A POST-TREATMENT OF THE HOMOGENIZATION METHOD FOR SHAPE OPTIMIZATION

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Abstract. We propose an alternative to the classical post-treatment of the homogenization method for shape optimization. Rather than penalize the material density once the optimal composite shape is obtained (by the homogenization method) in order to produce a workable shape close to the optimal one, we macroscopically project the microstructure of the former through an appropriate procedure that roughly consists in laying the material along the directions of lamination of the composite. We have tested our approach in the framework of compliance minimization in two-dimensional elasticity. Numerical results are provided.

Key words. shape optimization, homogenization, compliance, elasticity

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1. Introduction. Shape optimization consists in finding the optimal shape (represented by an open set) that minimizes a given cost-function (a mapping from the set of open sets into \( \mathbb{R} \)). The homogenization method in shape optimization extends the space of admissible shapes to composite shapes, that is, shapes containing micropores. This approach is motivated by theoretical as well as practical considerations and is now well established (see the books [1], [6], [8], and [10]). Theoretically, the problem does not, in general, admit solutions in a classical sense if the shape is not submitted to conditions; it is often useful to nucleate a multitude of tiny holes. Practically, optimization by the homogenization method allows the substitution of a nonstandard admissible space (the space of open sets) by a vector space (the parameters of the composite material). Solving the relaxed problem then leads to the obtention of a composite optimal shape.

However, more than an optimal shape, we are looking for a workable shape, i.e., a sufficiently smooth open set. A commonly used procedure consists in continuing the optimization, by the homogenization method, of the optimal composite solution obtained, albeit simultaneously penalizing the intermediate material densities. The main drawback of this method lies in the difficulty of controlling the level of detail with respect to the qualitative loss of optimality of the shape. Moreover, only the material density of the composite shape is explicitly used in the penalization procedure. The pattern of its microstructure is not directly exploited when the latter holds important information that can be turned into profit. In this paper, we propose an alternative method that consists in reconstituting a shape close to the optimal by straightforwardly reproducing the underlying microstructure at a macroscopic scale. A parameter allows us to control the desired degree of detail.

Let us mention that our work was prompted by another algorithm, which we have proposed in [13], that automatically combined the homogenization method with the boundary variation, and a topological criterion for nucleating holes. Indeed, we had
noticed that following the nucleation of a hole, the algorithm tended to produce areas of low density around the hole that were reminiscent of the underlying microstructure. Finally, note that our method is not a visual post-treatment intended to display the local microstructure of the optimal composite shape (see, for instance, [11]).

We first recall some classical results from the homogenization theory in elasticity. At this point, we present two important classes of composite materials: the periodic materials and the laminates. We have tested our procedure on the compliance minimization problem for an elastic structure in dimension two. It is a paradigm of the shape optimization problem, since it has the nonnegligible advantage of possessing an explicit relaxed formulation. The optimal shape can be attained in the special class of composite shapes made of laminates. This allows the numerical computation of (at least almost) optimal composite shapes. The compliance minimization problem as well as its relaxed version are presented in section 3. Lastly, in section 4, we introduce a new method that builds a sequence of shapes close to the optimal solution of the compliance minimization problem obtained through the homogenization method. This is illustrated by some numerical examples.

2. Preliminaries. The theory of homogenization. This section is a brief review of classical homogenization results in linear elasticity. We refer the reader to the monograph of Allaire [1] for more details on this topic (see also [6], [8], and [10]).

As stated in the introduction, the majority of problems arising in shape optimization are ill posed. The minimizing sequences do not converge to a classical shape. In effect, the nucleation of microscopic holes turns out to be profitable more often than not. The sequence of open sets thus obtained does not “converge” in the space of open sets. However, we can still define a notion of convergence for which these sequences are compact. The limits that may be attained in this fashion constitute the set of composite bodies. For simplicity, instead of considering bodies made of a mixture of material and void, we shall study composite bodies, occupying a fixed domain, made of two distinct materials. The theory of homogenization is actually better understood in this context. Numerically, void shall be approximated by a material of weak resistance, also called soft material.

2.1. Composite materials: Homogenization in elasticity. All the results presented in this section are classical. Their proofs can be found, for instance, in [1]. Let $\Omega$ be an open set in $\mathbb{R}^N$. An elastic body occupying the domain $\Omega$ is characterized in every point $x$ of the domain $\Omega$ by its Hooke law $A(x)$, a fourth-order tensor operating on $N \times N$ symmetric matrices. Let $\alpha$ and $\beta$ be two positive real numbers. We introduce the following subsets of Hooke laws:

$$\mathcal{M}_{\alpha,\beta} := \{ A \in \mathcal{M}_N^4 : A\xi : \xi \geq \alpha|\xi|^2 \text{ and } A^{-1}\xi : \xi \geq \beta|\xi|^2 \text{ for all } \xi \in \mathcal{M}_N^s \},$$

where $\mathcal{M}_N^4$ denotes the space of fourth-order tensors operating on symmetric matrices and $\mathcal{M}_N^s$ is the space of $N \times N$ symmetric matrices. We assume the body $\Omega$ to be clamped on its boundary, submitted to dead loads $f \in L^2(\Omega; \mathbb{R}^N)$. The elasticity system consists in determining the displacement $u$ of the structure, i.e., the unique solution to the boundary value problem:

$$\begin{cases} -\text{div}(Ae(u)) = f & \text{ in } \Omega, \\ u = 0 & \text{ on } \partial\Omega, \end{cases}$$

where $e(u)$ is the linearized metric tensor associated with the displacement $u$:

$$e(u) = \frac{1}{2}(\nabla u + \nabla u^T).$$
Consider a sequence of Hooke laws $A^\varepsilon \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. The sequence $A^\varepsilon$ is said to H-converge to $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ if and only if, for all $f \in L^2(\Omega; \mathbb{R}^N)$, the sequence of displacements $u_\varepsilon$ of the boundary value problems:

\[
\begin{cases}
-\text{div}(A^\varepsilon e(u_\varepsilon)) = f & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial\Omega,
\end{cases}
\]

converges in $L^2(\Omega; \mathbb{R}^N)$ to the solution $u$ of the boundary value problem:

\[
\begin{cases}
-\text{div}(A^* e(u)) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

A remarkable property of H-convergence is that every sequence in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ admits a convergent subsequence.

**Remark 1.** If a sequence $A^\varepsilon$ H-converges to $A^*$, the convergence result extends to elasticity problems with mixed boundary conditions.

Consider two elastic materials of Hooke laws $A$ and $B$. A composite body made of these materials is a linear elastic body whose Hooke law may be obtained as an H-limit of a sequence of elements $A^\varepsilon$ such that for all $x \in \Omega$, $A^\varepsilon(x)$ is equal to either $A$ or $B$. For all mapping $\theta \in L^\infty([0,1])$, we call $G_\theta$ the set of Hooke laws $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ derived using the materials $A$ and $B$ with respective local proportions $\theta(x)$ and $1 - \theta(x)$. In other words,

\[
G_\theta := \left\{ A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}) : \exists \chi^\varepsilon \in L^\infty(\Omega; \{0,1\}) \text{ such that } A^\varepsilon = \chi^\varepsilon A + (1 - \chi^\varepsilon)B \xrightarrow{H} A^* \text{ and } \chi^\varepsilon \xrightarrow{L^\infty} \theta \right\}.
\]

Another noteworthy property of H-convergence is that the set of Hooke laws $A^*(x)$ achieved at a point depends only on the value of $\theta(x)$. Hence, for all real $\theta \in [0,1]$, there exists a closed subspace $G_\theta$ of $\mathcal{M}_{\alpha,\beta}$ satisfying

\[
G_\theta = \left\{ A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta}) : A^*(x) \in G_{\theta(x)} \right\}.
\]

In particular, this allows us to be content with the study of homogeneous composite solids in order to determine the properties of composite materials.

**2.2. Periodic composites.** A special family of composites is obtained by arranging periodically the two types of material $A$ and $B$, as shown in Figure 1. Let $Y = [0,1]^N$. Let $\chi \in L^\infty(\mathbb{R}^N; \{0,1\})$ be a $Y$-periodic function, i.e., $\chi(x + f_i) = \chi(x)$ for all $x \in \mathbb{R}^N$ and all vector $f_i$ ($i = 1, \ldots, N$) of the canonical basis of $\mathbb{R}^N$. For all real $\varepsilon > 0$, we denote by $A^\varepsilon$ the element of $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ defined for all $x \in \Omega$ by

\[
A^\varepsilon(x) = \chi(x/\varepsilon)A + (1 - \chi(x/\varepsilon))B.
\]

The sequence $A^\varepsilon$ H-converges to a constant element $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ defined for all symmetric matrices of $N$th order by

\[
A^{-*}\sigma \cdot \sigma = \min_{\tau \in L^2_0(Y; \mathcal{M}_N)} \int_Y \left( \chi(x)A + (1 - \chi(x))B \right)^{-1}\tau \cdot \tau dx,
\]

where $L^2_0(Y; \mathcal{M}_N)$ is the set of $Y$-periodic $L^2$ functions with values in $\mathcal{L}_N^\varepsilon$. We call $P_\theta$ the set of periodic composites obtained using $A$ and $B$ with respective proportions $\theta$ and $1 - \theta$. Yet another remarkable property is the fact that $\overline{P}_\theta = G_\theta$; i.e., any composite may be approximated by a periodic material.
2.3. Sequential laminates. A rank-1 sequential laminate is obtained by succes-
successively layering the materials $A$ and $B$. Higher-rank laminates are obtained recursively by repeating the procedure with the lower-rank laminate and $A$ (or $B$). Figure 2 shows a rank-2 laminate. The significant property of laminates consists in the fact that, unlike periodic composites, which necessitate solving variational problems, there are explicit formulas that enable us to compute the associated homogenized Hooke laws. In what follows, we shall consider only rank-2 laminates. Let us consider the limit case $B = 0$. A laminate is determined by several parameters: the density $\theta$ of the material used, the successive directions of lamination $e_i$, and the proportions of lamination $m_i$, which represent the fraction of material with respect to the total quantity of material $\theta$ used at each step of the manufacture of the laminate. In particular, $\sum_i m_i = 1$. The Hooke law $A^{*}$ associated with such a composite is

$$A^{*-1} = A^{-1} + \frac{1 - \theta}{\theta} \left( \sum_{i=1}^{N} m_i f_{A}^c(e_i) \right)^{-1},$$

where for all unit vectors $e$ of $\mathbb{R}^2$, $f_A^c(e)$ is the fourth-order tensor defined for all symmetric matrix $\xi$ by the quadratic form

$$f_A^c(e) \xi \cdot \xi = A \xi \cdot \xi - \frac{1}{\mu} |A \xi|^2 + \frac{\mu + \lambda}{\mu(2\mu + \lambda)} ((A \xi) e \cdot e)^2.$$ 

For all densities $\theta \in [0, 1]$, we call $\mathcal{L}_{\theta,p}$ the set of all the Hooke laws generated by rank-$p$ laminates.

3. Compliance optimization. We restrict our analysis henceforth to the two-dimensional case, $N = 2$. Let $\Omega$, a connected open set of $\mathbb{R}^2$, be the reference configuration of a homogeneous and isotropic linearly elastic body. Assume that the boundary
The displacement of the structure $u$ is the unique element,

$$u \in V := \{ v \in H^1(\Omega) : u = 0 \; \text{a.e. on } \Gamma_D \},$$

where for all $v \in V$, we have

$$\int_{\Omega} Au(u) \cdot e(v) \, dx = \int_{\Gamma_N} g \cdot v \, dx.$$ 

In the above, $A$ is the used material’s Hooke law defined for all symmetric tensors $\xi$ by

$$A\xi = 2\mu\xi + \lambda \text{tr} \xi \text{Id},$$

where $\lambda$ and $\mu$ are the Lamé coefficients of the constitutive material of the solid $\Omega$, and $e(v)$ is the linearized metric tensor of $v$:

$$e(v) = \frac{1}{2} (\nabla v + \nabla v^T).$$

The compliance associated with the shape $\Omega$ is defined as the work of exterior forces exerted on the solid, that is,

$$J(\Omega) = \int_{\Gamma_N} g \cdot u \, dx = \int_{\Omega} 2\mu |e(u)|^2 + \lambda (\text{div } u)^2 \, dx.$$ 

The weaker the compliance, the more rigid the structure. Since we neglect the weight of the body, it is always advantageous to add material in order to strengthen the structure and reduce its compliance. We consider the compliance minimization problem that consists in determining $\Omega^* \in U_{ad}$ satisfying

$$(3.1) \quad J(\Omega^*) = \min_{\Omega \in U_{ad}} J(\Omega),$$

where $U_{ad}$ denotes the set of open sets whose boundaries contain $\Gamma_N$ and $\Gamma_D$, and whose volumes do not exceed a given maximal volume $V$:

$$U_{ad} = \{ \Omega \text{ open set in } \mathbb{R}^2 \text{ such that } \Gamma_N \cup \Gamma_D \subset \partial \Omega \text{ and } |\Omega| \leq V \}.$$ 

### 3.1. Relaxation by the homogenization method

Problem (3.1) is ill posed: it does not have an optimal solution. Minimizing sequences of $J$ consist of shapes $\Omega$ having an increasing number of holes, and do not converge to an element of $U_{ad}$. In order to solve this problem, it is necessary to enlarge the space of admissible designs by allowing for composite shapes. Such a structure is determined through the local density of the used material $\theta(x)$, and through its effective Hooke law $A^*(x)$ corresponding to its microstructure. In the particular case we are concerned with, an optimal solution may be obtained thanks to the particular class of composite materials that consists of the rank-2 sequential laminates. Should we impose upon the shape to be contained in a fixed box $D$, the relaxed minimization problem associated with the compliance minimization problem consists in solving the following minimization problem:

$$\min_{0 \leq \theta \leq 1} \left\{ J(\theta, A^*), \begin{cases} A^* \in L^{2,2} \\ \int_D \theta \, dx \leq V \end{cases} \right.$$
with
\[ J(\theta, A^*) = \int_{T_N} g \cdot u \, dx, \]
where
\[ u \in W := \{ v \in H^1(D)^2 : v = 0 \text{ a.e. on } \Gamma_D \}, \]
and the displacement of the structure satisfies for all \( v \in W, \)
\[ \int_D A^* e(u) \cdot e(v) \, dx = \int_{T_N} g \cdot v \, dx. \]

To conclude, let us recall that the above problem may be rewritten, using the dual (or complementary) energy principle, as the minimization of a functional that does not directly involve the solution of a variational problem. More precisely, we have

\[ J(\theta, A^*) = \min_{\{ \sigma \in L^2(D; M^2) : \text{div } \sigma = 0 \text{ in } D, \sigma_n = g \text{ on } \Gamma_N \}} \int_D A^{\ast -1} \sigma \cdot \sigma \, dx. \]

The asset of such a formulation is that it enables swapping the different steps of minimization. Now, for a given \( \theta \) and \( \sigma \), we may explicitly determine the tensor \( A^* \) of \( L_{\theta,2} \) that minimizes \( A^* \sigma \cdot \sigma \). In addition, we infer that the directions of lamination coincide with the eigenvectors of the matrix \( \sigma \) (in particular, they are orthogonal) and that the respective proportions of lamination are
\[ m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|} \quad \text{and} \quad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}. \]

Lastly, we have
\[ \min_{A^* \in L_{\theta,2}} A^{\ast -1} \sigma \cdot \sigma = A^{-1} \sigma \cdot \sigma + \frac{1 - \theta}{\theta} g^*(\sigma), \]
with
\[ g^*(\sigma) = \frac{\kappa + \mu}{2\mu\nu} (|\sigma_1| + |\sigma_2|)^2. \]

The relaxed problem (3.2) is classically solved by successive minimizations with respect to \((\theta, A^*)\) and \(\sigma\); minimizing with respect to \(\sigma\) requires, at each iteration, solving a variational problem.

4. Projection of a composite shape. In the previous section, we have recalled how the optimal solution to the compliance minimization problem is determined. Unfortunately, the solution obtained is a composite shape that is not workable in practice. To make up for this problem, it is natural to try to build up a sequence of shapes \( \Omega^\varepsilon \) that reproduce at a macroscopic scale the underlying microstructure of the optimal composite. This sequence shall be built so that its limit behavior is that of the optimal composite shape. However, this construction is difficult to achieve because of the different scales the construction of the shape involves: the size of the structure \( L \), and the two scales of the microstructure. Indeed, the principal microstructure (of
characteristic scale \( \varepsilon_1 \) itself contains a composite material (a micro-microstructure in a certain sense), namely, a rank-1 laminate (of characteristic scale \( \varepsilon_2 \)); see Figure 3. We may consider proceeding in two steps. First, reproduce a microstructure (at the macroscopic level) composed of a mixture of material, and a rank-1 laminate. Next, apply once again this procedure in order to end up with a noncomposite shape. Seemingly, three different scales have to be introduced. From a practical viewpoint, the smallest scale \( \varepsilon_2 \) is bounded from below by the size of the smallest workable detail. The other magnitude scale \( \varepsilon_1 \) is submitted to two contradictory constraints: it has to be small compared with the size of the structure \( L \), and it has to be large with respect to the former scale \( \varepsilon_2 \), i.e., the smallest details allowed for; see Figure 3. Details must therefore be of a size order twice as small as the structure. Since the size of the smallest details, as well as the scale of the structure, are both parameters of the problem, they do not necessarily agree with this constraint. Moreover, the relationship between the Hooke law and the microstructure is not univocal. The same Hooke law may be achieved by different microstructures. Hence, the choice of reproducing the microstructure of a rank-2 laminate at a macroscopic scale proves to be somewhat arbitrary, all the more so as, even in this subclass of composite materials, different microstructures lead to the same Hooke law. For instance, changing the order of lamination (\( e_1 \) with \( e_2 \), and \( m_1 \) with \( m_2 \)) produces different microstructures without changing the Hooke law. These observations have induced us into opting for a slightly different approach that consists in reproducing, at a macroscopic level, a locally periodic shape whose associated constitutive law remains close (without being identical, however) to that of a rank-2 laminate. Such a shape has (locally) only one scale parameter, that is, the period.

### 4.1. Construction of locally periodic elastic bodies

As stated in section 2, in the nondegenerate case \((A \text{ and } B \in M_{a,b})\), every solid, whose Hooke law is that of a homogeneous composite solid pointwise (i.e., belonging to \( G_\theta \)), may be obtained as the limit of solid bodies with Hooke laws taking their values in \( \{A, B\} \). However, we do not have at our disposal a similar result in the general case (in particular, if \( B = 0 \)). Therefore, it is not obvious that we shall manage to build up a sequence of shapes \( \Omega^c \) whose limit behavior converges to an optimal composite solid belonging to \( G_\theta \). Since we are not able to identify the set of composite solids that can be obtained with a unique elastic material \( A \) (combined with void), we set out to exhibit some of them. This shall enable us to project the optimal composite on an element of this subclass and accordingly build up a sequence of real shapes that converges to this element.

The constructions of homogeneous periodic solids and homogeneous sequential
laminates proposed in section 2 extend to the case \( B = 0 \). Homogeneous periodic solids are obtained as limits of the open sets:

\[
\Omega^\varepsilon = D \cap \omega^\varepsilon,
\]

where

\[
\omega^\varepsilon = \{ x \in \mathbb{R}^2 : \varepsilon^{-1} x \in \omega \},
\]

and \( \omega \) is a \( Y \)-periodic open set, i.e.,

\[
x \in \omega \iff x + f_1 \in \omega \iff x + f_2 \in \omega,
\]

where \((f_1, f_2)\) is the canonical basis of \( \mathbb{R}^2 \). It is easy to modify this definition in order to produce nonhomogeneous solids by allowing the open set \( \omega \) to depend on the considered point \( x \). Let \( \omega \) be a smooth enough mapping from \( D \) into the space of \( Y \)-periodic open sets of \( \mathbb{R}^2 \). The sequence of open sets

\[
\Omega^\varepsilon = \{ x \in D : \varepsilon^{-1} x \in \omega(x) \}
\]

converges to a composite solid whose Hooke law at every point \( x \) is that of a homogeneous periodic solid associated with \( \omega(x) \). The set of composite solids that we may build up, in this fashion, remains limited after all. In particular, all the cells of periodicity are square (when we may likewise use rectangular cells) and identically oriented. Let \( \varphi \) be a smooth mapping of \( D \) into \( \mathbb{R}^2 \); the sequence of open sets

\[
\Omega^{\varepsilon}_{\varphi} = \{ x \in D : \varepsilon^{-1} x \in \varphi^{-1}(\omega(x)) \}
\]
converges to a composite solid whose Hooke law $A^*$ is, at every point $x$, that of a periodic homogeneous body of period $D\varphi^{-1}(x)Y$, associated with the open set $D\varphi^{-1}(x)\omega(x)$; see Figure 4. It should be mentioned that Briane [9] proposes a different construction that is also based on the deformation of a periodic network.

4.2. An alternate composite material. We substitute the optimal composite shape by a composite whose behavior is close and that may be obtained by the construction (4.1) described in the previous section.

At each point $x$ of the domain $D$, the optimal laminate is determined by the orthogonal directions of lamination $e_1$ and $e_2$, the density of the material $\theta$, and the proportions of lamination $m_1$ and $m_2$. First of all, with all parameters $m_1$, $m_2$, and $\theta$, we associate the $Y$-periodic open set $\omega_{\theta,m}$ defined by

$$\omega_{\theta,m} \cap Y = \{ x \in Y : 2|x_1 - 1/2| > p_1 \text{ and } 2|x_1 - 1/2| > p_2 \},$$

where $p_1$ and $p_2$ are positive reals that satisfy

$$1 - p_1p_2 = \theta \quad \text{and} \quad (1 - p_1)m_2 = (1 - p_2)m_1.$$ 

We assume that the directions of lamination $e = (e_1, e_2)$ of the optimal shape constitute a regular field and that $D$ is simply connected. Furthermore, we introduce the set $\mathcal{F}_e$ of functions from $D$ into $\mathbb{R}^2$ defined by

$$(4.2) \quad \mathcal{F}_e = \left\{ \varphi : D \to \mathbb{R}^2 : \det(D_x\varphi) \neq 0, \quad D_x\varphi(e_1) \land f_1(x) = 0 \text{ and } D_x\varphi(e_2) \land f_2(x) = 0 \right\}.$$ 

The following lemma ensures that $\mathcal{F}_e$ is not empty.

**Lemma 4.1.** Let $D$ be a smooth connected bounded open set, and let $e_1 \in C^1(D; \mathbb{R}^2)$ be a vector field of unit norm defined in a neighborhood of $\Omega$; then the set $\mathcal{F}_e$ defined by (4.2) is not empty.

This result is standard. However, it is not easy to find in the literature under the current formulation. For the reader's convenience, a proof is given in the appendix.

**Remark 2.** Should the directions of lamination reveal pointwise singularities or should $D$ be not simply connected, $\mathcal{F}_e$ may be empty.
For all elements \( \varphi \in \mathcal{F}_e \), we call \( \Omega_{\varphi}^\varepsilon \) the sequence of open sets produced by the previous construction applied to \( \varphi \) and \( \omega = \omega_{\eta,m} \),
\[
\Omega_{\varphi}^\varepsilon = \left\{ x \in D : \varepsilon^{-1} x \in \varphi^{-1}(\omega_{\eta,m}(x)) \right\}.
\]
The sequence of open sets \( \Omega_{\varphi}^\varepsilon \) converges to a locally periodic composite whose density is that of the optimal laminate. Conditions \( D_2 \varphi(e_i) \land f_j(x) = 0 \) \((i = 1, 2)\) ensure that periodicity cells are oriented along the directions of lamination.

**Remark 3.** The limit behavior of the sequence of open sets \( \Omega_{\varphi}^\varepsilon \) is of the form \( \Omega_{\varphi}^\varepsilon = \Omega_{\varphi} \) is a linear mapping (more precisely, \( R = D\varphi^{-1} \)). Instead of solving the compliance minimization problem over rank-2 laminates, it is possible to perform only a partial relaxation by restricting the minimization to this class of composites. This approach was developed by Bendsoe and Kikuchi [7] for periodicity cells of the form above, where \( R \) is a rotation.

5. **Numerical applications.** It remains to determine an element \( \varphi \) of \( \mathcal{F}_e \) in order to infer the sequence of open sets \( \Omega_{\varphi}^\varepsilon \). We choose to determine the element \( \varphi \) of \( \mathcal{F}_e \) that minimizes
\[
I(\varphi) = \frac{1}{2} \int_D |\nabla \varphi_1 - e_1|^2 + |\nabla \varphi_2 - e_2|^2 \, dx.
\]
The shape \( \Omega_{\varphi}^\varepsilon \) is then defined by
\[
\Omega_{\varphi}^\varepsilon = \left\{ x \in D : \cos(\varphi_1(x)/\varepsilon) > \cos(\pi(1 - p_1)), \right. \\
\quad \text{and} \quad \left. \cos(\varphi_2(x)/\varepsilon) > \cos(\pi(1 - p_2)) \right\}.
\]
The choice made is somewhat arbitrary, but it entails a linear system that is easily solved. Other choices are conceivable. The selection of the element \( \varphi \) of \( \mathcal{F}_e \) has an influence on the relative size of the nucleated holes and on their (more or less elongated) shape. We may consider selecting \( \varphi \) in order to minimize the error when substituting the optimal composite shape by the limit composite of the sequence \( \Omega_{\varphi}^\varepsilon \). However, an explicit formula expressing a periodic material’s Hooke law is not available, so this seems to be quite delicate to carry out.

5.1. **Orientation of eigenvectors.** The procedure described here requires the orientation of eigenvalues in a consistent fashion. Nevertheless, we can avoid such a computation. Assume that such an orientation has been established. We introduce vectors \( v_1 \) and \( v_2 \) defined by \( v_1 = \varphi_1 e_1 \) and \( v_2 = \varphi_2 e_2 \). In this case, for \( i = 1, 2 \) and \( j = 1, 2 \), we have
\[
\nabla \varphi_i \cdot e_j = (\nabla v_i \cdot e_i) \cdot e_j.
\]
Hence, minimizers of \( I(\varphi) \) also minimize the functional
\[
I(v) = \frac{1}{2} \int_D |(\nabla v_1 \cdot e_1) \cdot e_1 - 1|^2 + |(\nabla v_2 \cdot e_2) \cdot e_2 - 1|^2 \, dx
\]
on the set of vectors \( v = (v_1, v_2) : D \to \mathbb{R}^2 \times \mathbb{R}^2 \) that satisfy
\[
\tag{5.1} v_1 \land e_1 = 0, \ v_2 \land e_2 = 0, \ (\nabla v_1 \cdot e_2) \cdot e_1 = 0, \ \text{and} \ (\nabla v_2 \cdot e_1) \cdot e_2 = 0.
\]
Moreover, we have \( |\varphi_1| = ||v_1|| \) and \( |\varphi_2| = ||v_2|| \). Now, the definition of \( \Omega_{\varphi}^\varepsilon \) depends only on the moduli of \( \varphi_1 \) and \( \varphi_2 \). Therefore, in order to fully determine \( \Omega_{\varphi}^\varepsilon \), it suffices to minimize \( I \) under the constraints (5.1), which is independent from the orientation of the eigenvectors \( e_1 \) and \( e_2 \).
5.2. Computing a minimizer of $I$. Note that the computations of $v_1$ and $v_2$ are disconnected so that they can be carried out separately. To compute $v_1$, we introduce the Lagrangian

$$
\mathcal{L}(v_1, p, P) = \frac{1}{2} \int_D |(\nabla v_1 \cdot e_1) \cdot e_1 - 1|^2 + |\nabla(v_1 \cdot e_2)|^2 + |(\nabla v_1 \cdot e_2) \cdot e_1|^2
+ \left| \frac{\sigma v_1 - \lambda v_1}{\lambda_1 - \lambda_2} \right|^2 + |v_1 \cdot (\nabla e_1 \cdot e_1)|^2 - |P \wedge e_1|^2 \, dx
+ \int_D (\nabla v_1 \cdot e_2) \cdot e_1 p + \frac{\sigma v_1 - \lambda_1 v_1}{\lambda_1 - \lambda_2} \cdot P \, dx,
$$

where the functions $p$ and $P$ are valued in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Minimizers of $I$ are saddle points of the Lagrangian. In order to determine a stationary point of the Lagrangian, we have used Laplace finite elements $P_l$ for all unknowns. An easy, but somewhat tedious, computation allows us to explicitly specify the different terms that appear in the Lagrangian $\mathcal{L}$ in terms of $v_1$ and $\sigma$. In particular,

$$
(\nabla v_1 \cdot e_1) \cdot e_1 = \delta^{-1} \left( (\lambda_1 - \sigma_{22}) \frac{\partial v_1}{\partial x_1} + \sigma_{12} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) + (\lambda_1 - \sigma_{11}) \frac{\partial^2 v_1}{\partial x_2^2} \right),
$$

$$
\nabla v_1 \cdot e_2 = \delta^{-1} \left( \sigma_{12} \left( \frac{\partial v_1}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) + (\lambda_1 - \sigma_{11}) \frac{\partial^2 v_1}{\partial x_1 \partial x_2} - (\lambda_1 - \sigma_{22}) \frac{\partial^2 v_1}{\partial x_2 \partial x_1} \right)
- \delta^{-3} \left[ \sigma_{12} (\lambda_1 - \sigma_{11}) \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_1} - \sigma_{12} \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_2} - (\lambda_1 - \sigma_{22})(\lambda_1 - \sigma_{11}) \frac{\partial \sigma_{12}}{\partial x_2} \right] v_1^1
- \sigma_{12} (\lambda_1 - \sigma_{11}) \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_1} + (\sigma_{22} - \sigma_{11})(\lambda_1 - \sigma_{22}) \frac{\partial \sigma_{12}}{\partial x_2} \right] v_1^2,
$$

$$
\nabla v_1 \cdot e_1 = \delta^{-1} \left( \sigma_{12} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) - (\lambda_1 - \sigma_{22}) \frac{\partial v_1}{\partial x_1} - (\lambda_1 - \sigma_{11}) \frac{\partial^2 v_1}{\partial x_2^2} \right)
- \delta^{-3} \left[ - \sigma_{12} \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_1} - \sigma_{12} (\lambda_1 - \sigma_{11}) \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_2} + (\sigma_{22} - \sigma_{11})(\lambda_1 - \sigma_{22}) \frac{\partial \sigma_{12}}{\partial x_2} \right] v_1^1
+ \sigma_{12} (\lambda_1 - \sigma_{22}) \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_1} + \sigma_{12} \frac{\partial \sigma_{22} - \sigma_{11}}{\partial x_2} - (\sigma_{22} - \sigma_{11})(\lambda_1 - \sigma_{22}) \frac{\partial \sigma_{12}}{\partial x_2} \right] v_1^2\right],
$$
and

\[(\nabla v_1 \cdot e_2) \cdot e_1 = \delta^{-1} \left( \sigma_{12} \left( \frac{\partial v_1^2}{\partial x_2} - \frac{\partial v_1^1}{\partial x_1} \right) + (\lambda_1 - \sigma_{22}) \frac{\partial v_1^1}{\partial x_2} - (\lambda_1 - \sigma_{11}) \frac{\partial v_2^1}{\partial x_1} \right),\]

where

\[\lambda_1 = \frac{1}{2} (\sigma_{11} + \sigma_{22} + \delta),\]

is the greatest eigenvalue with \(\delta = \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}\) and \(v_1 = (v_1^1, v_1^2)\). We have tested our approach for the compliance minimization of various structures: a power pylon, a cantilever, and several bridges. The results obtained are displayed in Figure 6. For each configuration, we show, on the one hand, the density of the optimal composite produced by the homogenization method, and, on the other, the projected shape \(\Omega_{\varepsilon}\), derived by our method. Each structure is clamped on a part of its boundary (represented by a hatched block) and submitted to dead surface loads on another part of it (applied forces are represented by arrows). The weight of the structures was not taken into account. Finally, let us mention that the optimization may be naturally pursued by a geometric optimization method. To this end, a level set method (see [5], [3], [4], [2], [14], [15], [12]) seems quite appropriate. Figure 7 displays the shapes \(\Omega_{\varepsilon}\)
obtained for different values of $\varepsilon$. In the case at hand, the compliance of the optimal composite is 16.28. We notice that the compliance decreases proportionally with the parameter $\varepsilon$.

5.3. Singularity of the field of eigenvectors. The proposed method does not allow for the projection of laminates whose directions of lamination (i.e., the eigenvectors of the stress tensor $\sigma$ of the optimal composite) show singularities. Generically, the singularities of the eigenvector field are made of a finite set of points for which $\sigma$ is proportional to the identity. In such an instance, the eigenvector field is not orientable: it performs u-turns around singularities. Two different kinds of singularities are likely to occur according to the direction of rotation of the eigenvector field around the singularity. A singularity is said to be positive if along a small circle around it the eigenvector field has the standard trigonometric orientation; otherwise, it is said to be negative. Such a singularity typically appears in the case of a bridge with two loads as displayed in Figure 6(d). Figure 8 shows the (nonorientable) eigenvector field of the constraint $\sigma$ of the optimal composite associated with the greatest eigenvalue $\lambda_1$ around the singularity (located between the two loads, slightly above the platform of the bridge). We have also plotted the level sets orthogonal to the eigenvector fields $e_1$ and $e_2$. We notice that the network thus obtained is not diffeomorphic to a square.
network. It has a defect; i.e., the cell containing the singularity has five right angles. The trick, introduced earlier, that consists in bringing in the vectors $v_1$ and $v_2$ allows us to circumvent the problem in the case of a single singularity. Actually, the set of vector fields $v_1$ of zero gradient and satisfying the constraints (5.1) is not empty, contrary to $F_\varepsilon$. Figures 9 and 10 display the eigenvector field $e_1$ associated with, respectively, a negative and a positive singularity, as well as the projection plotted for different values of the parameter $\varepsilon$ (with $\theta = 1/2$ and $m_1 = m_2 = 1/2$).
In the case of several singularities, the set of vector fields $v_1$ of zero gradient meeting the constraints (5.1) may be empty, and our algorithm is no longer adapted. Figures 11 and 12 display the eigenvector field $e_1$ associated with, respectively, negative and positive singularities, as well as the projection plotted for several values of the parameter $\varepsilon$ (with $\theta = 1/2$ and $m_1 = m_2 = 1/2$). The result of the projection is not satisfactory. In particular, it is different from the shape produced by symmetrizing shapes obtained for isolated singularities.

Our method cannot be applied as is if such a combination of singularities occurs. We may always consider partitioning the domain in order to isolate each singularity, apply our method to each part, then glue back the pieces. Yet, such a procedure is difficult to automate.

6. Conclusion. The post-treatment of the homogenization method presented here provides, in the framework of compliance minimization, quite interesting results compared to the classical penalization method. The main advantage is that it enables a sharp control of the size of the details of the final shape. Furthermore, the computational time it demands is negligible with respect to the homogenization procedure, whereas the material density penalization requires as much time as the latter. Several issues deserve to be investigated. First of all, this method should be coupled with a level set algorithm in order to sharpen the final shape. Besides, the suggested post-treatment of the homogenization method applies to the case where the directions of lamination exhibit at most one singularity. This is a strong limitation if we wanted to extend it to more general objective functions and/or complex geometries. In both
cases, the directions of lamination of the solutions computed by the homogenization method are likely to have more than one singularity. In the three-dimensional case, even more tricky situations may arise as lines (and not points) of singularities will have to be handled. Therefore, one needs to break free from this limitation. To conclude, this method potentially offers the prospect of an alternative to the classical topological gradient method by allowing us to nucleate a multitude of holes (or bars) in one iteration.

Appendix A. Proof of Lemma 4.1. We set out to exhibit an element of $\mathcal{F}_e$. First of all, we note that a function $\varphi = (\varphi_1, \varphi_2)$ belongs to $\mathcal{F}_e$ if and only if

\begin{align}
\nabla \varphi_1 \neq 0, \quad \nabla \varphi_1 \cdot e_2 = 0
\end{align}

and

\begin{align}
\nabla \varphi_2 \neq 0, \quad \nabla \varphi_2 \cdot e_1 = 0.
\end{align}

The conditions, to which $\varphi_1$ and $\varphi_2$ are submitted, are independent of one another. Therefore, it suffices to build a function $\varphi_1$ satisfying hypotheses (A.1), since the function $\varphi_2$ is produced by a similar procedure.

If $\varphi_1$ satisfies (A.1), the level sets are smooth curves tangent to $e_2$. For all elements $x$ in $D$, we denote by $X_2(x,t)$ the solution of the following equation:

\[
\begin{cases}
X_2(x,0) = x, \\
\frac{\partial X_2}{\partial t}(x,t) = e_2(X(x,t)).
\end{cases}
\]
Fig. 12. Two positive singularities of the eigenvector field.

Likewise, we define $X_1(x, t)$ (by changing $e_2$ into $e_1$). By the Cauchy–Lipschitz theorem, there exists a unique maximal solution to this equation, since $e_2$ is Lipschitzian. We call $T^-_x \in \mathbb{R} \cup \{-\infty\}$ and $T^+_x \in \mathbb{R} \cup \{+\infty\}$ the lower and upper bounds of the time interval in which the maximal solution is defined, and we call $S_x$ the set of points spanned by $X(x, t)$:

$$S_x = \{ y = X(x, t) \}.$$ 

Note that there exists $T_x > 0$ for which the mapping $G_x$ from $]-T_x, T_x[$ into $\Omega$ that maps $(t_1, t_2)$ onto $X_2(X_1(x, t_2), t_1)$ is defined. Moreover, for $T_x > 0$ small enough, $G_x$ is a diffeomorphism, since $D(G_x)(x) = (e_1(x), e_2(x))$. Now, $\mathcal{D}$ is a compact set included in the image of functions $G_x$, so there exists a finite subset $\mathcal{X}$ of elements of $\mathcal{D}$ such that $\mathcal{D} \subset \bigcup_{x \in \mathcal{X}} \text{Im}(G_x)$. We set $\omega = \bigcup_{x \in \mathcal{X}} \text{Im}(G_x)$. Should we replace $\Omega$ by
\(\omega\), we may assume that \(\Omega = \omega\). We shall prove, on the one hand, that \(S_x\) cannot be a closed curve (it necessarily has endpoints), and, on the other hand, that it is of finite length, that is, \(T_x^-\) and \(T_x^+\) are finite. Assume that this is not true; every component of the preimage of \(G_y\) (for an element \(y \in \chi\)) by the map \([T_x^-,T_x^+] \cap \Omega, t \mapsto X_2(x,t)\) has its diameter bounded from below (by \(2 \max_{s \in \chi} T_x\)). Thus, if the interval \([T_x^-,T_x^+]\) is not bounded, there exists an element \(y \in \chi\) such that \(X_2(x,\cdot)^{-1}(G_y)\) has an infinity of connected components. As we shall show, this is impossible. If this were the case, there would exist \(t_1\) and \(t_2\) in \([T_x^-,T_x^+]\) such that \(t_2 > t_1\),

\[
X_2(x,t_1) = G_y(T_y,h_1), \quad \text{and} \quad X_2(x,t_2) = G_y(-T_y,h_2).
\]

Without loss of generality, we may assume that \(t_1 = 0\).

In addition, should we alter the field \(e_1\) as it is illustrated in Figure 13, we may assume that \(h_1 = h_2\), and so \(S_x\) is diffeomorphic to a circle. However, all injections of the circle into \(\mathbb{R}^2\) are isotopic either to the canonical injection, or to the latter composed with a central symmetry. Therefrom, we infer the existence of a diffeomorphism \(F\) from the unit disc \(D^1\) onto the open set \(U\) included in \(D\), and satisfying \(F(S^1) = S_x\). Let \(f \in C^0(D^1;S^1)\) be defined by \(f(x) = DF^{-1}(e_1(F(x)))/|DF^{-1}(e_1(F(x)))|\). For all elements \(s \in S^1\), \(f(s)\) is nothing but \(s\) rotated by \(\pi/2\). Now, according to Brouwer’s theorem, such a field cannot be extended into a continuous field defined on the whole circle, which is exactly what \(f\) achieves. We have come to a contradiction, our initial hypothesis is accordingly false, and \(R_x\) is an open curve of finite length.

For all elements \(x\) in \(D\), we denote by \(H_x\) the mapping defined in a neighborhood of the origin of \(\mathbb{R}^2\) by \(H_x(t,h) = X_2(X_1(x,h),t)\). The restriction of \(H_x(t,h)\) to the axis \(h = 0\) is nothing but the injection \(t \mapsto X_2(x,t)\) of image \(S_x\). Furthermore, the map \(H_x\) is differentiable, and

\[
\frac{\partial H_x}{\partial t} = X_2(X_1(x,h),t) = e_2(X_2(X_1(x,h),t)) = e_2(X_1(x,h),t),
\]

\[
\frac{\partial H_x}{\partial h} = D_{X_1(x,h),t}X_2e_1(X_1(x,h)).
\]

Wherefrom, we deduce, in particular, that

\[
\frac{\partial}{\partial t} D_{h,t}H_x = D_{X_2(X_1(x,h),t)}e_2D_{h,t}H_x
\]

and that

\[
\text{det}(D_{(h,t)}H) = \text{Tr}(D_{X_2(X_1(x,h),t)}e_2)\text{det}(D_{(h,t)}H).
\]

However, \(D_{h=0,t=0}H_x = (e_1,e_2)\), so that \(\text{det}(D_{(h=0,t)}H) > 0\) for all \(t\). Let \(\delta_x > 0\) be small enough for the endpoints of \(X_2(x,|T_x^- + \delta_x, T_x^+ - \delta_x|)\) to belong to \(\omega \setminus \overline{D}\). According to the local inversion theorem, for \(h_x > 0\) small enough, the map \([T_x^- + \delta_x, T_x^+ - \delta_x]\) to \(\omega\), \((t,h) \mapsto H_x(t,h) = X_2(X_1(x,h),t)\) is a diffeomorphism onto its image, called \(V_x\).

Once this tubular neighborhood is built, it is somewhat easy to build a function \(\varphi_x\) having the aforementioned properties in a neighborhood of \(S_x \cap \overline{D}\). To do so, we first remark that \(D \setminus S_x\) consists of two distinct connected components (since \(\omega\) is simply connected). We call \(D_x^+\) the connected component of \(D \setminus S_x\) that contains \(H_x(t = 0,h_x)\), and \(D_x^-\) the one that contains \(H_x(t = 0, -h_x)\). We denote by \(\pi_2\)
the projection of $\mathbb{R}^2$ on the second coordinate. Let $T$ be an infinitely differentiable increasing mapping satisfying $T(x) = x$ in $]-1/2, 1/2[$, and $T(x)$ is constant if $|x| > 1$. We define the mapping $\varphi_x : \overline{D} \to \mathbb{R}$ by $\varphi_x(y) = T(\pi_2 \circ H_1^{-1}(y)/h_x)$ for all $y \in V_x \cap D$, $\varphi(x) = T(1)$ for all $y \in \overline{D} \setminus V_x$, and $\varphi(y) = T(-1)$ for all $y \in \overline{D} \setminus V_x$. The map $\varphi_x$ is $C^1$ and satisfies $\nabla \varphi_x \cdot e_2 = 0$, $\nabla \varphi_x \cdot e_1 \geq 0$. Finally, $\nabla \varphi_x \cdot e_1 > 0$ in a neighborhood $W_x$ of $D \cap S_x$ in $D$. Now, $\overline{D}$ is compact, which implies the existence of a finite family $\tau$ of elements $x \in D$ such that $\overline{D} = \bigcup_{x \in \tau} W_x$. The function $\varphi_1 = \sum_{x \in \tau} \varphi_x$ satisfies (A.1); thus $\mathcal{F}_x$ is not empty as claimed.

REFERENCES


