \alpha |\hat{w}(k) k|^2 - 4i\pi \overline{\hat{\chi}(k)} \hat{w}(k)\, k \cdot \eta \right)$$

$$= \sum_{k \in \mathbb{Z}^N} \left(4\pi^2 \alpha |k|^2 |\hat{w}(k)|^2 + 4\pi \mathcal{I}m \left(\overline{\hat{\chi}(k)} \hat{w}(k) \right) \eta \cdot k \right).$$

To minimiz in $w(y) \in H^1_\#(Y) \Leftrightarrow$ to minimize in $\hat{w}(k) \in \mathbb{C}$.

For $k \neq 0$ the minimum is achieved by

$$\hat{w}(k) = -\frac{i\hat{\chi}(k)}{2\pi\alpha|k|^2}\eta \cdot k,$$

and we deduce

$$g(\chi,\eta) = \left(\alpha^{-1} \sum_{k\in\mathbb{Z}^N,\ k\neq 0} |\hat{\chi}(k)|^2 \frac{k}{|k|} \otimes \frac{k}{|k|} \right) \eta \cdot \eta = \alpha^{-1}\theta(1-\theta)M\eta \cdot \eta\,,$$

where M is a symmetric non-negative matrix. Since, by Plancherel theorem, we have

$$\sum_{k\in\mathbb{Z}^N,\ k\neq 0} |\hat{\chi}(k)|^2 = \int_Y |\chi(y) - \theta|^2\, dy = \theta(1-\theta),$$

we deduce that the trace of M is equal to 1.

Regrouping terms yields, for any $\xi, \eta \in \mathbb{R}^N$,

$$A^*\xi \cdot \xi \geq \alpha|\xi|^2 + (1-\theta)\Big(2\xi\cdot\eta - (\beta-\alpha)^{-1}|\eta|^2\Big) - \alpha^{-1}\theta(1-\theta)M\eta\cdot\eta.$$

The minimum (in ξ) of this inequality is obtained when

$$\xi = (1-\theta)(A^*-\alpha)^{-1}\eta$$

We deduce

$$(1-\theta)(A^*-\alpha)^{-1}\eta\cdot\eta \leq (\beta-\alpha)^{-1}|\eta|^2 + \alpha^{-1}\theta M\eta\cdot\eta \quad \forall\, \eta \in \mathbb{R}^N.$$

$$\Leftrightarrow (1-\theta)(A^*-\alpha)^{-1} \leq (\beta-\alpha)^{-1}\,\mathrm{Id} + \alpha^{-1}\theta M$$

Taking the trace of this matrix inequality, and since $\mathrm{Tr}M = 1$, we obtain the lower Hashin-Shtrikman bound.

The proof of the upper bound is similar.

7.4 Homogenized formulation of shape optimization

The relaxed or homogenized optimization problem is

$$\min_{(\theta,A^*)\in\mathcal{U}^*_{ad}} J(\theta,A^*),$$

with an objective function

$$J(\theta,A^*)=\int_\Omega f\,u\,dx, \quad \text{or} \quad J(\theta,A^*)=\int_\Omega |u-u_0|^2 dx,$$

and an homogenized admissible set given by

$$\mathcal{U}^*_{ad}=\left\{(\theta,A^*)\in L^\infty\left(\Omega;[0,1]\times\mathbb{R}^{N^2}\right), A^*(x)\in G_{\theta(x)} \text{ in } \Omega, \int_\Omega \theta(x)\,dx=V_\alpha\right\},$$

where G_θ is explicitly characterized.

The homogenized state equation is

$$\begin{cases} -\operatorname{div}\left(A^*\nabla u\right)=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 7.19 (admitted). The homogenized formulation is actually a **relaxation** of the original shape optimization problem in the sense that:

☞ there exists, at least, one optimal composite shape (θ, A^*),

☞ any minimizing sequence of classical shapes χ converges, in the sense of homogenization, to a composite optimal solution (θ, A^*),

☞ any composite optimal solution (θ, A^*) is the limit of a minimizing sequence of classical shapes.

The minima of the original and homogenized objective functions coincide

$$\inf_{\chi \in \mathcal{U}_{ad}} J(\chi) = \min_{(\theta, A^*) \in \mathcal{U}^*_{ad}} J(\theta, A^*).$$

Remark.

☞ The shape optimization problem is thus not changed by relaxation.

☞ Close to any optimal composite shape, we are sure to find a quasi-optimal classical shape.

☞ This theorem is at the root of new numerical algorithms.

7.4.2 Optimality conditions

We now compute the gradient of the following objective function

$$J(\theta, A^*) = \int_\Omega |u - u_0|^2 dx,$$

where $u_0 \in L^2(\Omega)$. We introduce the adjoint state p, unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A^* \nabla p) = -2(u - u_0) & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 7.20. Let $\alpha > 0$ and $\mathcal{M}_\alpha$ be the set of symmetric positive definite matrices M such that $M \geq \alpha \,\mathrm{Id}$. The functional J is differentiable with respect to A^* in $L^\infty(\Omega; \mathcal{M}_\alpha)$, and its derivative is

$$\nabla_{A^*} J(\theta, A^*) = \nabla u \otimes \nabla p \,.$$

Remark. The partial derivative with respect to θ vanishes because θ appears only in the constraint $A^* \in G_\theta$.

Proof of Proposition 7.20

It is standard ! It became a parametric (sizing) shape optimization problem where A^* plays the role of a thickness.

We introduce the Lagrangian

$$\mathcal{L}(A^*, v, q) = \int_\Omega |v - u_0|^2 dx + \int_\Omega A^* \nabla v \cdot \nabla q \, dx - \int_\Omega f q \, dx$$

Its partial derivative with respect to q yields the state.

Its partial derivative with respect to v yields the adjoint.

Its partial derivative with respect to A^* yields the gradient

$$\nabla_{A^*} J(\theta, A^*) = \frac{\partial \mathcal{L}}{\partial A^*}(A^*, u, p) = \nabla u \otimes \nabla p \, .$$

Essential consequence

Theorem 7.21. Let (θ, A^*) be a global minimizer of J in $\mathcal{U}^*_{ad}$ which admits u and p as state and adjoint. There exists $(\tilde{\theta}, \tilde{A}^*)$, another global minimizer of J in $\mathcal{U}^*_{ad}$, which admits the same state and adjoint u and p, and such that $\tilde{A}^*$ is a rank-1 simple laminate.

Simplification: in the definition of $\mathcal{U}^*_{ad}$ the set G_θ can be replaced by its simpler subset of rank-1 simple laminates.

Remark.

- ☞ Optimality condition $\Rightarrow$ simplification of the problem.
- ☞ We actually use this simplification in the numerical algorithms.
- ☞ Simplification which holds true for other objective functions, but not for multiple loads optimization.

Proof. We fix θ and makes variations on A^* only. Remarking that G_θ is convex (not obvious), the optimality condition is an Euler inequality which is

$$\int_\Omega (A^0 - A^*)\nabla u \cdot \nabla p \, dx \geq 0$$

for any $A^0 \in G_\theta$, which is equivalent to

$$A^*\nabla u \cdot \nabla p = \min_{A^0 \in G_\theta} \left(A^0 \nabla u \cdot \nabla p \right) \quad \forall \, x \in \Omega.$$

If ∇u or ∇p vanishes, then any A^* is optimal. Otherwise, we define

$$e = \frac{\nabla u}{|\nabla u|} \quad \text{and} \quad e' = \frac{\nabla p}{|\nabla p|},$$

and we look for minimizers A^* of

$$\min_{A^0 \in G_\theta} 4A^0 e \cdot e' = A^0(e + e') \cdot (e + e') - A^0(e - e') \cdot (e - e').$$

A lower bound is easily obtained

$$\begin{aligned}\min_{A^0\in G_\theta} 4A^0 e\cdot e' &\geq \min_{A^0\in G_\theta} A^0(e+e')\cdot(e+e') - \max_{A^0\in G_\theta} A^0(e-e')\cdot(e-e') \\ &= \lambda_\theta^- |e+e'|^2 - \lambda_\theta^+ |e-e'|^2.\end{aligned}$$

This lower bound is actually the precise minimal value.

Indeed, choosing $A^0 = A^1$ which is a rank-1 simple laminate in the direction $e+e'$, orthogonal to $e-e'$, we get

$$A^1(e+e') = \lambda_\theta^-(e+e') \quad \text{and} \quad A^1(e-e') = \lambda_\theta^+(e-e')$$

and an easy computation shows that

$$4A^1 e\cdot e' = \lambda_\theta^- |e+e'|^2 - \lambda_\theta^+ |e-e'|^2$$

Thus

$$\min_{A^0\in G_\theta} 4A^0 e\cdot e' = \lambda_\theta^- |e+e'|^2 - \lambda_\theta^+ |e-e'|^2$$

If now A^* is any optimal tensor, then, as a rank-1 laminate, it satisfies

$$A^*(e+e') = \lambda_\theta^-(e+e') \quad \text{and} \quad A^*(e-e') = \lambda_\theta^+(e-e') \tag{1}$$

Indeed, if (1) does not hold true, one of the arithmetic and harmonic bounds would give a strict inequality

$$4A^*e\cdot e' = A^*(e+e')\cdot(e+e') - A^*(e-e')\cdot(e-e') > \lambda_\theta^-|e+e'|^2 - \lambda_\theta^+|e-e'|^2$$

which is a contradiction with the optimal character of A^*.

We deduce that any optimal A^* satisfies, like the rank-1 simple laminate A^1,

$$2A^*\nabla u = \quad 2A^1\nabla u = \quad \left(\lambda_\theta^+ + \lambda_\theta^-\right)\nabla u + \left(\lambda_\theta^+ - \lambda_\theta^-\right)\frac{|\nabla u|}{|\nabla p|}\nabla p$$

$$2A^*\nabla p = \quad 2A^1\nabla p = \quad \left(\lambda_\theta^+ + \lambda_\theta^-\right)\nabla p + \left(\lambda_\theta^+ - \lambda_\theta^-\right)\frac{|\nabla p|}{|\nabla u|}\nabla u,$$

Therefore any optimal tensor A^* can be replaced by this rank-1 simple laminate A^1 without changing u and p.

$$-\operatorname{div}\Big(A^*\nabla u\Big) = -\operatorname{div}\Big(A^1\nabla u\Big) = f$$

$$-\operatorname{div}\Big(A^*\nabla p\Big) = -\operatorname{div}\Big(A^1\nabla p\Big) = -2(u - u_0)$$

Parametrization of rank-1 simple laminates

In space dimension $N = 2$ (to simplify) a rank-1 laminate is defined by

$$A^*(\theta, \phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_\theta^+ & 0 \\ 0 & \lambda_\theta^- \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad \phi \in [0, \pi].$$

The admissible set is thus simply

$$\mathcal{U}_{ad}^L = \left\{ (\theta, \phi) \in L^\infty\left(\Omega; [0,1] \times [0,\pi]\right), \int_\Omega \theta(x)\, dx = V_\alpha \right\}.$$

Proposition 7.23. The objective function $J(\theta, \phi)$ is differentiable with respect to (θ, ϕ) in $\mathcal{U}_{ad}^L$, and its derivative is

$$\nabla_\phi J(\theta, \phi) = \frac{\partial A^*}{\partial \phi} \nabla u \cdot \nabla p \quad \text{and} \quad \nabla_\theta J(\theta, \phi) = \frac{\partial A^*}{\partial \theta} \nabla u \cdot \nabla p$$

7.4.3 Numerical algorithm

Projected gradient algorithm for the minimization of $J(\theta, \phi)$.

1. We initialize the design parameters θ_0 and ϕ_0 (for example, equal to constants).

2. Until convergence, for $k \geq 0$ we iterate by computing the state u_k and adjoint p_k, solutions with the previous design parameters (θ_k, ϕ_k), then we update these parameters by

$$\theta_{k+1} = \max\left(0, \min\left(1, \theta_k - t_k\left(\ell_k + \frac{\partial A^*}{\partial \theta}(\theta_k, \phi_k)\nabla u_k \cdot \nabla p_k\right)\right)\right)$$

$$\phi_{k+1} = \phi_k - t_k \frac{\partial A^*}{\partial \phi}(\theta_k, \phi_k)\nabla u_k \cdot \nabla p_k$$

with ℓ_k a Lagrange multiplier for the volume constraint (iteratively enforced), and $t_k > 0$ a descent step such that $J(\theta_{k+1}, \phi_{k+1}) < J(\theta_k, \phi_k)$.

The self-adjoint case

A first example: maximization of torsional rigidity (maximization of compliance).

$$\min_{(\theta,A^*)\in\mathcal{U}^L_{ad}} \left\{ J(\theta, A^*) = -\int_\Omega u(x)dx \right\},$$

where u is the solution of

$$\begin{cases} -\operatorname{div}(A^*\nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the adjoint state is just $p = u$.

We solve in the domain $\Omega = (0,1)^2$ with the phases $\alpha = 1$ and $\beta = 2$. We fix a 50% volume constraint of α. We initialize with a constant value of $\theta = 0.5$ and a constant zero lamination angle. We perform 30 iterations.

Self-adjoint case $p = u$.

$$\nabla_{A^*} J(\theta, A^*) = \nabla u \otimes \nabla u \geq 0\,.$$

To minimize J we have to decrease A^*.

Any optimal A^* satisfies

$$A^* \nabla u = \lambda_\theta^- \nabla u$$

thus the optimal composite is the **worst possible conductor**.

Consequence. We can eliminate the angle ϕ and it remains to optimize with respect to θ only !

Convexity

We rewrite the optimization problem thanks to the primal energy

$$-\int_\Omega u\,dx = -\int_\Omega \lambda_\theta^- |\nabla u|^2 dx = \min_{v\in H_0^1(\Omega)} \int_\Omega \lambda_\theta^- |\nabla v|^2 dx - 2\int_\Omega v\,dx$$

Thus, we obtain a double minimization

$$\min_{\theta, A^*=\lambda_\theta^-} J(\theta, A^*) = \min_{\theta, v} \int_\Omega \lambda_\theta^- |\nabla v|^2 dx - 2\int_\Omega v\,dx$$

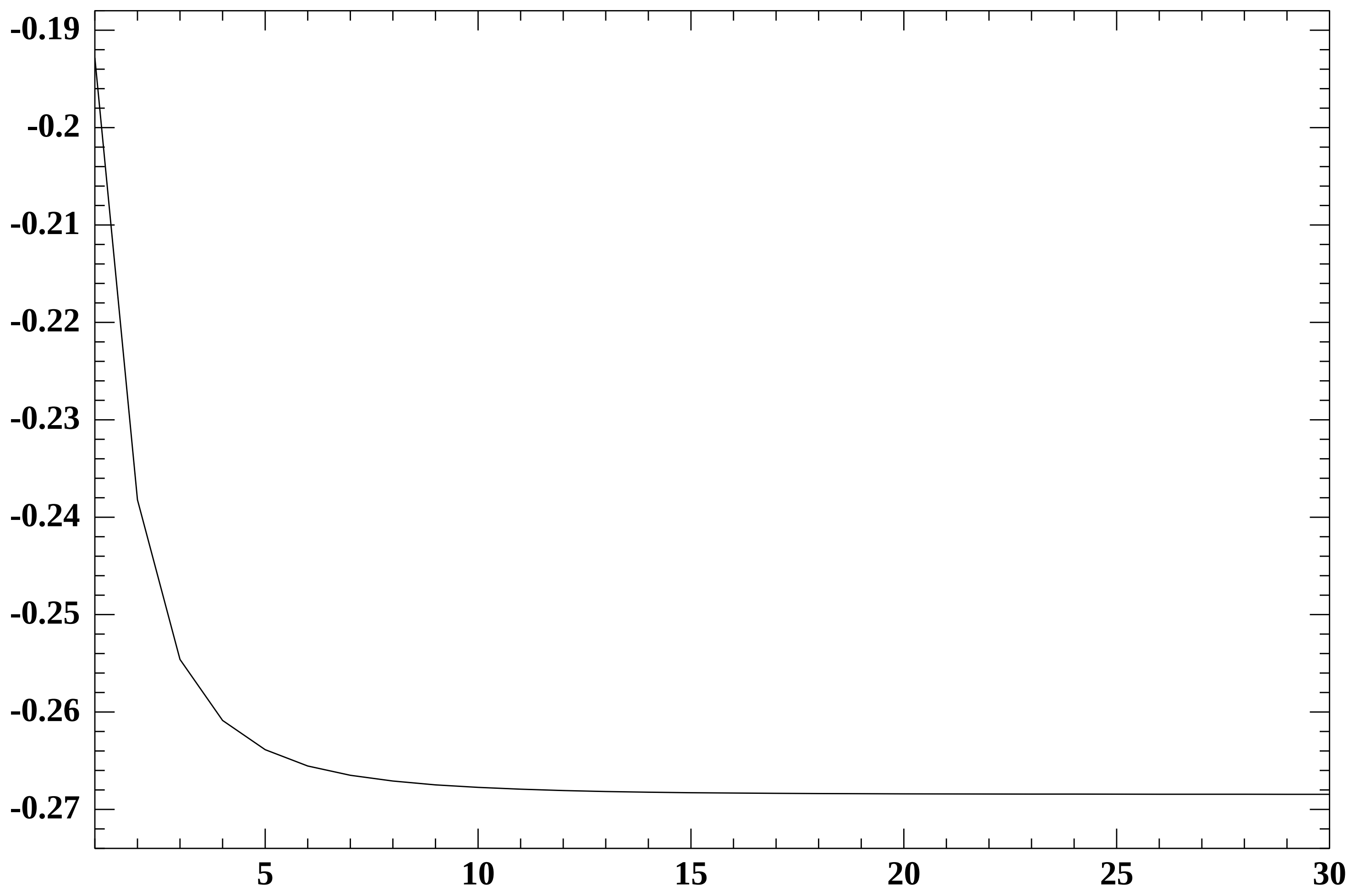
Remember: the function $(\theta, v) \to \lambda_\theta^- |\nabla v|^2$ is convex.

Consequence. There are only global minima !

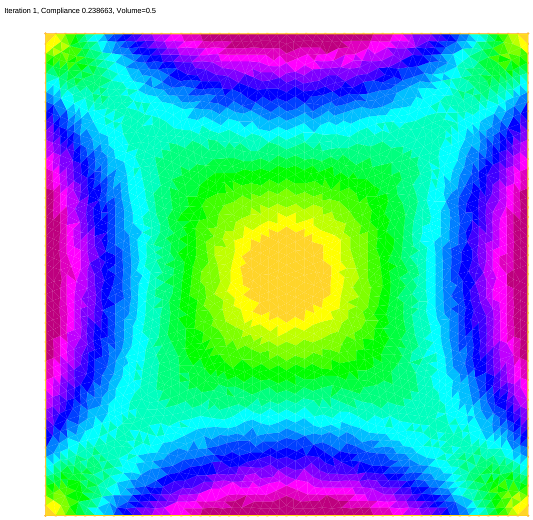
Numerically, we use an algorithm based on alternate direction minimization (see chapter 5).

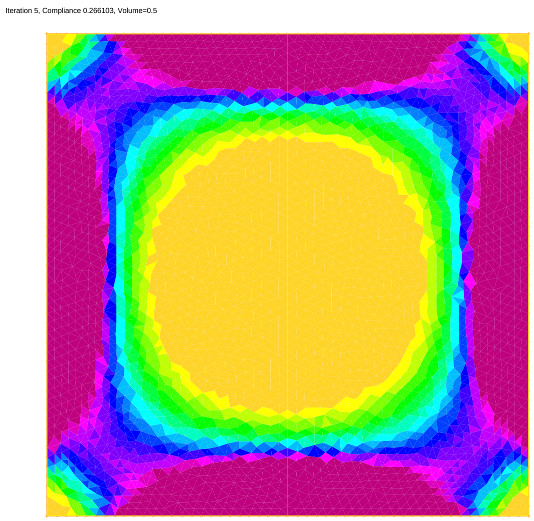
Convergence history:

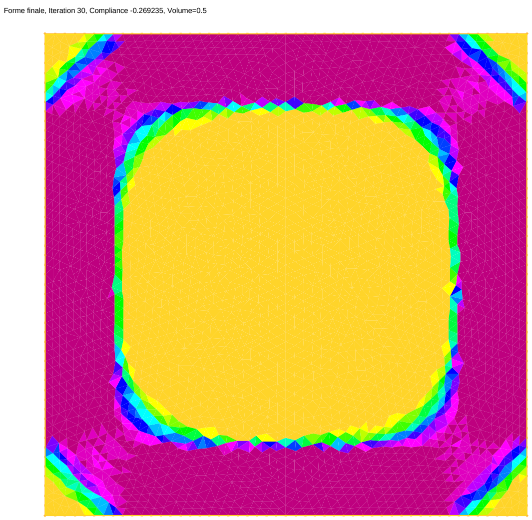
objective function in terms of the iteration number.



Volume fraction θ (iterations 1, 5, and 30)







A second self-adjoint example

Compliance minimization.

$$\min_{(\theta,A^*)\in\mathcal{U}_{ad}^L} \left\{ J(\theta,A^*) = \int_\Omega u(x)dx \right\},$$

where u is the solution of

$$\begin{cases} -\operatorname{div}(A^*\nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the adjoint state is just $p = -u$.

We solve in the domain $\Omega = (0,1)^2$ with the phases $\alpha = 1$ and $\beta = 2$. We fix a 50% volume constraint of α. We initialize with a constant value of $\theta = 0.5$ and a constant zero lamination angle. We perform 30 iterations.

Self-adjoint case $p = -u$.

$$\nabla_{A^*} J(\theta, A^*) = -\nabla u \otimes \nabla u \leq 0\,.$$

To minimize J we have to increase A^*.

Any optimal A^* satisfies

$$A^* \nabla u = \lambda_\theta^+ \nabla u$$

thus the optimal composite is the **best possible conductor**.

Consequence. We can eliminate the angle ϕ and it remains to optimize with respect to θ only !

Convexity

We rewrite the optimization problem thanks to the dual energy

$$\int_\Omega u\,dx = \min_{\substack{\tau \in L^2(\Omega)^N \\ -\operatorname{div}\tau = 1 \text{ in } \Omega}} \int_\Omega (\lambda_\theta^+)^{-1} |\tau|^2 dx \ .$$

Thus, we obtain a double minimization

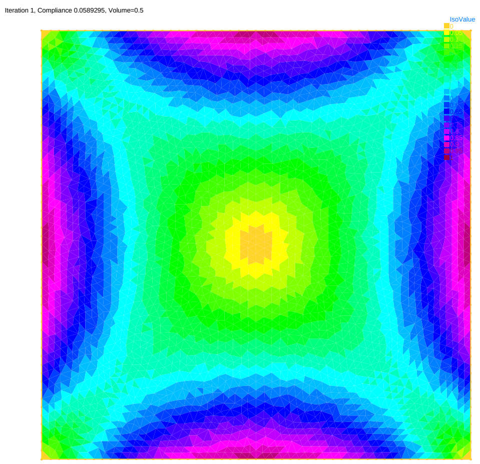
$$\min_{\theta, A^* = \lambda_\theta^+} J(\theta, A^*) = \min_{\theta, \tau} \int_\Omega (\lambda_\theta^+)^{-1} |\tau|^2 dx$$

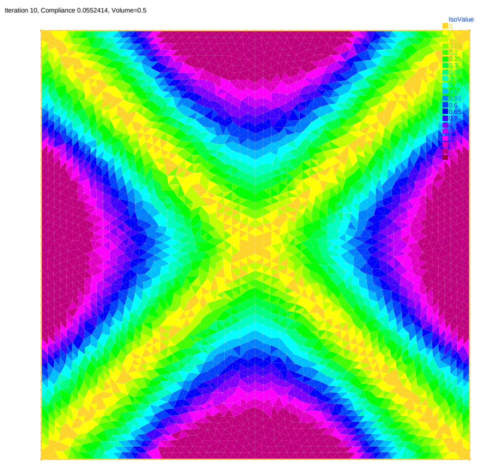
Remember: the function $(\theta, \tau) \rightarrow \dfrac{|\tau|^2}{\lambda_\theta^+}$ is convex.

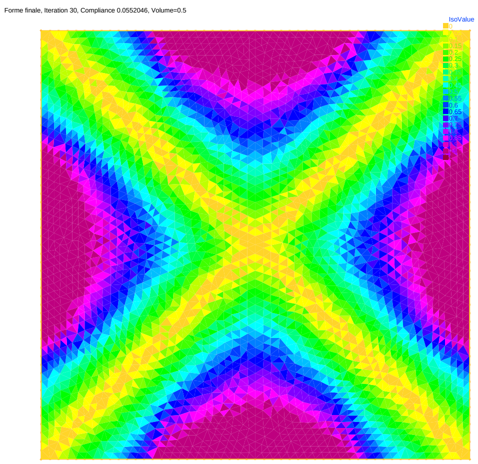
Consequence. There are only global minima !

Numerically, we use an algorithm based on alternate direction minimization (see chapter 5).

Minimal compliance membrane (iterations 1, 10, and 30)







Remarks

Convergence to a global minimum.

1. Numerical experiments with various initializations.
2. Convexity properties.

Shape optimization rather than two-phase optimization.

1. Numerically, holes can be mimicked by a very weak phase α ($\approx 10^{-3}\beta$).
2. Mathematically, when $\alpha \to 0$ we obtain Neumann boundary conditions on the holes boundaries.

Penalization

The previous algorithm compute composite shapes while we are rather interested by classical shapes.

Therefore we use a penalization process to force the density to take values close to 0 or 1.

Possible algorithms: after convergence to a composite shape,

1. either we add a penalization term to the objective function

$$J(\theta, A^*) + c_{pen} \int_\Omega \theta(1-\theta)\, dx,$$

2. either we continue the previous algorithm with a modified "penalized" density

$$\theta_{pen} = \frac{1-\cos(\pi\theta_{opt})}{2}.$$

If $0 < \theta_{opt} < 1/2$, then $\theta_{pen} < \theta_{opt}$, while, if $1/2 < \theta_{opt} < 1$, then $\theta_{pen} > \theta_{opt}$.

Example

Optimal radiator.

$$\begin{cases} -\operatorname{div}(A^*\nabla u) = 0 & \text{in } \Omega \\ A^*\nabla u \cdot n = 1 & \text{on } \Gamma_N \\ A^*\nabla u \cdot n = 0 & \text{on } \Gamma \\ u = 0 & \text{on } \Gamma_D. \end{cases}$$

We minimize the temperature where heating takes place

$$\min_{(\theta,A^*)\in\mathcal{U}_{ad}^L} \left\{ J(\theta,A^*) = \int_{\Gamma_N} u\,ds \right\}.$$

This is precisely the compliance ! Thus the problem is self-adjoint with $p = -u$.

Isotropic materials with conductivity $\alpha = 0.01$ and $\beta = 1$, in proportions $50, 50\%$, in the domain $\Omega = (0,1)^2$.

Optimal radiator