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STRUCTURAL OPTIMIZATION USING OPTIMAL MICROSTRUCTURES

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Abstract. In the context of shape optimization, the problem of minimizing the sum of the elastic compliance and of the weight of a plane structure under specified loading is considered. A relaxed formulation of the original problem is introduced, which allows microperforated composites as admissible designs. It is shown how the mathematical theory of optimal microstructures for composites can be used in practice to compute this relaxed formulation. In particular, the importance of so-called finite-rank sequentially laminated composites is emphasized. This approach leads to a new numerical algorithm for shape optimization.

Keywords. Optimal design, shape optimization, composite materials, finite-rank laminates.

INTRODUCTION

Since the pioneering work of Hashin and Shtrikman [1] many efforts have been devoted to the problem of bounding effective properties of composite materials, obtained by mixing two elastic components in fixed proportion. This is actually a problem of optimization of the microstructure (or arrangement of the components), so that the best possible bounds on effective properties are attained for so-called *optimal* microstructures. There is a great variety of such optimal microstructures, including the concentric spheres assemblage (see e.g. [1]), the periodic arrangement of properly shaped holes discovered by Vigdergauz [2], and the so-called finite-rank laminates introduced by Tartar [3] and Francfort and Murat [4]. By far, the third class (of finite-rank laminates) is the most general and easiest to use since, in particular, their effective properties may be computed explicitly.

The main goal of this paper is to show how this theory of optimal composites may be used in practice for some structural optimization problems. More precisely, we consider the model problem of optimal shape design for an elastic body : the design criteria are weight and compliance, the latter being a global measure of rigidity. For most loading configurations it turns out that there is no definite optimal shape, but, rather, a sequence of increasingly better designs obtained by removing from the initial body more and more, smaller and smaller, holes. The limit design is not a "classical" shape : it behaves like a composite material obtained by microperforation.

The content of this paper is the following : the optimal shape design problem is formulated in the first section, the second one is devoted to a brief review of the necessary results from the theory of optimal composites, and finally, in the third section, a so-called relaxed formulation of the problem is introduced, which leads to a new numerical algorithm for computing optimal shapes. The work reported here has been partially accomplished in collaboration with R.V. Kohn and G. Francfort (see [5], [6], [7]) ; it is a pleasure for me to acknowledge their help and friendship.

(I) OPTIMAL SHAPE DESIGN

The usual goal in structural optimization is to find the "best" structure which is, at the same time, of minimal weight and of maximum strength. Here, we consider a model problem of this type, in the context of linear elasticity with a single loading configuration. For simplicity, we work in two space dimensions, but most part of the analysis can be carried away in three space dimensions. We begin with a plane bounded domain Ω, occupied by a linearly elastic material with Hooke's law A_0, and loaded on its boundary by some known function f. Admissible designs are obtained by removing a subset $H \subset \Omega$, consisting of one or more holes (the new boundaries created this way are traction-free). The equations of elasticity for the resulting structure are

$$\begin{cases} \sigma = A_0 e(u), \quad e(u) = \frac{1}{2}(\nabla u + {}^t\nabla u) \\ div\ \sigma = 0 \quad in\ \Omega \backslash H \\ \sigma.n = f \quad on\ \partial\Omega,\ \sigma.n = 0 \quad on\ \partial H, \end{cases} \tag{1}$$

and the compliance is

$$c(\Omega \backslash H) = \int_{\partial\Omega} f.u = \int_{\Omega \backslash H} \langle A_0 e(u), e(u)\rangle = \int_{\Omega \backslash H} \langle A_0^{-1}\sigma,\sigma\rangle . \tag{2}$$

Introducing a positive Lagrange multiplier λ, the goal is to minimize, over admissible designs $\Omega \backslash H$, the weighted sum of the compliance and the weight, i.e.

$$Min_H \left[c(\Omega \backslash H) + \lambda |\Omega \backslash H| \right]. \tag{3}$$

The Lagrange multiplier λ has the effect of balancing the two contradictory objectives of rigidity and lightness of the optimal structure (increasing its value decreases the weight). As already said in the introduction, problem (3) may have no "classical" minimizer (i.e. there is no optimal shape $\Omega \backslash H$), since it is often advantageous to cut infinitely many small holes in a given design in order to decrease the functional (3). Thus, achieving the minimum may require a limiting procedure leading to a "generalized" design consisting of composite materials made by microperforation.

To take into account this physical behavior of nearly optimal shapes, we have to enlarge the space of admissible designs by permitting perforated composites from the start (this process is called *relaxation*). Such a structure is determined by two functions $\theta(x)$, the local volume fraction of material taking values between 0 and 1, and $A(x)$, the corresponding effective Hooke's law. The equations of elasticity now take place everywhere in the domain Ω

$$\begin{cases} \sigma = A(x)e(u), \quad e(u) = \frac{1}{2}(\nabla u + {}^t\nabla u) \\ div\ \sigma = 0 \quad in\ \Omega \\ \sigma.n = f \quad on\ \partial\Omega, \end{cases} \tag{4}$$

and the compliance is defined as

$$c(A) = \int_{\partial\Omega} f.u = \int_{\Omega} \langle A(x)e(u), e(u)\rangle = \int_{\Omega} \langle A(x)^{-1}\sigma,\sigma\rangle . \tag{5}$$

The weight is just the integral of the material volume fraction. Therefore the *relaxed* *formulation* of (3) is

$$Min_{\theta(x),A(x)} \left[c(A) + \lambda \int_{\Omega} \theta(x)\, dx \right], \tag{6}$$

where the minimization takes place over all perforated composites with density $\theta(x)$, and Hooke's law $A(x)$. A priori, the minimization of (6) is a formidable task since the set of all possible composite Hooke's law is not known. However, thanks to the precise form of the compliance (which is nothing but the stored elastic energy), it will be shown in section (II) that (6) always admits minimizers among a class of optimal composites, namely the rank-2 sequentially laminated composites. Furthermore, in section (III), part of this minimization will be done analytically, and (6) will be reduced to a non-linear minimization over statically admissible stresses which is thus easily amenable to numerical computation.

The advantage of the relaxed formulation (6) over the original one (3) is twofold (see [7] for details). On the one hand, problem (6) has always a solution. On the other hand, the minimum values of (3) and (6) are the sames, and each solution of (6) determines a minimizing sequence of classical designs for (3). At this point the use of composites might appear to be just a trick for proving existence theorems. In fact its importance goes much further. Indeed, it permits to separate the minimization of (6) in two different tasks : first, optimize locally the microstructure (this will be done analytically), second, minimize globally on the density $\theta(x)$. This has the effect of transforming the difficult "free-boundary" problem (3) into a much easier "sizing" optimization problem (6) in a fixed domain. This idea is at the root of the new numerical procedure proposed in section (III) for computing optimal shapes. Although the mathematical theory of relaxation by homogenization of microstructures is, by now, well-established (see Murat and Tartar [8], Lurie and al. [9], Kohn and Strang [10]), it is only recently that the first numerical applications have appeared thanks to the pioneering work of Bendsoe and Kikuchi [11]. However, the present work differs from their in one important respect : here, the use of *optimal* microstructures (i.e. rank-2 layered composites, see figure 1) is emphasized, while Bendsoe and Kikuchi considered *ad hoc* microstructures, namely square holes in squared cells, which are known to be sub-optimal.

(II) COMPOSITE MATERIALS AND OPTIMAL MICROSTRUCTURES

This section presents a brief review of some results from the theory of optimal bounds on composite materials, which will be used latter for computing the relaxed problem (6). Let us consider the problem of mixing two isotropic, linearly elastic, materials (with perfect bonding at the interface) in fixed proportion. Such a mixture behaves as an effective elastic medium, also called composite material. Since the arrangement (or the microstructure) of the two phases is not prescribed, different effective Hooke's laws (possibly anisotropic) may arise for different geometries of the mixture. Unfortunately, there is no simple algebraic characterization of such effective Hooke's laws. In other words, the set of all possible composite materials is not known a priori (of course, this set is included between the Voigt and Reuss bounds, i.e. the harmonic and arithmetic means of the components Hooke's laws, but it is always strictly smaller). Yet, we have a partial knowledge of the boundaries of this set. More precisely, in some cases we know what is the extremal rigidity or elasticity of all possible composite materials, and which microstructures are extremal. For a general account of this theory the reader is referred to [1], [4], [12], [13], [14] and the bibliographies therein.

Here, we are going to specialize the theory to the case of composites (with Hooke's law denoted by A) obtained by mixing an isotropic material (with Hooke's law A_0) with void. From a physical point of view, these composites A are nothing but the original material A_0 perforated by holes with traction-free boundaries. The proportion of material is fixed and is denoted by θ. Let us also fix a constant stress tensor σ. In view of the structural optimization problem considered in section (I) (where the compliance (5) plays a key role), we seek a composite material A (not necessarily unique) which minimizes the complementary energy $< A^{-1}\sigma,\sigma >$, i.e. which is the most rigid under the given stress σ. This question has been worked out in [7] and [14], and we recall their main results without proof. It turns out that such an optimal composite can always be choosen in the class of so-called *rank-2 layerings* (in two space dimensions). In other words, denoting by G_θ (resp. L_θ) the set of all possible perforated composites (resp. rank-2 layerings) with density θ, we have

$$\mathrm{Min}_{A\in G_\theta} < A^{-1}\sigma,\sigma > \;=\; \mathrm{Min}_{A\in L_\theta} < A^{-1}\sigma,\sigma >. \tag{7}$$

Furthermore, the parameters of the optimal rank-2 layering and the value of the minimum in (7) can be explicitly computed. Before doing this, let us describe precisely what is a rank-2 layering.

A rank-2 layering of material A_0 and void, with an overall density θ, is obtained by two successive laminations. It is characterized by four parameters : the layers normals e_1 and e_2, and the proportions m_1 and m_2 which satisfies $m_1+m_2 = 1$, $0 < m_1 < 1$, and $0 < m_2 < 1$. In a first step, a proportion $m_1\theta/(1-m_2\theta)$ of material A_0 is layered with void, in the direction e_1. In a second step, a proportion $m_2\theta$ of material A_0 is layered with the previous layering (obtained in the first step), in the direction e_2. Thus, for the final rank-2 layered composite, the parameters m_i are exactly the proportion of material distributed in layers normal to e_i. Of course each step involves a limiting procedure, i.e. the thickness of the layers goes to zero, but further, the length scale of the second step has to be much larger than that of the first step. As a consequence, this microstructure appears physically as the original material A_0 perforated with very thin and long holes (see figure 1). One of the main advantage of rank-2 layerings is the existence of an explicit algebraic formula for their effective Hooke's law (see proposition 4.2 in [4]).

We now turn to the characterization of the optimal rank-2 layering in (7) which is the strongest, or the most rigid, perforated composite supporting the stress σ (see [5] for details). Denoting by σ_1 and σ_2 the eigenvalues of the stress tensor σ (a two-by-two matrix in two space dimensions), the layers normals are simply the eigendirections of σ, and the amount of material in each layer is proportional to the corresponding eigenvalue, i.e.

$$m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|}, \quad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}. \tag{8}$$

Furthermore, the minimum value in (7) is just

$$\mathrm{Min}_{A\in G_\theta} < A^{-1}\sigma,\sigma > \;=\; < A_0^{-1}\sigma,\sigma > + \frac{(\kappa+\mu)(1-\theta)}{4\kappa\mu\theta}\Big[|\sigma_1| + |\sigma_2|\Big]^2 \tag{9}$$

where κ and μ are the bulk and shear moduli of material A_0. Remark that if we take σ equal to the identity, and if we assume that the composite A is isotropic, formula (9) is nothing but the well-known Hashin-Shtrikman upper bound on the effective bulk modulus. This bound is also attained by a microstructure obtained with the concentric spheres assemblage (see [1]). Thus, optimal microstructures are not necessarily unique. We favor rank-2 layerings for two reasons : there is an explicit algebraic formula for their effective Hooke's law, and they are optimal for any value of the stress σ. (Actually, their generalizations, so-called finite-rank laminates, are optimal for almost all known bounds ; see e.g. [4], [13], and [14].)

:x) dx] , (6)

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(III) COMPUTATION OF THE RELAXED FORMULATION AND NUMERICAL RESULTS

We are now equipped to reformulate the relaxed problem (6) in a way suitable for numerical computations. The starting point is the principle of minimum complementary energy which gives the value of the compliance (5) as

$$c(A) = \int_{\partial\Omega} f.u = \operatorname{Min}_{\substack{div\ \tau = 0 \ in\ \Omega \\ \tau.n = f \ on\ \partial\Omega}} \int_{\Omega} < A^{-1}(x)\tau,\tau > dx. \tag{10}$$

Thus, the relaxed formulation (6) appears as a double minimization over perforated composite materials and statically admissible stresses. The next step is to interchange the order of minimization, and to put the optimization in θ and A inside the integral since it is subject only to local constraints. Thus, the relaxed problem (6) is equivalent to

$$\operatorname{Min}_{\substack{div\ \tau = 0 \ in\ \Omega \\ \tau.n = f \ on\ \partial\Omega}} \int_{\Omega} \operatorname{Min}_{\substack{0 \le \theta \le 1 \\ A}} \left[< A^{-1}\tau,\tau > + \lambda\, \theta \right] dx \ . \tag{11}$$

The main advantage of formulation (11) over (6) is that we can restrict the class of admissible composites A to that of optimal composites in the sense of section (II), i.e. to rank-2 layerings. The equivalence between (6) and (11) reflects the two possible approaches of shape optimization. In (6), one fixes a design, then solves an elasticity problem, then adjusts the design to improve its performance. In (11), a statically admissible stress is choosen, then the optimal design is found for this stress, then the stress is adjusted to achieve kinematic admissibility. We favor this last approach since, as shown in section (II), we know explicitly an optimal microstructure for any given stress.

A new numerical algorithm for computing optimal shapes is thus deduced from the relaxed formulation (11). It amounts to iterate until convergence the following procedure : (*i*) having a candidate stress field, compute (analytically) the parameters of the corresponding optimal rank-2 layering ; (*ii*) solve a linear (possibly anisotropic) elasticity problem for this design and deduce the new stress field. Convergence is detected when almost no changes occur between two successive stress fields. For details about the practical implementation of this algorithm, the reader is referred to [5] and [7].

We present some numerical results for the so-called cantilever problem. The original shape (the domain Ω) is a rectangle which is fixed on its left side (zero displacement). Its other sides are traction free except on the middle point of the right side where a constant unit force is applied vertically (parallel to the edge) (see figure 2). The density θ of the computed optimal design is plotted on figures 3, 4, and 5 (void is obtained for $\theta = 0$, pure material for $\theta = 1$, and composites for intermediate densities). The first result (figure 3) holds for a rectangle, the heigth of which is twice its width. The optimal shape looks like two bars connected at right angle. This design is very similar to what could be obtained with the celebrated Michell truss approach [15]. It is not a surprise since, in the limit where the Lagrange multiplier λ goes to infinity, the relaxed formulation (11) is essentially equivalent to a continuous Michell truss problem (see [7] for details). Remark that the design includes almost no composites (up to errors of discretization).

The second result (figure 4) holds for a rectangle, the heigth of which is half its width. Here, the optimal shape includes a lot of composite regions (which is fair since the relaxed formulation allows them). However, from a practical point of view, perforated composites are difficult to manufacture, and one would rather suppress them if possible. This can easily be done by penalizing intermediate densities in our numerical algorithm, i.e. by forcing θ to be closed to 0 or 1 (see [5] for details). For the same configuration as in figure 4, figure 5 displays the result of our penalized procedure which is a "near-optimal" structure, i.e. very close to the optimal design drawn on figure 4 (the objective function (11) has increased of less than 5%), but with almost no composites.

The new numerical algorithm, which has been briefly presented above, has an essential advantage : it places no restriction on the topology of the optimal design. Indeed, the computation takes place in a fixed domain, and no "front-tracking" procedure is used to follow the boundaries of the optimal shape. Thus, this type of algorithm may be seen as a topology optimization method which can be used as a pre-processor for a more conventional code. In other words, an optimal design produced by this algorithm is a good candidate for an initial guess of a classical optimization code which would smooth out its edges without changing its topology (for striking examples, see [16]).

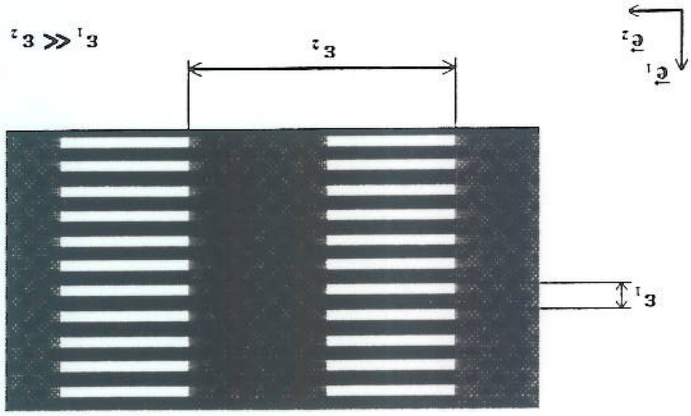


Figure 1
a rank-2 layering (the grey area is the material part)

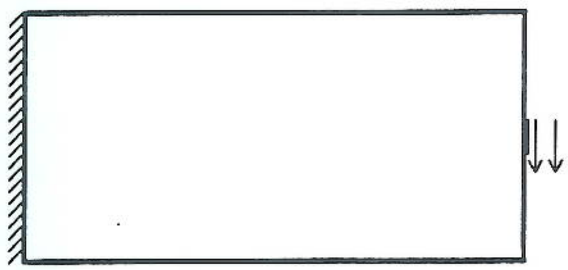


Figure 2
loading configuration of the cantilever problem

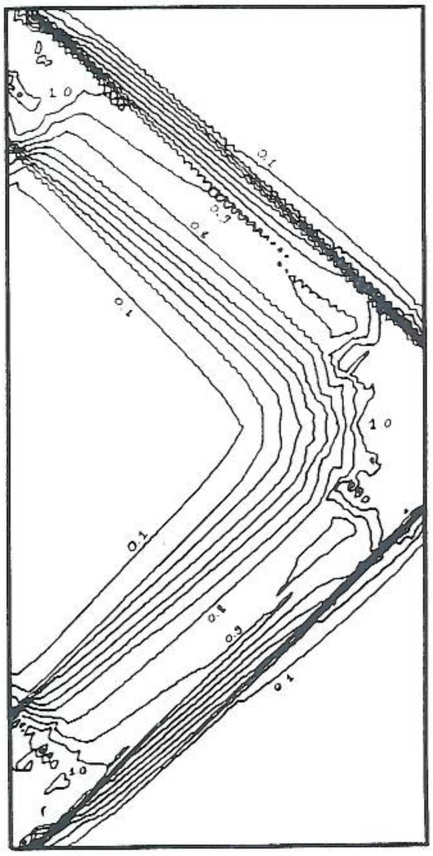
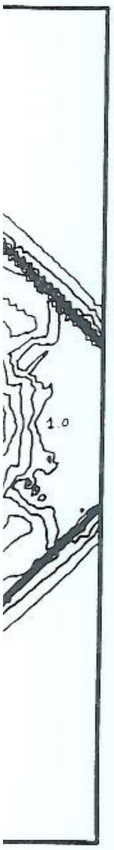


Figure 3
first cantilever problem : 70% of material removed



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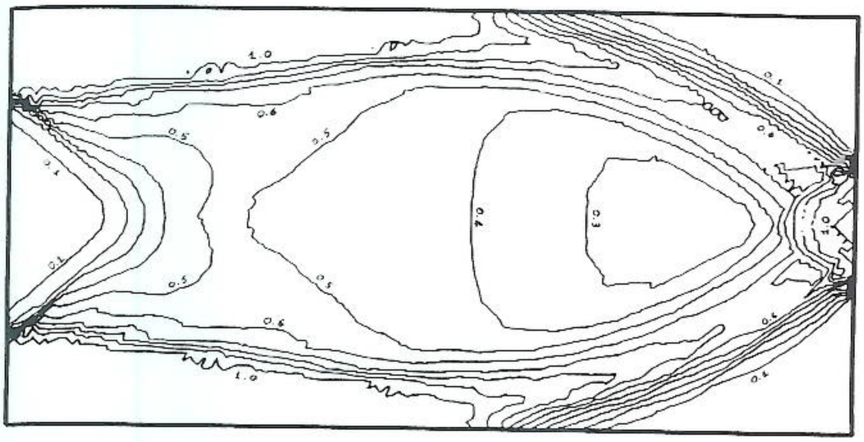

Figure 4
second cantilever problem : 48% of material removed

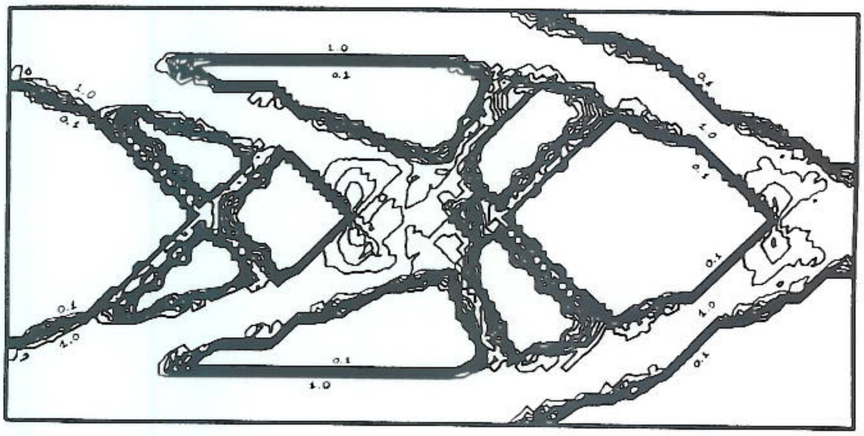

Figure 5
second cantilever problem : after penalization, 51% of material removed

References :

[1] Hashin Z. and Shtrikman S., *A Variational Approach to the Theory of the Elastic Behavior of Multiphase Materials*, J. Mech. Phys. Solids 11, pp.127-140 (1963).

[2] Vigdergauz S., *Effective Elastic Parameters of a Plate with a Regular System of Equal-Strength Holes*, Mechanics of Solids (transl. of Mekh. Tver. Tela.), 21, pp.162-166 (1986).

[3] Tartar L., *Estimations Fines des Coéfficients Homogénéisés*, Ennio de Giorgi colloquium, P. Krée ed., Pitman Research Notes in Math. 125, pp.168-187 (1985).

[4] Francfort G. and Murat F., *Homogenization and Optimal Bounds in Linear Elasticity*, Arch. Rat. Mech. Anal. 94, pp.307-334 (1986).

[5] Allaire G. and Francfort G., *A Numerical Algorithm for Topology and Shape Optimization*, Topology design of structures, Nato ASI Series E, M. Bendsoe et al. eds., Kluwer, Dordrecht (1993).

[6] Allaire G. and Kohn R.V., *Topology Optimization and Optimal Shape Design Using Homogenization*, Topology design of structures, Nato ASI Series E, M. Bendsoe et al. eds., Kluwer, Dordrecht (1993).

[7] Allaire G. and Kohn R.V., *Optimal Design for Minimum Weight and Compliance in Plane Stress Using Extremal Microstructures*, to appear in Europ. J. Mech..

[8] Murat F. and Tartar L., *Calcul des Variations et Homogénéisation*, in Les Méthodes de l'Homogénéisation: Théorie et Applications en Physique, Eyrolles, pp.319-369 (1985).

[9] Lurie K. and Cherkaev A. and Fedorov A., *Regularization of Optimal Design Problems for Bars and Plates I,II*, J. Optim. Th. Appl. 37, pp.499-521, 523-543 (1982).

[10] Kohn R.V. and Strang G., *Optimal Design and Relaxation of Variational Problems I-III*, Comm. Pure Appl. Math. 39, pp.113-137, 139-182, 353-377 (1986).

[11] Bendsoe M. and Kikuchi N., *Generating Optimal Topologies in Structural Design Using a Homogenization Method*, Comp. Meth. Appl. Mech. Engrg. 71, pp.197-224 (1988).

[12] Milton G., *On Characterizing the Set of Possible Effective Tensors of Composites: the Variational Method and the Translation Method*, Comm. Pure Appl. Math., XLIII, pp.63-125 (1990).

[13] Avellaneda M., *Optimal Bounds and Microgeometries for Elastic Two-Phase Composites*, SIAM J. Appl. Math. 47, pp.1216-1228 (1987).

[14] Allaire G. and Kohn R.V., *Optimal Bounds on the Effective Behavior of a Mixture of Two Well-Ordered Elastic Materials*, to appear in Q. Appl. Math..

[15] Michell A.G.M., *The Limits of Economy of Material in Frame Structures*, Phil. Mag. S6, vol. 8, 47, pp.589-597 (1904).

[16] Olhoff N. and Bendsoe M. and Rasmussen J., *On CAD-Integrated Structural Topology and Design Optimization*, Comp. Meth. Appl. Mech. Engrg., 89, pp.259-279 (1991).