Introduction to portfolio insurance

Portfolio insurance

- Maintain the portfolio value above a certain predetermined level (floor) while allowing some upside potential.
- Performance may be compared to a stock market index, or may be guaranteed explicitly in terms of this index.
- Usually implementented via *strategic allocation* between the benchmark index, risk-free account and (possibly) option on the benchmark index.

Portfolio insurance example (equity)

Example: Hawaii 3 fund marketed by BNP Paribas:

- At maturity, the value of the fund will be greater or equal to the largest of:
  - 105% of the initial value. $\Leftarrow$ less than the risk-free return over the holding period.
  - 85% of the highest value attained by the Fund between 23/01/2007 and 3/07/2013 $\Leftarrow$ floor can be adjusted throughout the life of the portfolio
- The portfolio protection is valid only at maturity.

Portfolio insurance example (cont'd)

- Objective: benefit from the performance of a basket (DJ Euro STOXX 50, S&P 500 and Nikkei 225) while ensuring minimum annual performance of 0.7%.
- Danger of monetarization: to satisfy the insurance constraint, the exposure to risky asset may become and remain zero.
- Even if the Fund performance depends partially on the Basket, it can be different due to capital insurance.
- Strategy: The Fund will be actively managed using portfolio insurance techniques.

Portfolio insurance techniques

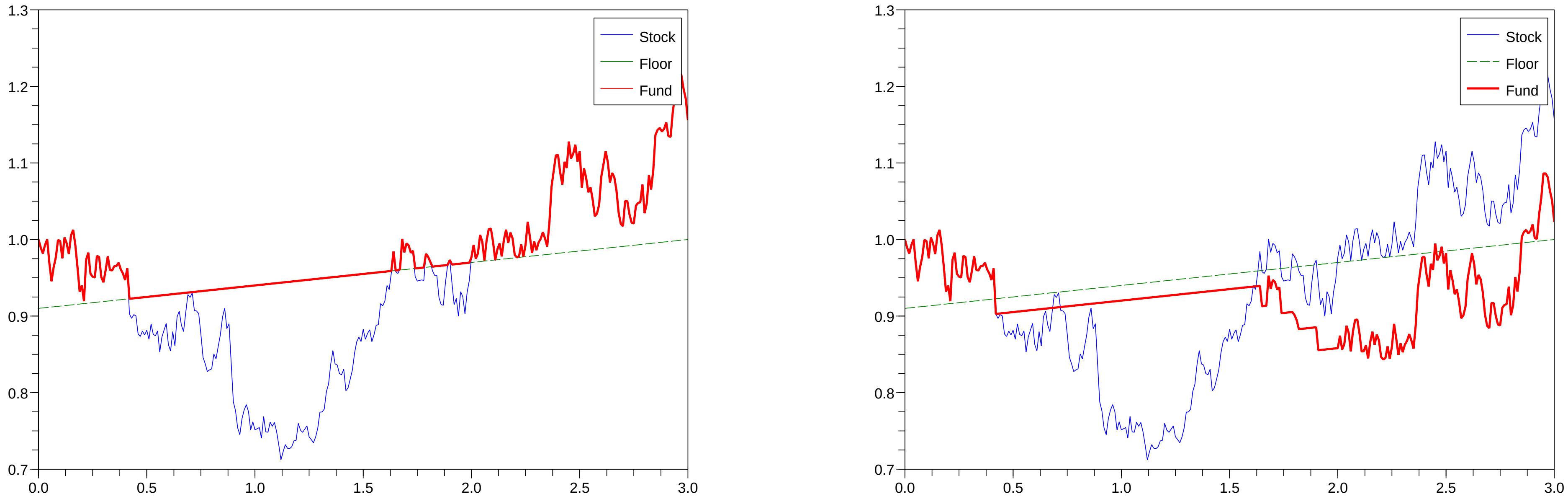
- Stop-loss (for someone who doesn't know stochastic calculus).
- Option-based portfolio insurance (OBPI).
- OBPI with option replication.
- Constant proportion portfolio insurance (CPPI).

Stop-loss strategy

Stop-loss strategy

- The simplest and the most intuitive strategy but its cost is difficult to quantify in practice.
- The entire portfolio is initially invested into the risky asset.
- As soon as the risky asset S_t drops below the floor F_t, the entire position is rebalanced into the risk-free asset.
- If the market rebounds above the floor, the fund is reinvested into risky assets.

Stop-loss strategy



Stop-loss strategy in ’theory’ (left) and in practice.

The loss is not stopped

The cost of stop-loss can be quantified via the Itô-Tanaka formula:

$$\max(S_t, F) = \int_0^t 1_{S_s \geq F} dS_s + \frac{1}{2} L_t,$$

where L is the local time of S at F (increasing process).

- The price (risk-neutral expectation) of the loss equals to the price of an at the money call option on the index.

Option-based portfolio insurance

Basic strategy with European guarantee

- Let K be the floor (with $KB(0,T) < 1$).
- Invest a fraction λ of the fund into the index S.
- Use the remainder to buy a Put on λS.
- The total cost is

$$f(\lambda) = \lambda + P_{\lambda S}(T, K)$$

increasing function with $f(0) = KB(0,T) < 1$ and $f(1) = 1 + P_S(T, K) > 1$.

$\Rightarrow$ There exists a unique $\lambda^* \in (0,1)$, realizing the put-based strategy.

Optimality of the put-based strategy

Let $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and let S_T be the optimal unconstrained portfolio:

$$E[u(S_T)] = \max E[u(X_T)] \quad \text{subject to} \quad X_0 = 1.$$

Then the put-based strategy is the optimal strategy subject to the floor constraint (El Karoui, Jeanblanc, Lacoste '05).

Equivalent strategy using calls

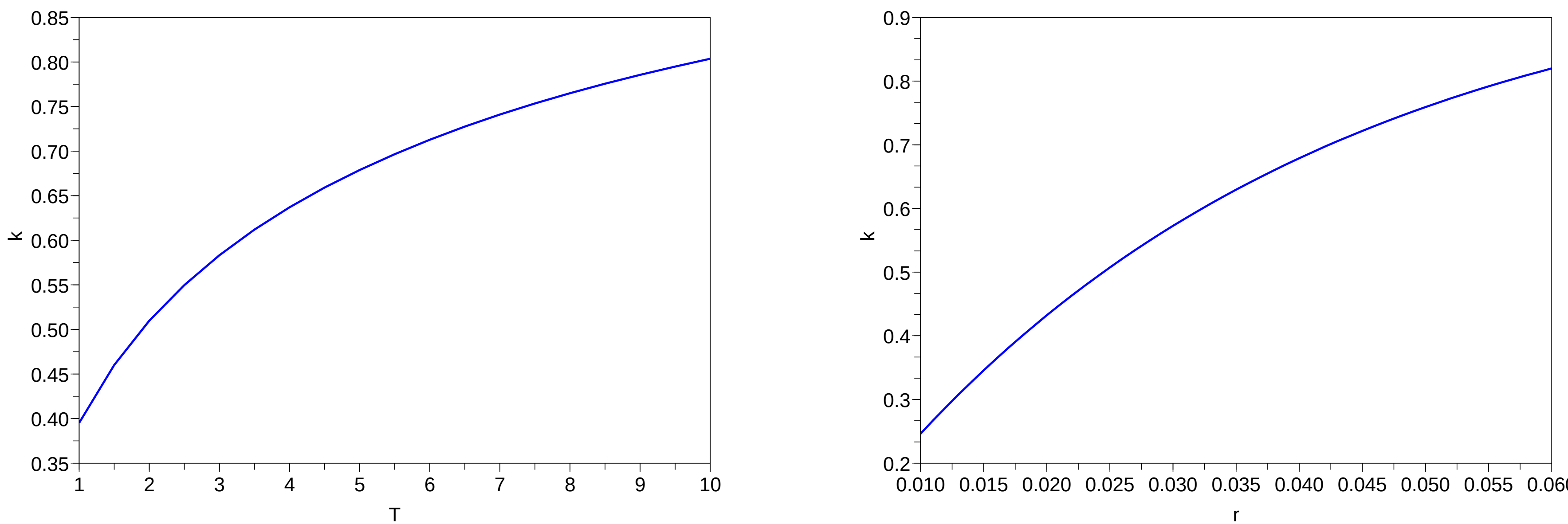
By put-call parity, the put-based strategy is equivalent to:

- Buy zero-coupon with notional K to lock in the capital at maturity
- Use the remainder to buy a call on $\lambda^* S_T$ with strike K (or anything else!).
- Often, at-the-money calls are used; fund's performance is then proportional to the risky asset performance:

$$V_T = 1 + k(S_T - 1)^+, \quad k = \frac{1 - B(0,T)}{C(T)} < 1.$$

k is called gearing or indexation.

The gearing factor



Dependence of the indexation k on the time to maturity (left) and the interest rate (right). Other parameters: $K = S_0$, $\sigma = 0.2$, $r = 4\%$ (left) and $T = 5$ years (right).

American capital guarantee

- One cannot simply buy and hold an American put because it is not self-financing.
- The correct strategy is dynamic trading in S and American puts on S:

$$V_t = \lambda_t S_t + P^a(t, \lambda_t S_t), \quad \lambda_t = \lambda_0 \vee \sup_{u \le t} \left(\frac{b(u)}{S_u} \right),$$

  where $b(t)$ is the exercise boundary and λ_0 is chosen from the budget constraint.
- This self-financing strategy, satisfies $V_t \geq K, 0 \leq t \leq T$ and is optimal for power utility in complete markets (EKJL).

Replicating options

Danger of the OBPI approach: absence of liquid options for long maturities (especially in the credit world)

- Counterparty risk if the option is bought over-the-counter
- Marking-to-market difficult at intermediate dates

Common solution: replicate the option with a self-financing portfolio containing $\Delta(S_t)$ stocks.

OBPI with option replication

Advantages:

- No need to structure a long-dated option
- The portfolio is easy to mark to market and liquidate

Drawbacks:

- The replication is only approximate, especially in incomplete markets
- Transaction costs may be high
- Model-dependent

Constant proportion portfolio insurance

The basic CPPI strategy

- Introduced by Black and Jones (87) and Perold (86).
- A fixed amount N is guaranteed at maturity T.
- At every t, a fraction is invested into risky asset S_t and the remainder into zero-coupon bond with maturity T and nominal N (denoted by B_t).
- If $V_t > B_t$, the risky asset exposure is $mC_t \equiv m(V_t - B_t)$, with $m > 1$.
- If $V_t \leq B_t$, the entire portfolio is invested into the zero-coupon.

Features and extensions

- Model-independent (for continuous processes).
- Maturity-independent, open-entry and open-exit.
- Greater upward potential than OBPI: while in OBPI the exposure is limited to the indexation $k < 1$, the CPPI exposure in bullish markets is only limited by the multiplier.
- Variable floor (ratchet) easily incorporated.

Analysis of CPPI: Gaussian setting

Suppose that the interest rate r is constant and

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Then the fund's evolution is given by

$$dV_t = m(V_t - B_t)\frac{dS_t}{S_t} + (V_t - m(V_t - B_t))r dt.$$

C_t satisfies the Black-Scholes SDE:

$$\frac{dC_t}{C_t} = (m\mu + (1-m)r)dt + m\sigma dW_t.$$

Analysis of CPPI: Gaussian setting

In the Black-Scholes model, CPPI strategy is equivalent to

- Buying a zero-coupon with nominal N to guarantee the capital at maturity (superhedging the floor);
- Investing the remaining sum into a risky asset which has m times the excess return and m times the volatility of S and is perfectly correlated with S.

Analysis of CPPI: Gaussian setting

The portfolio value is explicitly given by

$$V_T = N + (V_0 - Ne^{-rT}) \exp\left(rT + m(\mu - r)T + m\sigma W_T - \frac{m^2\sigma^2 T}{2} \right).$$

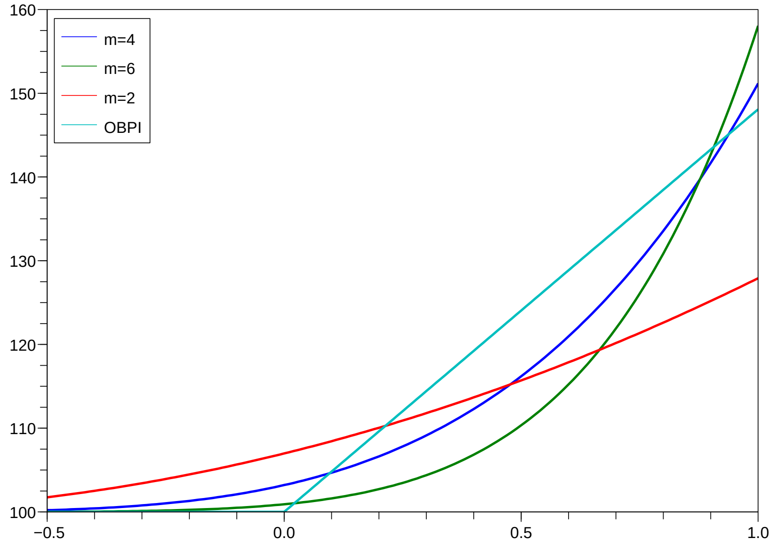
which can be rewritten as

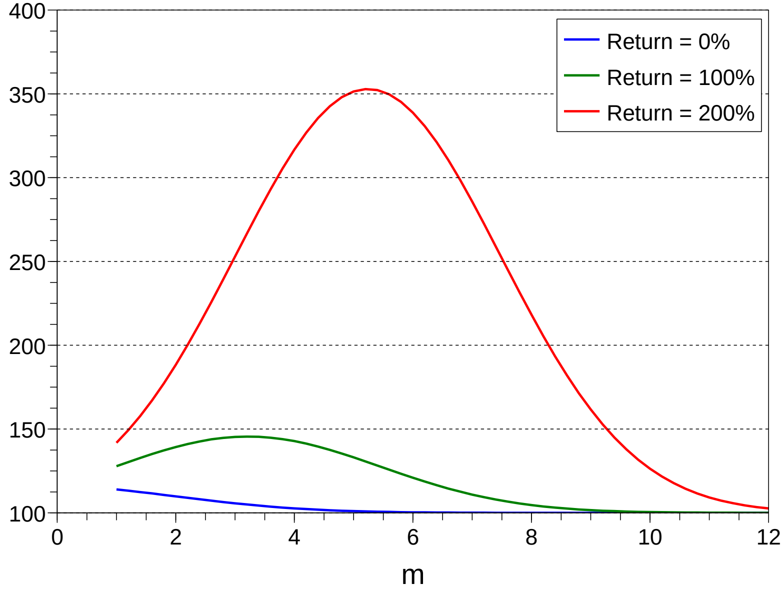
$$V_T = N + (V_0 - Ne^{-rT}) C_m \left(\frac{S_T}{S_0} \right)^m,$$

where

$$C_m = \exp\left(-(m-1)rT - (m^2 - m)\frac{\sigma^2}{2}T \right).$$

Gain profiles of the CPPI strategy





Left: CPPI portfolio as a function of stock return. Right CPPI portfolio return as a function of multiplier for given stock return. Parameters are $r = 0.03$, $\sigma = 0.2$, $T = 5$.

Optimality of CPPI

The CPPI strategy can be shown to be optimal in the context of *long-term risk-sensitive* portfolio optimization (Grossman and Vila '92, Sekine '08):

$$\sup_{\pi \in \mathcal{A}} \limsup_{T \to \infty} \frac{1}{\gamma T} \log E \left(X_T^{x,\pi} \right)^\gamma \qquad (RS)$$

- The optimal strategy π and the value function do not depend on the initial value $x > 0$.
- In the Black-Scholes setting, the Merton strategy $\pi^* \equiv \frac{\mu - r}{\sigma^2 (1-\gamma)}$ is optimal.

Optimality of CPPI

For the problem (RS) under the constraint $X_t^{x,\pi} \geq K_t$ for all t, an optimal strategy is described by

- Superhedge the floor process with any portfolio $\bar{K}$ satisfying $\bar{K}_0 < x$.
- Invest $x - \bar{K}_0$ into the unconstrained optimal portfolio.

In the Black-Scholes model $\Rightarrow$ classical CPPI with multiplier given by the Merton portfolio π^*.

- Extension by Grossman and Zhou '93 and Cvitanic and Karatzas '96: the CPPI strategy with stochastic floor is optimal for (RS) in case of drawdown constraints.

Optimality of CPPI: critique

- Merton's multiplier may be too low: it results from the unconstrained problem which takes into account both gains and losses, and under the floor constraint investors accept greater risks to maximize gains.
- In models with jumps, the positivity constraint often implies $0 \leq \pi \leq 1$, which is not sufficient for CPPI $\Rightarrow$ one may want to authorise some gap risk $\Rightarrow$ optimisation under VaR constraint.
- Market practice is to use basic CPPI with m fixed as function of the VaR constraint.

Some explicit computations for CPPI with jumps

Introducing jumps

Suppose that S and B may be written as

$$\frac{dS_t}{S_{t-}} = dZ_t \quad \text{and} \quad \frac{dB_t}{B_t} = dR_t,$$

where Z is a semimartingale with $\Delta Z > -1$ and R is a continuous semimartingale.
This implies

$$B_t = B_0 \exp\left(R_t - \frac{1}{2}[R]_t\right) > 0.$$

Example: $R_t = rt$ and Z is a Lévy process.

Stochastic differential equation

Let $\tau = \inf\{t : V_t \leq B_t\}$. Then, up to time τ,

$$dV_t = m(V_{t-} - B_{t-})\frac{dS_t}{S_{t-}} + \{V_{t-} - m(V_{t-} - B_{t-})\}\frac{dB_t}{B_{t-}},$$

which can be rewritten as

$$\frac{dC_t}{C_{t-}} = m dZ_t + (1-m)dR_t.$$

where $C_t = V_t - B_t$ is the cushion.

Solution via change of numeraire

Writing $C_t^* = \frac{C_t}{B_t}$ and applying Itô formula,

$$\frac{dC_t^*}{C_{t-}^*} = m(dZ_t - d[Z,R]_t - dR_t + d[R]_t) := m dL_t,$$

which can be written as

$$C_t^* = C_0^* \mathcal{E}(mL)_t,$$

where $\mathcal{E}$ denotes the stochastic exponential:

$$\mathcal{E}(X)_t = X_0 e^{X_t - \frac{1}{2}[X]_t^c} \prod_{s \le t, \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s}.$$

Solution via change of numeraire

After time τ, the process C^* remains constant. Therefore, the portfolio value can be written explicitly as

$$C_t^* = C_0^* \mathcal{E}(mL)_{t\wedge\tau},$$

or again as

$$\frac{V_t}{B_t} = 1 + \left(\frac{V_0}{B_0} - 1\right) \mathcal{E}(mL)_{t\wedge\tau}.$$

Probability of loss

Proposition Let $L = L^c + L^j$, with L^c continuous and L^j independent Lévy process with Lévy measure ν. Then

$$P[\exists t \in [0,T] : V_t \leq B_t] = 1 - \exp\left(-T \int_{-\infty}^{-1/m} \nu(dx)\right).$$

- In Lévy models, the basic CPPI has constant loss probability per unit time.

Probability of loss

Proof: $V_t \le B_t \iff C_t^* \le 0 \iff$

$$C_t^* = C_0^* \mathcal{E}(mL)_t \le 0.$$

But since

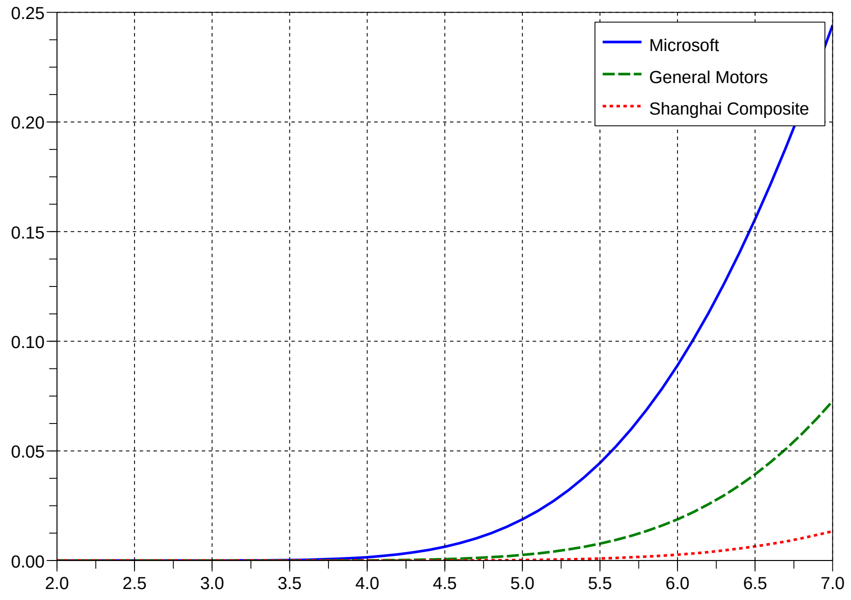
$$\Delta \mathcal{E}(X)_t = \mathcal{E}(X)_{t-}(1 + \Delta X_t),$$

this is equivalent to $\Delta L_t^j \le -1/m$.

For a Lévy process L^j, the number of such jumps in the interval $[0, T]$ is a Poisson random variable with intensity $T \int_{-\infty}^{-1/m} \nu(dx)$.

Example: Kou's model

$$\nu(x) = \frac{\lambda(1-p)}{\eta_+} e^{-x/\eta_+} 1_{x>0} + \frac{\lambda p}{\eta_-} e^{-|x|/\eta_-} 1_{x<0}.$$



Loss probability over $T = 5$ years as function of the multiplier.

Stochastic volatility and variable multiplier strategies

Stochastic volatility via time change

The traditional stochastic volatility model $\frac{dS_t}{S_t} = \sigma_t dW_t$ can be equivalently written as

$$S_t = X(v_t) \quad \text{where} \quad v_t = \int_0^t \sigma_s^2 ds \quad \text{and} \quad \frac{dX(t)}{X(t)} = dW_t.$$

Similarly, Carr et al.(2003) construct stochastic volatility models with jumps from a jump-diffusion model:

$$S_t = \mathcal{E}(L)_{v_t}, \quad v_t = \int_0^t \sigma_s^2 ds, \quad L \text{ is a jump-diffusion.}$$

- The stochastic volatility determines the intensity of jumps

The Heston parameterization

The volatility process most commonly used is the square root process

$$d\sigma_t^2 = k(\theta - \sigma_t^2)dt + \delta\sigma_t dW.$$

The Laplace transform of integrated variance v is known:

$$\mathcal{L}(\sigma, t, u) = \frac{\exp\left(\frac{k^2\theta t}{\delta^2}\right)}{\left(\cosh\frac{\gamma t}{2} + \frac{k}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2k\theta}{\delta^2}}} \exp\left(-\frac{2\sigma_0^2 u}{k + \gamma\coth\frac{\gamma t}{2}}\right)$$

where $\mathcal{L}(\sigma, t, u) := E[e^{-uv_t}|\sigma_0 = \sigma]$ and $\gamma := \sqrt{k^2 + 2\delta^2 u}$.

Loss probability with stochastic vol

If the volatility is stochastic, the loss probability

$$P[\exists s \in [t,T] : V_s \leq B_s | \mathcal{F}_t] = 1 - \exp\left(-(T-t)\int_{-\infty}^{-1/m} \nu(dx)\right)$$

becomes volatility dependent

$$P[\exists s \in [t,T] : V_s \leq B_s | \mathcal{F}_t] = 1 - \mathcal{L}(\sigma_t, T-t, \int_{-\infty}^{-1/m} \nu(dx))$$

- Crucial for long-term investments: a two-fold increase in volatility may increase the loss probability from 5% to 20%.

Managing the volatility exposure

The vol exposure can be controlled by varying the multiplier m_t: the loss probability is

$$P[\tau \leq T] = 1 - E\left[\exp\left(-\int_0^T dt\,\sigma_t^2 \int_{-\infty}^{1/m_t} \nu(dx)\right)\right].$$

The loss event is characterized by hazard rate λ_t, interpreted as the probability of loss "per unit time":

$$\lambda_t = \sigma_t^2 \int_{-\infty}^{1/m_t} \nu(dx)$$

Managing the volatility exposure

The fund manager can control the local loss probability and the local VaR by choosing m_t as a function of σ_t to keep the hazard rate λ_t constant:

$$\sigma_t^2 \int_{-\infty}^{1/m_t} \nu(dx) = \sigma_0^2 \int_{-\infty}^{1/m_0} \nu(dx),$$

where m_0 is the initial multiplier fixed according to the desired loss probability level. If the jump size distribution is α-stable, the above formula amounts to $m_t = m_0 \left(\frac{\sigma_t}{\sigma_0}\right)^{-2/\alpha}$.