High-Frequency Trading in a Limit Order Book

Sasha Stoikov (with M. Avellaneda)

Cornell University

February 9, 2009
The limit order book

<table>
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<tr>
<th>Order Book</th>
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- Two main categories of traders
  1. Liquidity taker: buys at the ask, sell at the bid
  2. Liquidity provider: waits to buy at the bid, sell at the ask
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  1. Making the bid/ask spread
  2. Managing their risk by adjusting the quantities/prices
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  2. Managing their risk by adjusting the quantities/prices
- Factors affecting the optimal bid/ask prices:
  1. Inventory risk
    - The stock mid price: $S$
    - The stock volatility: $\sigma$
    - The risk aversion: $\gamma$
    - The liquidity: $\lambda(\cdot)$
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  2. Adverse selection risk
Outline

1 Optimization
   - The maximal utility problem
   - Optimal bid and ask prices
   - Some approximations
   - P&L profiles of the optimal strategy
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   - Modeling the order book
   - Estimating model parameters
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   • A market making algorithm
   • Autocorrelation in the order flow
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4 Conclusion
The mid price of the stock

- Brownian motion

\[ dS_t = \sigma dW_t \]
The mid price of the stock

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- Geometric Brownian motion
  \[ \frac{dS_t}{S_t} = \sigma dW_t \]
The mid price of the stock

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- Trading at the mid-price is not allowed. However, we may quote limit orders \( p^b \) and \( p^a \) around the mid-price.
The arrival of buy and sell orders

- Controls: $p_t^a$ and $p_t^b$
The arrival of buy and sell orders

- Controls: $p_t^a$ and $p_t^b$
- Number of stocks bought $N_t^b$ is Poisson with intensity $\lambda^b(p^b - s)$, an increasing function of $p^b$
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- The wealth in cash

$$dX_t = p^a dN_t^a - p^b dN_t^b$$

The inventory

$$q_t = N_t^b - N_t^a$$
The market maker’s objective

- Maximize exponential utility

\[ u(s, x, q, t) = \max_{p_t^a, p_t^b, 0 \leq t \leq T} E_t \left[ -e^{-\gamma(X_T + q_T S_T)} \right] \]
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- Mean/variance objective

\[ \nu(s, x, q, t) = \max_{p_t^a, p_t^b, 0 \leq t \leq T} \mathbb{E}_t \left[ (X_T + q_T S_T) \right] - \frac{\gamma}{2} \text{Var} \left[ (X_T + q_T S_T) \right] \]
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- Mean/variance objective

\[ v(s, x, q, t) = \max_{p^a_t, p^b_t, 0 \leq t \leq T} E_t \left[ (X_T + q_T S_T) - \frac{\gamma}{2} \text{Var} [(X_T + q_T S_T)] \right] \]

- Infinite horizon exponential utility

\[ w(x, s, q) = \max_{p^a_t, p^b_t} E \left[ \int_0^{\infty} -\exp(-\omega t) \exp(-\gamma(X_t + q_t S_t)) dt \right] . \]
The market maker’s objective

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- Other objectives: minimizing shortfall risk, value at risk, etc...
The HJB equation

\[ u(x, s, q, t) \text{ solves} \]

\[
\begin{aligned}
  u_t + \frac{1}{2} \sigma^2 u_{ss} & \\
  + \max_{p^b} \lambda^b(p^b) \left[ u(s, x - p^b, q + 1, t) - u(s, x, q, t) \right] & \\
  + \max_{p^a} \lambda^a(p^a) \left[ u(s, x + p^a, q - 1, t) - u(s, x, q, t) \right] & = 0 \\
  u(S, x, q, t) & = -\exp(-\gamma(x + qS)).
\end{aligned}
\]
The indifference or reservation prices

Definition
The indifference bid price $r^b$ (relative to a book of $q$ stocks) is given implicitly by the relation

$$u(x - r^b(s, q, t), s, q + 1, t) = u(x, s, q, t).$$

The indifference ask price $r^a$ solves

$$u(x + r^a(s, q, t), s, q - 1, t) = u(x, s, q, t).$$
The optimal quotes

**Theorem**

The optimal bid and ask prices $p^b$ and $p^a$ are given by the implicit relations

$$p^b = r^b - \frac{1}{\gamma} \ln \left( 1 + \gamma \frac{\lambda^b}{\partial \frac{\partial \lambda^b}{\partial p}} \right)$$

and

$$p^a = r^a + \frac{1}{\gamma} \ln \left( 1 - \gamma \frac{\lambda^a}{\partial \frac{\partial \lambda^a}{\partial p}} \right).$$
The “Frozen-Inventory” Approximation

- If we assume there is no arrival of orders

\[ \nu(x, s, q, t) = E_t[-\exp(-\gamma(x + qS_T))] \]

\[ = -\exp(-\gamma x) \exp(-\gamma qs) \exp \left( \frac{\gamma^2 q^2 \sigma^2 (T-t)}{2} \right) \]
The “Frozen-Inventory” Approximation

- If we assume there is no arrival of orders
  \[ \nu(x, s, q, t) = E_t[- \exp(-\gamma(x + qS_T))] \]
  \[ = - \exp(-\gamma x) \exp(-\gamma qs) \exp\left(\frac{\gamma^2 q^2 \sigma^2 (T-t)}{2}\right) \]

- The indifference price of a stock, given an inventory of \( q \) stocks is
  \[ r(s, q, t) = s - q\gamma \sigma^2 (T-t) \]
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- The indifference price of a stock, given an inventory of \( q \) stocks is

\[ r(s, q, t) = s - q \gamma \sigma^2 (T-t) \]

- This is an approximation to \( r^a \) and \( r^b \) for the problem with order arrivals
The “Econophysics” Approximation

1. The density of market order size is

\[ f^Q(x) \propto x^{-1-\alpha} \]

Gabaix et al. (2006)
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\[ \Delta p \propto \ln(Q) \]

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3. Constant frequency of order arrivals \( \Lambda \)
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Potters and Bouchaud (2003)

3. Constant frequency of order arrivals \( \Lambda \)

- Imply that arrival rates are exponential

\[ \lambda^a = A \exp(-k(p^a - s)) \quad \text{and} \quad \lambda^b = A \exp(-k(s - p^b)) \]
The optimal quotes

• Step one: the indifference price

\[ r(s, q, t) = s - q \gamma \sigma^2 (T - t) \]
The optimal quotes

- Step one: the indifference price

\[ r(s, q, t) = s - q\gamma\sigma^2(T - t) \]

- Step two: the bid/ask quotes

\[ p^b = r - \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \]

and

\[ p^a = r + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right). \]

\( k \) is a measure of the liquidity of the market.
A stock price simulation for $\gamma = 0.1$
P&L profile for $\gamma = 0.5$

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Spread</th>
<th>Profit</th>
<th>std(Profit)</th>
<th>std(Final q)</th>
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Table: 1000 simulations with $\gamma = 0.5$
P&L profile for $\gamma = 0.1$

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Table: 1000 simulations with $\gamma = 0.1$
P&L profile for $\gamma = 0.01$

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Table: 1000 simulations with $\gamma = 0.01$
A market order

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- The sizes of market and limit orders are random.
- The above events are mutually independent.
The simulation pseudocode

At each time step, generate the next event:

- Probability of a market buy order
  
  \[
  \frac{\mu^a}{\mu^b + \mu^a + \sum_d (\lambda^b(d) + \lambda^a(d)) + \sum_d \theta(d)Q^b_t(d) + \sum_d \theta(d)Q^a_t(d)}
  \]

  Draw order size (in shares) from empirical distribution.
The simulation pseudocode

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\[ \mu^a \]
\[ \mu^b + \mu^a + \sum_d (\lambda^b(d) + \lambda^a(d)) + \sum_d \theta(d)Q^b_t(d) + \sum_d \theta(d)Q^a_t(d) \]

Draw order size (in shares) from empirical distribution.

- Probability of a limit buy order \( i \) ticks away from the best ask

\[ \lambda^b(i) \]
\[ \mu^b + \mu^a + \sum_d (\lambda^b(d) + \lambda^a(d)) + \sum_d \theta(d)Q^b_t(d) + \sum_d \theta(d)Q^a_t(d) \]

Draw order size from empirical distribution.
The simulation pseudocode

- Probability of a cancel buy order $i$ ticks away from the best ask

$$\frac{\theta(i)Q^b_t(i)}{\mu^b + \mu^a + \sum_d (\lambda^b(d) + \lambda^a(d)) + \sum_d \theta(d)Q^b_t(d) + \sum_d \theta(d)Q^a_t(d)}$$

If there are $j$ orders at that price, cancel one of them with uniform probability.

... same procedure for the sell side of the book.
The market statistics

The simulation parameters

- Ticker: AMZN
- Number of events (market, limit, cancel): 50,000
- Number of market orders: $\mu^a + \mu^b = 2.371$
- Number of limit orders within a 2 dollar window:
  $$\sum \lambda^a(d) + \lambda^b(d) = 24.221$$
- Number of cancel orders within a 2 dollar window:
  $$\sum_d \theta(d)Q^b_t(d) + \sum_d \theta(d)Q^a_t(d) = 22.613$$
The market statistics

The distribution of limit orders

as a function of the distance to the opposite best quote $\lambda(d)$
The cancel rates per order as a function of the distance to the opposite best quote $\theta(d)$
The market statistics

The market order size distribution

Market Share Distribution

replay totals, zero totals
The market statistics

The limit order size distribution
The market statistics

The zero market

The average book shape

Book Shape

replay Average, zero average

![Graph showing the average book shape with lines for replay Average and zero average](image-url)
The zero market

Sample paths

![Graph showing mid-price sample paths with a legend indicating 'replay MidPrice' and 'zero MidPrice'.]
The Trump agent controls inventory by lowering the quotes after he buys, and raising the quotes after he sells. His properties include the following parameters:

- A start time (e.g. right after the book is seeded)
- A premium around the market spread (e.g. bid minus $\delta_b = 2$ cents, ask plus $\delta_a = 2$ cents)
- A position limit (e.g. 500 shares)
- A lot size (e.g. 100 shares)
- An aggressiveness parameter for inventory control
Individual agent pseudocode

Trump agent operates by:

1. **Condition**: If time > start time and Trump does not have two outstanding limit orders

2. **The action**: cancel outstanding orders and submit two limit orders at the prices

\[
p_b = m_b - \delta_b + \delta_b \frac{q}{\text{floor}} \times \text{aggr}
\]

and

\[
p_a = m_a + \delta_a - \delta_a \frac{q}{\text{ceiling}} \times \text{aggr}
\]

where the first term is the market bid or ask, the second term is the bid and ask premium and the third term controls the inventory. If the floor is reached, there is no ask quote. If the ceiling is reached, there is no bid quote.
The individual’s statistics

Trump in Zero

- Capital = 10,000$, Position limited to ±40,000$, AMZN price = 79$
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- Market: 50,000 events in AMZN Zero (roughly 1 hour of clock time)
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- Results: 82,831 shares traded, 3.3% market participation, 862$ profit
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- \(( \text{avg premium 4.8 cents}) \times 82,831 = 3,964 \) $
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- (avg premium 4.8cents) * 82,831 = 3,964 $
- Adverse selection loss = 3,101 $
The individual’s statistics

Trump in Zero

![Graphs showing price, inventory, and P&L over time.](image-url)
The rho market

• The zero market picks the type of market orders (BUY/SELL) independently of past market orders
Autocorrelation in the order flow

The rho market

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- Empirically, the market data has long sequences of BUY (resp. SELL) orders
Autocorrelation in the order flow

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The rho market

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- Empirically, the market data has long sequences of BUY (resp. SELL) orders.
- We implement autocorrelation:
  1. Label $X_i = 1$ for a buy order and $X_i = 0$ for a sell order.
  2. Run a regression

\[
X_i = \alpha + \beta_1 X_{i-1} + \ldots + \beta_{10} X_{i-10}
\]

3. In the simulation, enforce

\[
P(X_i = 1) = \alpha + \beta_1 X_{i-1} + \ldots + \beta_{10} X_{i-10}
\]
The rho market
The rho market
Autocorrelation in the order flow

Trump in Rho

- Agent: Premium = 4 cents, Limit 500 shares, Lot size 100 shares, Aggressiveness = 1
Autocorrelation in the order flow

Trump in Rho

- Agent: Premium = 4 cents, Limit 500 shares, Lot size 100 shares, Aggressiveness = 1
- Market: 50,000 events in Rho
Trump in Rho

- Agent: Premium = 4 cents, Limit 500 shares, Lot size 100 shares, Aggressiveness = 1
- Market: 50,000 events in Rho
- Results: 103,862 shares traded, 4.15% market participation, 151$ profit
Autocorrelation in the order flow

**Trump in Rho**

- Agent: Premium = 4 cents, Limit 500 shares, Lot size 100 shares, Aggressiveness = 1
- Market: 50,000 events in Rho
- Results: 103,862 shares traded, 4.15% market participation, 151$ profit
- (avg premium 4.7 cents) * 103,862 = 4,853$
Trump in Rho

- Agent: Premium = 4 cents, Limit 500 shares, Lot size 100 shares, Aggressiveness = 1
- Market: 50,000 events in Rho
- Results: 103,862 shares traded, 4.15% market participation, 151$ profit
- (avg premium 4.7cents) * 103,862=4,853$
- Adverse selection loss = 4701 $
Autocorrelation in the order flow

**Trump in Rho**

![Graphs of Price, Inventory, and P&L over time](image)

- **Price**
- **Inventory**
- **P&L**
Summary

- Prices depends on the trader’s inventory
Summary

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- The indifference price relative to the inventory is given by:

\[ r(s, q, t) = s - q \gamma \sigma^2 (T - t) \]
Summary

- Prices depends on the trader’s inventory
- The indifference price relative to the inventory
  \[ r(s, q, t) = s - q \gamma \sigma^2 (T - t) \]
- Compute the optimal bid and ask prices
  \[ p^b = r - \frac{1}{\gamma} \ln \left( 1 + \gamma \frac{\lambda^b}{\partial \lambda^b / \partial p} \right) \quad p^a = r + \frac{1}{\gamma} \ln \left( 1 - \gamma \frac{\lambda^a}{\partial \lambda^a / \partial p} \right) \]
Summary

- Prices depends on the trader’s inventory
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\[ r(s, q, t) = s - q \gamma \sigma^2 (T - t) \]

- Compute the optimal bid and ask prices

\[
\begin{align*}
p^b &= r - \frac{1}{\gamma} \ln \left( 1 + \gamma \frac{\lambda^b}{\partial r} \right) \\
p^a &= r + \frac{1}{\gamma} \ln \left( 1 - \gamma \frac{\lambda^a}{\partial r} \right)
\end{align*}
\]

- Order book simulations:
  - We model an order book as a continuous-time Markov chain
  - The simulation environment allows us to test market makers in different market environments
Current and future research

1. Generalize the market maker’s problem for
   - Multiple stocks
   - Multiple options
Current and future research

1. Generalize the market maker’s problem for
   - Multiple stocks
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2. Problems where the market maker can
   - Adjust quantities at the bid and the ask
   - Submit orders strategically
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2. Problems where the market maker can
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3. Modeling adverse selection:
   - Jumps in stock price
   - Correlation between the stock returns and inventory positions
   - Autocorrelation in the sign of market orders