Portfolio Optimization with Stochastic Volatilities: A Semi Linear PDE Approach

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Outline

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Portfolio optimization: a fundamental concern when investors trade between a large number of risky assets

- Merton (1971): Maximizing expected utility of terminal wealth
  He derived a closed formula.

**Maximizing utility problems** lead to **HJB Equations**
An efficient algorithm: **Howard algorithm** computing two sequences the optimal strategy and the value function. **Not possible when risky assets number > 3**.
Zariphopoulou (2001): maximizing utility in a market with one risky asset and a stochastic volatility model. Using a power transformation: the value function is a solution of a linear P.D.E.

Pham (2002): multidimensional model. A power transformation leads to a semi linear parabolic equation.

Mnif (2007): multidimensional model, stochastic volatility model, exponential utility, constraints on amounts, jump diffusion process. When there is no correlation between the risky asset and the factor volatility, the optimal investment strategy is determined by solving a static optimization problem.
Problem formulation

\((\Omega, \mathcal{F}, \mathcal{P})\) filtered probability space

financial market: bond \(S^0\) and \(n\) risky assets \(S\)

\[
S^0 \equiv 1
\]

\[
dS_t = \text{diag}(S_t^-) \left( b(\Lambda_t) dt + \sigma(\Lambda_t) dW_t + \bar{\sigma}(\Lambda_t) d\bar{W}_t \right) + \int_{\mathbb{R}^n \setminus \{0\}} \gamma(\Lambda_t, u) \tilde{\mu}(dt, du)
\]

\(W\) d-dimensional standard Brownian motion
\(\bar{W}\) m-dimensional standard Brownian motion independent of \(W\)

\(\mu\) a Poisson random measure and
\(\tilde{\mu}(dt, du) = (\tilde{\mu}(dt, du))_{1 \leq i \leq n} = (\mu_i(dt, du) - q_i(du) dt)_{1 \leq i \leq n}\) is
the compensated Poisson random measure
Problem formulation

$q_i(du)$ is the Lévy measure

$$\int_{\mathbb{R}^n \setminus \{0\}} q_i(du) < \infty.$$ 

Λ is a $d$ dimensional stochastic factor

$$d\Lambda_t = \eta(\Lambda_t)dt + dW_t,$$

Assumption (H1) $\eta$ is Lipschitz.

We denote

$$\Sigma(\lambda) = (\sigma(\lambda), \bar{\sigma}(\lambda)),$$

$$\alpha(\lambda) = \inf_{\pi \in \mathbb{R}^n, \pi \neq 0} \frac{|\Sigma(\lambda)^* \pi|^2}{|\pi|^2}, \lambda \in \mathbb{R}^d,$$
Assumption *(H2)* There exists some positive constants $C$ such that for a.e. $(\lambda, u) \in \mathbb{R}^d \times \mathbb{R}^n \setminus \{0\}$:

\[
\inf_{\lambda \in \mathbb{R}^d} \alpha(\lambda) > 0
\]

\[
|b(\lambda)| + \|\sigma(\lambda)\| \leq C.
\]

\[
\|\bar{\sigma}(\lambda)\| \leq C(1 + |\lambda|)
\]

\[
\gamma_{ij}(\lambda, u) > -1 \text{ for all } 1 \leq i, j \leq n
\]

\[
\|\gamma(\lambda, u)\| \leq C
\]
Problem formulation

A strategy $\pi$ is said admissible if it is $F$-predictable and

$$\pi_t \in \Delta := \{(\pi_1, \ldots, \pi_n) \in [0, 1]^n, \sum_{i=1}^{n} \pi_i \leq 1\} \text{ a.s. for all } 0 \leq t \leq T.$$ 

$\Delta$ the set of admissible controls.

$$X_t = x + \int_0^t X_{s^-} \left( \pi_s^* b(\Lambda_s) ds + \pi_s^* \sigma(\Lambda_s) dW_s + \pi_s^* \bar{\sigma}(\Lambda_s) d\bar{W}_s \right)$$

$$+ \int_{R^n \setminus \{0\}} \pi^*_s \gamma(\Lambda_s, u) \tilde{\mu}(ds, du)$$

The utility function: $U(x) = \frac{x^\delta}{\delta}, x \in R_+$, The objective of the agent

$$v(t, x, \lambda) = \sup_{\pi \in \mathcal{A}} E[U(X_T)|X_t = x, \Lambda_t = \lambda], \quad (t, x, \lambda) \in [0, T] \times R_+ \times R^d.$$
Semi-linear PDE

\[
\begin{align*}
\frac{\partial v}{\partial t} + \eta(\lambda)^* D_\lambda v &+ \frac{1}{2} \Delta_\lambda v + \max_{\pi \in \Delta} \left\{ x \pi^* b(\lambda) \frac{\partial v}{\partial x} ight\} \\
&+ x^2 \frac{1}{2} |\Sigma(\lambda)^* \pi|^2 \frac{\partial^2 v}{\partial x^2} + x \pi^* \sigma(\lambda) D_{x\lambda}^2 v \\
&+ \sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus \{0\}} (v(t, x(1 + \pi^* \gamma_i(\lambda, u)), \lambda) \\
&- v(t, x, \lambda) - x \pi^* \gamma_i(\lambda, u) \frac{\partial v}{\partial x} q_i(du) \right\} = 0,
\end{align*}
\]

for a.e. \((t, x, \lambda) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}^d\)

\[v(T, x, \lambda) = \frac{x^\delta}{\delta},\]
Due to the power utility function, we look for a candidate of HJB equation in the form:

$$v(t, x, \lambda) = \frac{x^\delta}{\delta} \exp\left(-\phi(t, \lambda)\right).$$

By differentiation, HJB implies

$$- \frac{\partial \phi}{\partial t} - \frac{1}{2} \triangle \phi + H(\lambda, D\phi) = 0, \quad (t, \lambda) \in [0, T) \times \mathbb{R}^d,$$

with terminal condition

$$\phi(T, \lambda) = 0, \quad y \in \mathbb{R}^d,$$
Semi-linear PDE

\[ H(\lambda, p) = \frac{1}{2} |p|^2 - p^* \eta(\lambda) \]

\[ + \max_{\pi \in \Delta} \left\{ \delta(\pi^* b(\lambda) - \pi^* \sigma(\lambda)p) - \frac{1}{2}\delta(1 - \delta)|\Sigma(\lambda)^* \pi|^2 \right\} \]

\[ + \sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus \{0\}} \left( (1 + \pi^* \gamma_i(\lambda, u))^\delta - 1 - \delta \pi^* \gamma_i(\lambda, u) \right) q_i(du) \]
Semi-linear PDE

**Theorem**

Let assumptions (H1) and (H2) hold. Suppose that there exists a solution \( \phi \in C^{1,2}([0, T) \times \mathbb{R}^d) \cap C^0([0, T] \times \mathbb{R}^d) \) to the semi-linear equation (1) with the terminal condition (2). We also assume that \( |D\phi(t, \lambda)| \) has a linear growth condition in \( \lambda \) uniformly in \( t \). Then the value function of (1) is given by:

\[
\nu(t, x, \lambda) = \frac{x^\delta}{\delta} \exp \left( -\phi(t, \lambda) \right), \quad (t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d.
\]
Theorem

The optimal portfolio is given by the Markov control
\[ \hat{\pi}_t = \hat{\pi}(t, \Lambda_t), 0 \leq t \leq T \] where
\[ \hat{\pi}(t, \lambda) \in \arg \min_{\pi \in \Delta} \{ -\delta(\pi^* b(\lambda) - \pi^* \sigma(\lambda) D\phi(t, \lambda)) \]
\[ + \frac{1}{2} \delta(1 - \delta) |\Sigma(\lambda)^* \pi|^2 \]
\[ - \sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus \{0\}} \left( (1 + \pi^* \gamma_i(\lambda, u))^\delta - 1 - \delta \pi^* \gamma_i(\lambda, u) \right) q_i(du) \}. \]
(H3)

(i) \( \inf_{y \in \mathbb{R}^d} \alpha(y) > 0 \) (uniform elliptic volatility).

(ii) \( \eta \) is \( C^1 \) and Lipschitz, \( b \) is \( C^1 \) and bounded and \( Db \) is bounded.

(iii) \( \Sigma \Sigma^* \) is \( C^1 \) and bounded and \( \| D(\Sigma \Sigma^*) \| \) is bounded.

(iv) \(-1 < \gamma_{ij}(\lambda, u) = \gamma_{ij}(u) \leq M \) for all \( (\lambda, u) \in \mathbb{R}^d \times \mathbb{R}^n \setminus \{0\} \) where \( M > 0 \).
Regularity of the value function

Theorem

Under Assumption (H3), there exists a solution \( \phi \in C^{1,2}([0, T] \times \mathbb{R}^d) \cap C^0([0, T] \times \mathbb{R}^d) \) to the semi-linear equation (1) with terminal condition (2) and linear growth condition in \( \lambda \) uniformly in \( t \) on the derivative \( D\phi \).
\[
\frac{dS_t^i}{S_t^i} = b_i dt + \nu_i(\Lambda_{it}) \sum_{j=1}^{i} \rho_{ij} (\rho_j dW_t^j + \sqrt{1 - \rho_j^2} d\bar{W}_t^j) \\
+ \sum_{j=1}^{n} \int_{R^n \setminus \{0\}} \gamma_{ij}(u) \tilde{\mu}_j (dt, du),
\]
\[
d\Lambda_{it} = (a_i - \theta_i \Lambda_{it}) dt + dW_t^i, \text{ for all } i \in \{1, ..., n\}.
\]

\((\nu_i)_{1 \leq i \leq n}\) bounded \(C^1\) functions with bounded derivatives and lower bounded by a positive constant \(\epsilon_i > 0\) for all \(i \in \{1, ..., n\}\). \(a_i\) and \(\theta_i\) are constants, \(\rho_{ij}\) is the constant correlation between the two Brownian motions of \(S^i\) and \(S^j\), \(\rho_j\) is the constant correlation between \(S^i\) and its volatility and \(\gamma_{ij}(u) > -1\) and is bounded.
The optimal portfolio is given by:

\[
\hat{\pi}(t, \Lambda_t) \in \arg \min_{\pi \in \Delta} \left[ -\delta \sum_{i=1}^{n} \pi_i b_i 
+ \frac{\delta(1 - \delta)}{2} \sum_{i,j=1}^{n} \pi_i \pi_j \left( \nu_i(\Lambda_{it}) \nu_j(\Lambda_{jt}) \sum_{k=1}^{\inf(i,j)} \rho_{ik} \rho_{jk} \right) 
+ \delta \sum_{i=1}^{n} \sum_{j=1}^{i} \pi_i \rho_{ij} \rho_j(\nu_i(\Lambda_{it})) \frac{\partial \phi}{\partial \lambda_j}(t, \Lambda_t) 
- \sum_{i=1}^{n} \int_{R^n \setminus \{0\}} \left( (1 + \pi^* \gamma_i(u))^\delta - 1 - \delta \pi^* \gamma_i(u) \right) q_i(du) \right].
\]
We define two processes \((Y, Z)\) by

\[ Y_t := \phi(t, \Lambda_t), \quad Z_t := D\phi(t, \Lambda_t) \quad \text{for all } t \in [0, T]. \]

**Proposition**

Suppose that Assumptions (H1), (H2) and (H3) hold. We define the processes \((\bar{Y}, \bar{Z})\) as follows

\[ \bar{Y}_t := \exp(Y_t), \quad \bar{Z}_t := \bar{Y}_t Z_t, \quad \text{for all } t \in [0, T], \]

then the couple \((\bar{Y}, \bar{Z})\) is a solution of the following BSDE

\[ -d\bar{Y}_t = \bar{g}(\Lambda_t, \bar{Y}_t, \bar{Z}_t)dt - \bar{Z}_t dW_t, \]

with terminal condition
Proposition

\[
\tilde{Y}_T = 1
\]

where

\[
\bar{g}(\lambda, \bar{y}, \bar{z}) = -\min_{\pi \in \Delta} \left\{ -\delta(\pi^* b(\lambda)\bar{y} - \pi^* \sigma(\lambda)\bar{z}) \right. \\
+ \left. \bar{y} \frac{\delta(1-\delta)}{2} |\Sigma(\lambda)^* \pi|^2 \right. \\
- \left. \bar{y} \sum_{i=1}^{n} \int_{R^n \setminus \{0\}} \left( (1 + \pi^* \gamma_i(u))^\delta - 1 - \delta \pi^* \gamma_i(u) \right) q_i(du) \right\}.
\]
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Connection with BSDEs and numerical study

Discretization and simulation of the decoupled FBSDE

Step 1. Problem discretization. \((t_k := kh = \frac{kT}{N})_{0 \leq k \leq N}\).

\[
\Lambda_{l,t_0}^N = \Lambda_{l,0} \\
\Lambda_{l,t_{k+1}}^N = \Lambda_{l,t_k}^N + \eta(\Lambda_{l,t_k}^N)h + \Delta W_{l,k},
\]

\[
\Delta W_{l,k} = W_{l,t_{k+1}} - W_{l,t_k}, 1 \leq l \leq d.
\]

\[
\bar{Y}_{t_N}^N = 1 \\
\bar{Z}_{l,t_k}^N = \frac{1}{h} E_k(\bar{Y}_{t_{k+1}}^N \Delta W_{l,k}),
\]

\[
\bar{Y}_{t_k}^N = E_k(\bar{Y}_{t_{k+1}}^N) + hE_k(\bar{g}(\Lambda_{t_k}^N, \bar{Y}_{t_{k+1}}^N, \bar{Z}_{t_k}^N)),
\]

where \(E_k(\cdot) = E(\cdot | \mathcal{F}_{t_k})\).
Remark

Since $Y^N_{t_N}$ is a constant and $\Lambda^N$ is a Markov Chain, it is easy to see that $\bar{Y}^N_{t_k} = \bar{y}^N_k(\Lambda^N_{t_k})$ and $\bar{Z}^N_{t_k} = \bar{z}^N_k(\Lambda^N_{t_k})$ where $\bar{y}^N_k$ and $\bar{z}^N_k$ are unknown regression functions defined in a backward manner.

Zhang (2004), Bouchard and Touzi (2004), Gobet et al. (2005) proved For $h$ small enough

Theorem

$$\max_{0 \leq k \leq N} E|\bar{Y}_{t_k} - \bar{Y}^N_{t_k}|^2 + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} E|\bar{Z}_t - \bar{Z}^N_{t_k}|^2 dt \leq C \left(1 + |\Lambda_0|^2\right) h.$$
Step 2. Localization. Localization of the Brownian increments and the function $\bar{g}$

- $[\triangle W_{l,k}]_w = -R_0 \sqrt{(h)} \lor \triangle W_{l,k} \land R_0 \sqrt{(h)}$
- $\bar{g}^R(\lambda, \bar{y}, \bar{z}) = \bar{g}^R(-R_1 \lor \lambda_1 \land R_1, ..., -R_d \lor \lambda_d \land R_d, \bar{y}, \bar{z})$

The localization modifies the numerical scheme. We define $(\bar{Y}^{N,R}, \bar{Z}^{N,R})$ by

$$
\begin{align*}
\bar{Y}_{t_N}^{N,R} &= 1 \\
\bar{Z}_{l,t_k}^{N,R} &= \frac{1}{h} E_k(\bar{Y}_{t_{k+1}}^{N,R}[\triangle W_{l,k}]_w), \\
\bar{Y}_{t_k}^{N,R} &= E_k(\bar{Y}_{t_{k+1}}^{N,R}) + hE_k(\bar{g}^R(\Lambda_{t_k}^{N}, \bar{Y}_{t_{k+1}}^{N,R}, \bar{Z}_{t_k}^{N,R})).
\end{align*}
$$
Remark

The main interest of the localization is to provide bounded regression functions $\bar{y}_{k}^{N,R}$ and $\bar{z}_{k}^{N,R}$. One has $||\bar{y}_{k}^{N,R}||_{\infty} \leq C_{y}(R)$ and $||\bar{z}_{k}^{N,R}||_{\infty} \leq C_{z}(R)$ (See Proposition 1 (Lemor Gobet and Warin (2006)))

Step 3. Function bases.

- We approximate $\bar{Y}_{t_{k}}^{N,R}$ and $\bar{Z}_{l,t_{k}}^{N,R}$ for all $1 \leq l \leq d$ by a projection on a finite-dimensional function bases $p_{0,k}(\Lambda_{t_{k}}^{N,R})$ and $p_{l,k}(\Lambda_{t_{k}}^{N,R})$ for all $1 \leq l \leq d$.

- We set $\alpha_{l,k}$ for all $0 \leq l \leq d$ the projection coefficients of $\bar{Y}_{t_{k}}^{N,R}$ and $\bar{Z}_{l,t_{k}}^{N,R}$, $1 \leq l \leq d$ on the function bases $(p_{l,k})_{0 \leq l \leq d}$. We denote by $K_{l,k}$ the size of $p_{l,k}$.

- We simulate $M$ independent Monte Carlo simulations of $(\Lambda_{t_k}^N)_{0 \leq k \leq N}$ and $(\triangle W_k)_{0 \leq k \leq N-1}$. We denote $(\Lambda_{t_k}^N,m)_{1 \leq m \leq M, 0 \leq k \leq N}$ and $(\triangle W_k^m)_{1 \leq m \leq M, 0 \leq k \leq N-1}$ these simulations.

- We write $p_{l,k}(\Lambda_{t_k}^N,m) = p_{l,k}^m$ and $B_{l,k}^M$ the matrix of size $M \times K_{l,k}$ which rows are $(p_{l,k}^m)^*$. We denote by $K_{l,k}^M$ the rank of $B_{l,k}^M$.

Step 5. Truncations. We force our approximation of $\bar{Y}_{N,R}$ and $\bar{Z}_{N,R}$ to be bounded by $C_y(R)$ and $C_z(R)$. For a function $\psi$, we define the truncations by $[\psi]_y(x) = -C_y(R) \lor \psi(x) \land C_y(R)$ and $[\psi]_z(x) = -C_z(R) \lor \psi(x) \land C_z(R)$. 

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Portfolio Optimization
The algorithm

→ $\hat{Y}^{N,R,M}_{t_N} = 1$

→ iteration: for $k = N - 1, \ldots, 0$, assume that $\hat{y}^{N,R,M}(\Lambda_{t_{k+1}}^N)$ is known, then we solve the $d + 1$ least-squares problems:

$$
\alpha_{l,k}^M = \arg \inf_{\alpha} \frac{1}{M} \sum_{m=1}^{M} \left| \hat{y}^{N,R,M}(\Lambda_{t_{k+1}}^N) \frac{[\triangle W^m_{l,k}] w}{h} - \alpha \cdot p_{l,k}^m \right|^2
$$

$$
\alpha_{0,k}^M = \arg \inf_{\alpha} \frac{1}{M} \sum_{m=1}^{M} \left| \hat{y}^{N,R,M}(\Lambda_{t_{k+1}}^N) + h\bar{g}^R(\Lambda_{t_{k+1}}^N, \hat{y}^{N,R,M}(\Lambda_{t_{k+1}}^N), [\alpha_{l,k}^M \cdot p_{l,k}^m] z) - \alpha \cdot p_{0,k}^m \right|^2
$$

→ $\hat{y}^{N,R,M}_k(\cdot) = [\alpha_{0,k}^M \cdot p_{0,k}]y(\cdot)$ and $\bar{z}^{N,R,M}_k(\cdot) = [\alpha_{l,k}^M \cdot p_{l,k}] z(\cdot)$. 
The algorithm

**Assumption (H4)** \((p_{l,k})_{0 \leq l \leq d, 0 \leq k \leq N}\) is a complete orthonormal system with respect to the empirical scalar product \(<\cdot,\cdot>_{k,M}\)

\[
<\psi_1, \psi_2>_{k,M} = \frac{1}{M} \sum_{m=1}^{M} \psi_1(\Lambda_{t_k}^N, m) \psi_2(\Lambda_{t_k}^N, m)
\]

\[
\to \frac{(B_{l,k}^M)^* (B_{l,k}^M)}{M} = Id
\]

\[
\alpha_{l,k}^M = \frac{1}{M} \sum_{m=1}^{M} p_{l,k}^m \bar{y}_{k+1}^{N,R,M}(\Lambda_{t_k+1}^N, m) [\triangle_\frac{W_{l,k}^m}{h}]w
\]

\[
\alpha_{0,k}^M = \frac{1}{M} \sum_{m=1}^{M} p_{0,k}^m \left( \bar{y}_{k+1}^{N,R,M}(\Lambda_{t_k+1}^N, m) + h\bar{g}^R(\Lambda_{t_k}^N, m, \bar{y}_{k+1}^{N,R,M}(\Lambda_{t_k+1}^N, m), [\alpha_{l,k}^M \cdot p_{l,k}^m]z) \right)
\]
The algorithm

- For the convergence, we refer to Lemor, Gobet Warin (2006)
- The error induced by the localization has a contribution of order $h$
- We choose the hypercube basis. We choose a domain $D$ and we partition it into small hypercubes of edge $\kappa$
- To get a global (squared) error of order $h^\beta$, $\beta \in (0, 1]$, we have to choose $\kappa \approx h^{\frac{\beta+1}{2}}$ (i.e. $K \approx Ch^{-\frac{d(\beta+1)}{2}}$) and $M \approx Ch^{-d(\beta+1)-(\beta+2)\log(h^{-\frac{d(\beta+1)}{4}}-\frac{\beta+1}{2})}$
Numerical Approach

We simulate the $d$-dimensional stochastic factor $\Lambda = (\Lambda_t)_{t \in [0, T]}$ by using Monte-Carlo method. We denote by $\Lambda_{N,m}^{t_k} \in \mathbb{R}^d$ the $m$-th approximation of $\Lambda_t$. We determine $d_{i,k}^{\text{max}} = \max_{1 \leq m \leq M} \Lambda_{N,m}^{i,t_k}$ and $d_{i,k}^{\text{min}} = \min_{1 \leq m \leq M} \Lambda_{N,m}^{i,t_k}$. We consider

$$D_k = \left\{ \lambda \in \mathbb{R}^d, \lambda_i \in [d_{i,k}^{\text{min}}, d_{i,k}^{\text{max}}], \text{ s.t. for all } 1 \leq i \leq d \right\}.$$ 

We partition into small hypercubes of edge $\kappa$ the domain $D_k$. 

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Two risky assets model results

\[
\begin{align*}
\frac{dS^1_t}{S^1_t} &= b_1 \, dt + \nu_1(\Lambda^1_t)(\rho_1 \, dW^1_t + \sqrt{1 - \rho_1^2} \, d\bar{W}^1_t), \\
\frac{dS^2_t}{S^2_t} &= b_2 \, dt + \nu_2(\Lambda^2_t) \left( \rho(\rho_1 \, dW^1_t + \sqrt{1 - \rho_1^2} \, d\bar{W}^1_t) \\
&\quad + \sqrt{1 - \rho^2}(\rho_2 \, dW^2_t + \sqrt{1 - \rho_2^2} \, d\bar{W}^2_t) \right), \\
\end{align*}
\]

\[
\begin{align*}
d\Lambda^1_t &= (a_1 - \theta_1 \Lambda^1_t) \, dt + dW^1_t, \quad \Lambda^{10} = 0, \\
d\Lambda^2_t &= (a_2 - \theta_2 \Lambda^2_t) \, dt + dW^2_t, \quad \Lambda^{20} = 0,
\end{align*}
\]

where \( \nu_i(\lambda) = \epsilon_i + \frac{1}{\sqrt{\lambda^2 + \beta_i^2}}, \quad i \in \{1, 2\} \). Our model parameters are given in Table 1.
Two risky assets model results

We determine $\bar{Y}_0$, then $\nu_1(0, x, \Lambda_0) := \frac{x^\delta}{\delta} \exp(-\bar{Y}_0)$. We compute $\nu_2(0, x, \Lambda_0) := \frac{1}{M} \sum_{m=1}^{M} U(X_T^m)$, where

$$X_T^m = x + \sum_{k=0}^{N-1} X_{t_k}^m \hat{\pi}_t^m \frac{S_{t_{k+1}}^m - S_{t_k}^m}{S_{t_k}^m}.$$
Two risky assets model results

Table: Results for the value function

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>6000</th>
<th>8000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_1(0, 10, \Lambda_0)$</td>
<td>6.391</td>
<td>6.413</td>
<td>6.415</td>
<td></td>
</tr>
<tr>
<td>$\nu_2(0, 10, \Lambda_0)$</td>
<td>6.391</td>
<td>6.412</td>
<td>6.416</td>
<td></td>
</tr>
</tbody>
</table>

The optimal investment strategy at $t = 0$ is given by $\hat{\pi}_0^* = (0, 0.56, 0.44)$. 
We consider an exponential utility function.

We denote by \( \pi = ((\pi^1_t, ..., \pi^n_t)_{t \in [0, T]})^* \) the \( \mathcal{F}_t \)-predictable process of the amount invested in the \( n \) risky assets \( S \).

We denote by \( K \) a closed convex set in \( \mathbb{R}^n \) containing the origin for modeling reasonable constraints on portfolio.

We assume that there is no correlation between \( S \) and the stochastic factor \( \Lambda \).
Power utility maximization problem

Theorem

\[ \hat{\pi}(\Lambda_t) \in \arg\min_{\pi \in K} \left[ -\delta \sum_{i=1}^{n} \pi_i b_i \right. \]

\[ + \frac{\delta^2}{2} \sum_{i,j=1}^{n} \pi_i \pi_j \left( \sqrt{\nu_i(\Lambda_{it})} \sqrt{\nu_j(\Lambda_{jt})} \inf_{i,j} \sum_{k=1}^{n} \rho_{ik} \rho_{jk} \right) \]

\[ + \sum_{i=1}^{n} \int_{R^n \setminus \{0\}} \left( \exp \left( -\delta \pi^* \gamma_i(z) \right) - 1 + \delta \pi^* \gamma_i(z) \right) q_i(dz) \left. \right] \]

a.s., \(0 \leq t \leq T\).
\( \nu(dt, dz) \) is a Poisson process with constant intensity \( q \).
All the claims have the same size denoted by \( \gamma \).
\( K = \mathbb{R}^n_+ \).

\[
\min_{\pi \in K} f(\pi),
\]

where
\[
f(\pi) := -\delta \pi^* b + \frac{\delta^2}{2} |\Sigma(\lambda)^* \pi|^2 + \sum_{i=1}^{n} q_i (\exp(-\delta \pi^* \gamma_i) - 1 + \delta \pi^* \gamma_i).
\]
Power utility maximization problem

The algorithm

**Initialization:** Choose $\pi^0 \in \mathbb{R}^n_+$. Set $k = 0, k_{cycle} := 1$, and

$$l_0 := \{1 \leq i \leq n; \pi^0_i = 0\}; \quad J_0 := \{1, \ldots, n\} \setminus l_0.$$  

**Step 1:** Compute the reduce Newton direction $d^k$, defined by

$$d^k := - \left( D_{\pi^{J_k} \pi^{J_k}}^2 f(\pi^k) \right)^{-1} D_{\pi^{J_k}} f(\pi^k),$$

and the maximum non-negative feasible step $\rho_{Max}^k$ along $d^k$, not greater than 1:

$$\rho_{Max}^k := \min \left( 1, \min \left\{ -\frac{\pi^k_i}{d^k_i}; \; d^k_i < 0, i \in J_k \right\} \right).$$
Problem formulation in the case of power utility
Characterization of the optimal investment strategy
Problem formulation in the case of exponential utility

Numerical study

Step 2: Perform an Armijo type linesearch, i.e., choose the greatest step \( \alpha_k \in (0, \rho_k^{Max}) \), of the form \( \beta j_k \rho_k^{Max} \) for some \( j_k \in N \), for which it holds that

\[
    f(\pi^k + \alpha^k d^k) \leq f(\pi^k) + w\alpha_k D_{\pi} f(\pi^k) d^k.
\]

Step 3: Update \( J_k \) and \( I_k \): Set \( I_{k+1} := I_k \) and \( J_{k+1} := J_k \), and for each \( i \in J_k \) such that \( \pi_i + \alpha_k d_i^k = 0 \), do \( J_{k+1} := J_k \setminus \{i\} \) and \( I_{k+1} := I_k \cup \{i\} \).

Step 4: Update of current point: set \( \pi^{k+1} := \pi^k + \alpha_k d^k \), and \( k := k + 1 \).

Step 5: Test of end of cycle: if

\[
    \left\| D_{\pi} f(\pi^k) \right\| \leq \epsilon_1 / 2^{k_{cycle}},
\]

go to step 6. Otherwise, go to step 1.
Step 6: Stopping test: let $i_0 \in l_k$ be such that

$$D_{\pi_{i_0}} f(\pi^k) = \min \left\{ \left( D_{\pi_i} f(\pi^k) \right) ; \ i \in l_k \right\} .$$

If $D_{\pi_{i_0}} f(\pi^k) \geq -\varepsilon_2$, stop.

Step 7: Start a new cycle. Set $k_{cycle} := k_{cycle} + 1$ and compute

$$J_k := J_k \cup \{i_0\}; \quad l_k := l_k \setminus \{i_0\}; \quad \hat{d} := -\left( D_{\pi_{J_k}}^2,\pi_{J_k} \right) f(\pi)^{-1} D_{\pi_{J_k}} f(\pi).$$

If $\hat{d}_{i_0} < 0$, set $l_k := l_k \cup \{i_0\}$ and $J_k := J_k \setminus \{i_0\}$. Go to step 1.
Numerical results

The matrix of correlation is given by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & 0.86 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.25 & 0.25 & 0.93 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.86 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.86 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.86 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.86 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.86 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.86 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.86 & 0.5 \\
\end{pmatrix}
\]
The matrix parameterizing the jumps is given by

\[
\begin{pmatrix}
0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 \\
0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4
\end{pmatrix}
\]
The intensity of jumps is given by
\[ q = (0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1) \ast. \]
\[ b = (0.1 \ 0.1 \ 0.02 \ 0.02 \ 0.02 \ 0.1 \ 0.02 \ 0.03 \ 0.15 \ 0.05) \ast. \]
The values of the parameters of the volatility are given by
\[ \beta = (0.1 \ 0.6 \ 0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.5 \ 0.5 \ 0.1) \ast, \]
\[ \epsilon_j = 0.01 \text{ for all } i = 1 \ldots n \text{ and} \]
For the factors, the values are given by
\[ \Lambda = (20 \ 4 \ 50 \ 40 \ 30 \ 35 \ 45 \ 6 \ 5 \ 40) \ast. \]
The optimal strategy to invest in risky assets is given by

<table>
<thead>
<tr>
<th>\pi_1</th>
<th>\pi_2</th>
<th>\pi_3</th>
<th>\pi_4</th>
<th>\pi_5</th>
<th>\pi_6</th>
<th>\pi_7</th>
<th>\pi_8</th>
<th>\pi_9</th>
<th>\pi_{10}</th>
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<tr>
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<td>0.13</td>
<td>0.24</td>
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<td>0</td>
<td>0.23</td>
<td>0.03</td>
<td>5.10</td>
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