Implied volatility at long maturities

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The price of a European call option with strike $K$ and maturity $T$

$$C_t(K, T) = S_t \Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

- $S_t$ underlying stock price (no dividends)
- $r$ spot interest rate
- $d_\pm = \frac{\log \left( \frac{S_t}{K} \right)}{\sigma \sqrt{T-t}} + \left( \frac{r}{\sigma} \pm \frac{\sigma}{2} \right) \sqrt{T-t}$
- $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \, dt$
\( \sigma \) is the volatility of the underlying stock

Unlike other parameters, not directly observed

\[ \sigma^2 = \frac{\text{Var}(\log S_T)}{T} \]

Liquid options priced by market already

Options often quoted in terms of implied volatility
The assumptions

- No arbitrage
- Calls of all maturities and strikes liquidly traded
- Zero interest rate

Let \((S_t)_{t\geq0}\) be a non-negative martingale.

Price of European call option with strike \(K\) and maturity \(T\)

\[
C_t(K, T) = \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t]
\]
Black–Scholes call price function

\[ BS(k, \nu) = \Phi \left( -\frac{k}{\sqrt{\nu}} + \frac{\sqrt{\nu}}{2} \right) - e^k \Phi \left( -\frac{k}{\sqrt{\nu}} - \frac{\sqrt{\nu}}{2} \right) \]

Definition

The random variable \( \Sigma_t(k, \tau) \) is defined on \( \{S_t > 0\} \) by

\[
\mathbb{E} \left[ \left( \frac{S_{t+\tau}}{S_t} - e^k \right)^+ \mid \mathcal{F}_t \right] = BS(k, \tau \cdot \Sigma_t(k, \tau)^2)
\]
Take \( t = 0 \), and write \( \Sigma = \Sigma_0 \). With no loss, let \( S_0 = 1 \).

**Theorem**

Assume \( S_t \to 0 \) almost surely as \( t \uparrow \infty \). Then

\[
\tau \Sigma(k, \tau)^2 = 8 \left| \log \mathbb{E}(S_\tau \land e^k) \right| - 4 \log \left| \log \mathbb{E}(S_\tau \land e^k) \right| - 4k - 4 \log \pi + \epsilon(k, \tau)
\]

where

\[
\sup_{-M \leq k \leq M} \left| \epsilon(k, \tau) \right| + \sup_{-M \leq k_1 < k_2 \leq M} \frac{\left| \epsilon(k_2, \tau) - \epsilon(k_1, \tau) \right|}{k_2 - k_1} \to 0
\]

for all \( M > 0 \).
The condition $S_t \rightarrow 0$ almost surely is natural since the following are equivalent:

- $S_t \rightarrow 0$ almost surely.
- There exists a $k \in \mathbb{R}$ such that $\tau \Sigma(k, \tau)^2 \uparrow \infty$
- $\tau \Sigma(k, \tau)^2 \uparrow \infty$ for all $k \in \mathbb{R}$.
- $C(K, T) \uparrow S_0$ for all $K > 0$.
- There exists a $K > 0$ such that $C(K, T) \uparrow S_0$
Warnings:

- If $S$ is a strictly local martingale, then

$$S_t \to 0 \text{ a.s.} \Rightarrow \lim_{\tau \uparrow \infty} \mathbb{E}[(S_\tau - K)^+] < S_0$$

In particular, $\sup_{\tau > 0} \tau \Sigma(k, \tau)^2 < \infty$

- If the interest rate is non-zero, but deterministic, everything holds by changing to prices discounted by the bank account.
Corollary

Let $\mathbb{Q}$ be the measure locally equivalent to $\mathbb{P}$ with density
\[
\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = S_t.
\]
Then
\[
\frac{\partial}{\partial k} \tau \Sigma(k, \tau)^2 = 4 \left( \frac{\mathbb{Q}(S_\tau < e^k) - e^k \mathbb{P}(S_\tau \geq e^k)}{\mathbb{Q}(S_\tau < e^k) + e^k \mathbb{P}(S_\tau \geq e^k)} \right) + \epsilon'(k, \tau)
\]
if the distribution of $S_\tau$ continuous at $e^k$. In particular,

\[
\limsup_{\tau \uparrow \infty} \sup_{k \in [-M, M]} \left| \frac{\partial}{\partial k} \tau \Sigma(k, \tau)^2 \right| \leq 4
\]
For comparison:

- \( \frac{\partial}{\partial k} \Sigma(k, \tau)^2 < \frac{4}{\tau} \) for all \( k \geq 0 \)
- \( \frac{\partial}{\partial k} \Sigma(k, \tau)^2 > -\frac{4}{\tau} \) for all \( k \leq 0 \)
- \( \frac{\partial}{\partial k} \Sigma(k, \tau)^2 < \frac{2}{\tau} \) for all \( k \geq k_+(\tau) \), for some \( k_+ \)
- \( \frac{\partial}{\partial k} \Sigma(k, \tau)^2 \geq -\frac{2}{\tau} \) for all \( k \leq k_-(\tau) \) for some \( k_- \)

Let

$$\psi_t(p) = \log \mathbb{E}(S_t^p).$$

**Theorem**

*Suppose $S_t > 0$ a.s. for all $t > 0$. If there is a $p^* \in (0, 1)$ and $C > 0$ such that*

$$\psi_t \left( p^* + i \frac{\theta}{\sqrt{t}} \right) - \psi_t(p^*) \to -C \theta^2 / 2$$

*and if a technical condition holds, then*

$$t \sum(k, t)^2 = -8\psi_t(p^*) + 4k(2p^* - 1) + 4 \log \left( \frac{2[p^*(1 - p^*)]^2 Ct}{-\psi_t(p^*)} \right) + \delta(k, t)$$
Let

\[ \phi_t(\theta) = \exp \left[ \psi_t \left( p^* + i \frac{\theta}{\sqrt{t}} \right) - \psi_t(p^*) \right] \]

Proof depends on the identity:

\[
\mathbb{E}(S_t \wedge e^k) = \frac{e^{k + \psi_t(p)}}{2\pi \sqrt{t}} \int_{-\infty}^{\infty} \frac{\phi_t(\theta) e^{ik\theta/\sqrt{t}}}{p(1 - p) + \theta^2/t + i(1 - 2p)\theta/\sqrt{t}} d\theta
\]
A sufficient condition:

\[ \int_{-\infty}^{\infty} \sup_{t>0} \frac{|\phi_t(\theta)|}{1 + \theta^2/t} \, d\theta < \infty \]
Long implied volatility can never fall

Theorem (Rogers–T 2008)

For any $k_1, k_2 \in \mathbb{R}$ we have

$$\limsup_{\tau \uparrow \infty} \Sigma_t(k_1, \tau) - \Sigma_s(k_2, \tau) \geq 0$$

for $t \geq s \geq 0$. There exist examples for which the inequality is strict.
Theorem (Dybvig–Ingersoll–Ross 1996)

Let $f_t(\tau)$ be the instantaneous forward interest rate with long rate

$$
\limsup_{\tau \uparrow \infty} f_t(\tau) = \ell_t.
$$

Then

$$
\ell_s \leq \ell_t
$$

for $0 \leq s \leq t$.

Theorem (Rogers–T. 2008)

Suppose

\[ \Sigma_t(k, \tau) = \Sigma_0(k, \tau) + \xi_t \]

for some process \((\xi_t)_{t \geq 0}\). Then \(\xi_t \geq 0\).

Furthermore, if there is a \(p \in \mathbb{R}\) such that

\[ t \mapsto \frac{1}{p(p - 1)} \psi_t(p) \]

is sublinear, then \(\xi_t = 0\).
**Theorem (Balland 2002)**

*If* $\xi_t = 0$ *for all* $t \geq 0$ *and* $S_t \to 1$ *in probability as* $t \downarrow 0$ *then* $(S_t)_{t\geq 0}$ *is an exponential Levy process.*