Importance Sampling and Particles for Portfolio Credit Risk

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GT ‘événements rares’
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Outline

1. Markovian Credit Models Numerics
2. Importance Sampling vs Particles
3. Numerical Benchmark
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Prices, Greeks and loss distribution computable by numerical resolution of the Kolmogorov equations

<table>
<thead>
<tr>
<th>Deterministic numerical schemes</th>
<th>Monte Carlo methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Matrix exponentiation formulas (for time-homogeneous models)</td>
<td>• The only reasonable alternative for ( d ) greater than a few units</td>
</tr>
<tr>
<td>• Discretization schemes (stiff ODE solvers)</td>
<td>• Slow</td>
</tr>
<tr>
<td>• Precluded by the curse of dimensionality for large ( d )'s</td>
<td></td>
</tr>
</tbody>
</table>
Numerics in Markovian Credit Models [Betal, FB*, LCF, H..]

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Variance Reduction Methods in Markovian Credit Models

Importance Sampling (IS) [G]
- **Explicit Girsanov** change of measure favoring the ‘important’ (‘rare’) events of interest
- Sometimes not clear which change of measure will reduce the variance
- Original measure sometimes not available (‘black box’ sampler)

Interacting Particle Systems (IPS) [DMG05-DM04]
- **Implicit Feynman–Kac** change of measure forcing the process into the events of interest
- All one needs is a sampler under the original distribution
- Knowledge of the original distribution not really necessary

Structural vs Intensity Model of Credit Risk

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Markov Chain Set-Up

- Discrete time filtered probability space \((\Omega, \mathcal{F}, \mathbb{Q})\)
- (possibly time-inhomogeneous) Markov chain \((X_i)_{0 \leq i \leq n}\) with transition kernel \(K(X_{i-1}, \cdot)\) at time step \(i = 1, \ldots, n\)
- Computation of small probabilities of events and related expectations of the form \(I = \mathbb{E}f(X_0, \ldots, X_n)\)
Compute $I = \mathbb{E}\{f(X_0, \ldots, X_n)\}$: Importance Sampling

Weight functions $w_i = w_i(x_0, \ldots, x_i)$ st $\mathbb{E}(w_i(X_0, \ldots, X_i) | \mathcal{F}_{i-1}) = 1$

Twisted probability measure $\tilde{Q}$ such that $\frac{d\tilde{Q}}{dQ} |_{\mathcal{F}_i} = \prod_{1 \leq l \leq i} w_l(X_0, \ldots, X_l)$

**IS Dynamics ($\tilde{\xi}_0 = x_0$)**

For every time step $i = 1, \ldots, n$ and trajectory $j = 1, \ldots, m$, draw

$$\tilde{\xi}_i \sim \tilde{K}(\tilde{\xi}_{i-1}, \cdot)$$

$\tilde{K}$ $\tilde{Q}$-transition kernel of $X$

$$I = \mathbb{E}f(X_0, \ldots, X_n) = \tilde{\mathbb{E}}\left(\frac{f(X_0, \ldots, X_n)}{\prod_{1 \leq i \leq n} w_i(X_0, \ldots, X_i)}\right) \approx \frac{1}{m} \sum_{j=1}^{m} \frac{f(\tilde{\xi}_0^j, \ldots, \tilde{\xi}_n^j)}{\prod_{1 \leq i \leq n} w_i(\tilde{\xi}_0^j, \ldots, \tilde{\xi}_i^j)} = I^#$$
Compute $I = \mathbb{E}\{f(X_0, \ldots, X_n)\}$: Importance Sampling

Weight functions $w_i = w_i(x_0, \ldots, x_i)$ s.t. $\mathbb{E}(w_i(X_0, \ldots, X_i) \mid F_{i-1}) = 1$

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Twisted probability measure \( \tilde{Q} \) such that \( \frac{d \tilde{Q}}{d Q} \mid_{F_i} = \prod_{1 \leq l \leq i} w_l(X_0, \ldots, X_i) \)

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For every time step \( i = 1, \ldots, n \) and trajectory \( j = 1, \ldots, m \), draw

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Compute $I = \mathbb{E}\{f(X_0, \ldots, X_n)\}$: Particles

(Path-)Particle $j$ at time $i$ $\xi^j_i = (\xi^j_{0,i}, \xi^j_{1,i}, \ldots, \xi^j_{i,i})$

Weight functions $w_i = w_i(x_0, \ldots, x_i)$

Selection/Mutation IPS Dynamics ($\xi_0 = x_0$)

For every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, draw

$$(\xi^j_{0,i}, \xi^j_{1,i}, \ldots, \xi^j_{i-1,i}) \sim \sum_{l=1}^{m} w_{i-1}(\xi^l_{i-1}) \delta_{\xi^l_{i-1}}, \quad \xi^j_{i,i} \sim K(\xi^j_{i-1,i}, \cdot)$$

$I = \mathbb{E}f(X_0, \ldots, X_n) \approx$

$$\left( \frac{1}{m} \sum_{j=1}^{m} \frac{f(\xi^j_n)}{\prod_{1 \leq i < n} w_i(\xi^j_{0,n}, \ldots, \xi^j_{i,n})} \right) \left( \prod_{1 \leq i < n} \frac{1}{m} \sum_{j=1}^{m} w_i(\xi^j_i) \right) = I^b$
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Selection/Mutation IPS Dynamics (\( \xi_0 = x_0 \))

For every \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), draw

\[
(\xi^j_{0,0}, \xi^j_{0,1}, \ldots, \xi^j_{i-1,1}) \sim \sum_{\ell=1}^m w_{i-1}(\xi^\ell_{i-1}) \delta_{\xi^\ell_{i-1}}, \xi^j_{i,i} \sim K(\xi^j_{i-1,1}, \cdot)
\]

\[
\begin{array}{ccccccc}
\xi^1_{0,0} & \xi^1_{0,1} & \xi^1_{1,1} & \ldots & \xi^1_{0,n-1} & \ldots & \xi^1_{n-1,1} \\
\xi^m_{0,0} & \xi^m_{0,1} & \ldots & \xi^m_{0,n-1} & \ldots & \xi^m_{n-1,1} \\
\end{array}
\]

\[
l = \mathbb{E}f(X_0, \ldots, X_n) \approx \left( \frac{1}{m} \sum_{j=1}^m \frac{f(\xi^j_{n})}{\prod_{1 \leq i < n} w_i(\xi^j_{0,n}, \ldots, \xi^j_{i,n})} \right) \left( \prod_{1 \leq i < n} \frac{1}{m} \sum_{j=1}^m w_i(\xi^j_{i}) \right) = I^b
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Properties of the Estimator

Unbiased at fixed $m$
Asymptotically convergent as $m \to \infty$

To minimize the variance of the estimators, use weight functions $w_i$'s.

In the case of IS
- $\text{st } \prod_{1 \leq i \leq n} w_i(X_1, \ldots, X_i) \propto |f(X_0, \ldots, X_n)|$
- ‘not too extreme’
- easy to implement

In the case of IPS
- favoring the occurrence of the event of interest
- without involving too large normalizing constants
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S Crépey, Evry University
IS and IPS for Portfolio Credit Risk
A Fully-Homogeneous model of Credit Risk

Markovian Portfolio Loss Model

- \( n = 125 \) credit names
- individual pre-default intensity processes \( \tilde{\lambda}_t = a \exp(bN_t/n) \)
  - \( b = 0 \) independent obligors
  - \( b > 0 \) defaults contagion
- Portfolio loss process (number of defaults) \( (L_t)_{t \in [0,T]} \)
- Exact loss distribution at the time horizon \( T \) by exponentiation of the model generator \((126 \otimes 126 \text{ matrix } A)\)

\[
p(T) = \exp(TA^*) \delta_0
\]
IS/IPS Algorithms

Mapping with the IS/IPS Markov Chain Set-Up

\[ X_i = (t_i, L_{t_i}), \quad 0 \leq i \leq n \]

- \( t_i \) \( i^{th} \) jump time of \( L \), capped at \( T \) (\( t_0 = 0 \))
- \( L_{t_i} = i \iff t_i < T \)

Parameterized families of IS/IPS variance reduction schemes

For IS, use Markovian changes of measures for MPP
- intensity functions bumped by a constant factor \( \alpha \), or
- intensity functions turned into a constant \( \alpha \)

For IPS, use weight functions \( w_i(x_0, \ldots, x_i) = \)
- \( \exp(\alpha(l_i - l_{i-1})) \) (‘favoring losses’), or
- \( \exp(-1_{t_i < T} \arctan(l_i - \alpha)) \) (‘favoring the loss level \( L_T = \alpha \)’).
IS/IPS Algorithms

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- \(\exp(-1_{t_i < T} \arctan(l_i - \alpha))\) ('favoring the loss level \(L_T = \alpha\)').
$\bar{\lambda}_t = a \exp(bN_t/n)$. Left pane: Independent obligors ($a = 0.01, b = 0$);
Right pane: Extreme contagion ($a = 0.01, b = 13$)
IS Good for Independent Obligors

Portfolio loss log-probabilities in a case of independent obligors: exact vs IS (left) and IPS (right) ($m = 10^4$ draws)
IS Good for Independent Obligors (Cont’d)

IS and IPS maps of losses in the case of independent obligors
Contagion: IPS Saves the Day!

Same as previously in the contagion case: IS (left) vs IPS (right) ($m = 10^4$)
Contagion: IPS Saves the Day! (Cont’d)

IS and IPS maps of losses in the contagion case
IS Hardly Affected by Dimension

Left: IS ‘as usual’; Right: IS with $d = 5-HGM$ encoding of the local intensity model (independent obligors, $m = 5000$)
IPS Hardly Affected by Dimension too

Same as previously but for IPS \((m = 10^5)\)
Conclusion: A Case-by-Case Approach

**IS ‘can do wonders’ when**
- a pertinent Girsanov like transformation can be identified
- the corresponding densities are easily computable and ‘not too extreme’ along the samples

**IPS useful substitute when**
- no obvious Girsanov change of measure is available
- no obvious measure favoring the event of interest
- distribution of the chain not known (sampler given as a black box)

**Model-dependent performances (and detailed spec.) of either method**
- No good for model calibration
- Good for complex pricing, Credit VaR..
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Appendix

IS vs IPS for Diffusions: Compute $I = \mathbb{E}\{f(W_{[0,T]})\}$

### Importance Sampling

- Use Girsanov to twist the distribution in path space

$$I = \tilde{\mathbb{E}} \left\{ f(W_{[0,T]}) \exp \left[ - \int_0^T \nabla h(W_t) d\tilde{W}_t - \frac{1}{2} \int_0^T |\nabla h(W_t)|^2 dt \right] \right\}$$

where $\frac{d\tilde{P}}{dP} = \mathcal{E} \left( \int_0^T \nabla h(W_t) dW_t \right)$ and $\tilde{W}_t = W_t - \int_0^t \nabla h(W_s) ds$

- $I^\sharp \leftarrow$ Monte Carlo samples from the twisted distribution $\tilde{P}$

### Interacting Particles System

- Use Feynman-Kac twisted distributions

$$I = \tilde{\mathbb{E}}\{f(W_{[0,T]}))e^{-\int_0^T V(W_s)ds}\} \mathbb{E}\{e^{\int_0^T V(W_s)ds}\} \text{ where } \frac{d\tilde{P}}{dP} = \frac{\mathbb{E}\{e^{\int_0^T V(X_s)ds}\}}{\mathbb{E}\{e^{\int_0^T V(W_s)ds}\}}$$

- $I^\flat \leftarrow$ Monte Carlo samples from the original distribution $P$
IS vs IPS for Diffusions: Compute $I = \mathbb{E}\{f(W_{[0,T]})\}$

**Importance Sampling**

- Use Girsanov to twist the distribution in path space

$$I = \widetilde{\mathbb{E}}\left\{f(W_{[0,T]})\exp\left[-\int_0^T \nabla h(W_t) d\widetilde{W}_t - \frac{1}{2} \int_0^T |\nabla h(W_t)|^2 dt\right]\right\}$$

where $\frac{d\widetilde{P}}{dP} = \mathcal{E}\left(\int_0^T \nabla h(W_t) dW_t\right)$ and $\widetilde{W}_t = W_t - \int_0^t \nabla h(W_s)ds$

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$$I = \widetilde{\mathbb{E}}\{f(W_{[0,T]})e^{-\int_0^T V(W_s)ds}\}\mathbb{E}\{e^{\int_0^T V(W_s)ds}\}$$

where $\frac{d\widetilde{P}}{dP} = \frac{e^{\int_0^T V(X_s)ds}}{\mathbb{E}\{e^{\int_0^T V(X_s)ds}\}}$

- $I^b \leftarrow$ Monte Carlo samples from the original distribution $P$
Armageddon event: everyone defaulted in the portfolio by the time of maturity $T$

**Independent obligors**
- Exponential right-tail decay
  - Armageddon Probability $= 1.044507e^{-164}$
- Variance reduction methods of complete necessity for computing by simulation high-losses related quantities
  - Probability of high loss levels or price of a super-senior CDO tranche

**Extreme contagion**
- No extremely rare levels of the loss any more
  - Armageddon Probability $= 7.106e^{-03} \approx 1\%$
  - The less likely loss level is the level $i = 115$, with a probability of $1.108e^{-06}$
- Variance reduction methods not strictly needed
# Homogeneous Groups Model

$d$ classes of $\nu - 1 = \frac{n}{d}$ homogeneous obligors [FB07,BCJR]

- Groups loss processes (number of defaults) $L_t^l$, $l = 1, \ldots, d$, jointly modeled as a $d$-variate Markov point process $\Lambda = (L_1, \ldots, L_d)$
- $\mathbb{F}^\Lambda$-intensity of $L^l$ given as $\lambda_t^l = (\nu - 1 - L_t^l)\tilde{\lambda}^l(\Lambda_t)$
- *Pre-default individual intensity functions* $\tilde{\lambda}^l = \tilde{\lambda}^l(\nu)$ for $l = 1, \ldots, d$, where $\nu = (i_1, \ldots, i_d) \in \{0, 1, \ldots, \nu - 1\}^d$
- Generator of $\Lambda = (\text{very sparse}) \nu^d \otimes \nu^d$ matrix $A$
- $L = \sum_{l=1}^d L^l$
- From pure top models for $d = 1$ to pure bottom-up models for $d = n$
- Reducibility to $d = 1$ whenever $\tilde{\lambda}^l(\nu) = \hat{\lambda}(\sum_{1 \leq \ell \leq d} i_\ell)$ (fully homogenous case)
- $t_i$'s ordered jump times of $\Lambda$, capped at a time horizon $T$ ($t_0 = 0$)
- Markov Chain $X_i = (t_i, \Lambda_{t_i})$, $0 \leq i \leq n$