Approximation of a generalized CSBP with interaction

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CSBP

CSBP

Let $\beta \in \mathbb{R}$, $c \geq 0$ and μ is a σ-finite measure on $(0, \infty)$ which satisfies

$$\textbf{(H1)} : \quad \int_0^\infty (1 \wedge z^2)\mu(dz) < \infty$$

Definition (CSBP) Jirina 1958

To each triplet (β, c, μ), we can associate a CSBP Y^x, which is the unique non negative strong solution of a SDE

$$\begin{aligned} Y_t^x =& x + \beta \int_0^t Y_r^x dr + \sqrt{2c} \int_0^t \int_0^{Y_{r-}^x} W(dr, du) \\ &+ \int_0^t \int_0^1 \int_0^{Y_{r-}^x} z \, \overline{M}(ds, dz, du) + \int_0^t \int_1^\infty \int_0^{Y_{r-}^x} z \, M(ds, dz, du) \end{aligned} \tag{1}$$

CSBP

where

- W is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}_+$,
- $M(dr, dz, du)$ is a Poisson random measure with mean measure $ds\mu(dz)du$ independent of W.

The branching mechanism is given by

$$\psi(\lambda) = -\beta\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}})\mu(dr).$$

We assume that Y^x does not explode, which equivalent to

$$\textbf{(H2)} : \quad \int_{0+} \frac{d\lambda}{|\psi(\lambda)|} = +\infty.$$

CSBP

In this work, we prove that a properly renormalized continuous time branching process with interaction converges to a generalized CSBP solution of the SDE

$$
\begin{aligned}
Z_t^x = & x + \int_0^t f(Z_r^x)dr + \sqrt{2c}\int_0^t \int_0^{Z_r^x} W(dr, du) \\
& + \int_0^t \int_0^1 \int_0^{Z_{r-}^x} z\,\overline{M}(ds, dz, du) + \int_0^t \int_1^\infty \int_0^{Z_{r-}^x} z\,M(ds, dz, du),
\end{aligned} \tag{2}
$$

where f is a nonlinear function satisfies :
Assumption(H3) : $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R})$, $f(0) = 0$, and

$$f(x+y) - f(x) \le \beta y, \quad \forall x, y \ge 0.$$

The model

The model

Let ν be a finite measure on $\mathbb{Z}_+ = \{0, 1, 2, ...\}$, satisfying $\nu(1) = 0$.

- Consider a population evolving in continuous time with *m* ancestors at time $t = 0$, in which :
  - each individual lives for an exponential time with parameter $\nu(\mathbb{Z}_+)$,
  -And is replaced by a random number of children according to the probability $\nu(\mathbb{Z}_+)^{-1}\nu$.
- For each individual we superimpose additional birth and death rates due to interactions.

The model

The total mass process X^m, which starts from m at time $t = 0$, is a Markov process whose evolution can be described as follows. X_t^m jumps from k to

$$\begin{cases} k+\ell-1, \ \text{at rate } \nu(\ell)k \ + \ \mathbf{1}_{\{\ell=2\}} \sum_{j=1}^{k}[\Delta(f(j))]^+, \ \ell \geq 2; \\ \\ k-1, \quad \text{at rate } \nu(0)k \ + \ \sum_{j=1}^{k}[\Delta(f(j))]^-, \end{cases}$$

where $\Delta(f(j)) = f(j) - f(j-1)$.

Renormalization

Preliminaries

- Let us define ψ_1 and $\psi_2 \in C([0,+\infty))$ by

$$\psi_1(u) = \int_0^1 (e^{-uz} - 1 + uz)\mu(dz) \text{ and } \psi_2(u) = \int_1^\infty (e^{-uz} - 1)\mu(dz),$$

where μ satisfies **(H1)**.

- We set

$$h_{1,N}(s) = s + \frac{\psi_1(N(1-s))}{N\psi_1'(N)} =: \sum_{\ell \geq 0} \pi_{1,N}(\ell) s^\ell$$

and

$$h_{2,N}(s) = s - \frac{\psi_2(N(1-s))}{N\psi_2'(N)} =: \sum_{\ell \geq 0} \pi_{2,N}(\ell) s^\ell,$$

$|s| \leqslant 1$.

Preliminaries

- Let us define

$$d_{1,N} = \psi_1'(N), \quad d_{2,N} = -\psi_2'(N) \quad \text{and} \quad d_N = 2cN + d_{1,N} + d_{2,N}.$$

- Let π be the probability measure defined by $\pi = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$.
- For any $\ell \geq 0$, we define

$$\nu_N(\ell) = \frac{1}{d_N}\left[2cN\pi(\ell) + d_{1,N}\pi_{1,N}(\ell) + d_{2,N}\pi_{2,N}(\ell)\right].$$

Renormalization

- We now choose $m = [Nx]$, and $\nu(\ell) = d_N \nu_N(\ell)$ for all $\ell \geq 0$
- We multiply f by N and divide its argument by N
- We attach to each individual in the population a mass equal to $1/N$.

Renormalization

The total mass process $Z^{N,x}$, which starts from $[Nx]/N$ at time $t = 0$, is a Markov process whose evolution can be described as follows. $Z^{N,x}$ jumps from k/N to

$$\begin{cases} \frac{k+\ell-1}{N} \text{ at rate } d_N \nu_N(\ell) k + N \mathbf{1}_{\{\ell=2\}} \sum_{i=1}^{k} [\Delta(f(j/N))]^+, \ \ell \geq 2; \\ \frac{k-1}{N} \quad \text{at rate } d_N \nu_N(0) k \ + \ N \sum_{i=1}^{k} [\Delta(f(j/N))]^-. \end{cases}$$

where $\Delta(f(j)) = f(j) - f(j-1)$.

Renormalization

From this, $Z^{N,x}$ can be expressed as

$$
\begin{aligned}
Z_t^{N,x} =& \frac{[Nx]}{N} + \frac{1}{N}\int_0^t \int_{\mathbb{Z}_+} \int_0^{NZ_{r^-}^{N,x}} (z-1) M^N(dr, dz, du) \\
&+ \frac{1}{N} P_1\left(\int_0^t \left\{ N \sum_{i=1}^{NZ_r^{N,x}} \Big(\Delta(f(j/N)) \Big)^+ \right\} dr \right) \\
&- \frac{1}{N} P_2\left(\int_0^t \left\{ N \sum_{i=1}^{NZ_r^{N,x}} \Big(\Delta(f(j/N)) \Big)^- \right\} dr \right).
\end{aligned}
\tag{3}
$$

M^N is a Poisson random measure on $(0,\infty) \times \mathbb{Z}_+ \times (0,\infty)$, with mean measure $d_N ds \nu_N(dz) du$, P_1 and P_2 are two standard Poisson processes, such that M^N, P_1 and P_2 are independent.

Convergence Result

Main result

Theoreme

Suppose that Assumptions **(H1)**, **(H2)** and **(H3)** are satisfied. Then for all $n \geq 1$, $0 < x_1 < x_2 < \cdots < x_n$,

$$(Z_\cdot^{N,x_1}, Z_\cdot^{N,x_2}, \cdots, Z_\cdot^{N,x_n}) \Rightarrow (Z_\cdot^{x_1}, Z_\cdot^{x_2}, \cdots, Z_\cdot^{x_n})$$

in $\mathcal{D}([0,\infty);\mathbb{R}^n)$, as $N \to \infty$, where $\{Z_t^x,\ t \geq 0,\ x \geq 0\}$ is the unique solution of the SDE (2).

Let us discuss only the convergence of $Z^{N,x}$ for a fixed $x > 0$.

Idea of proof (Tightness)

Let $\{\tau_N,\ N \geq 1\}$ be an arbitrary sequence of $[0, T]$-valued stopping times.

A) For all $T, \epsilon > 0$, there exists $k_\epsilon > 0$ such that

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} Z_t^{N,x} > k_\epsilon\right) \leq \epsilon.$$

B) For any $T > 0$, and $\eta,\ \epsilon > 0$, there exists $\theta_0 > 0$ such that

$$\sup_{N\geq 1}\ \sup_{0\leq\theta\leq\theta_0} \mathbb{P}\left(\left|Z_{\tau_N+\theta}^{N,x} - Z_{\tau_N}^{N,x}\right| \geq \eta\right) \leq \epsilon.$$

- The only difficulty is to prove *A*, then *B* is easily deduced.

Idea of proof (Tightness)

Let $Y^{N,x}$ be the Markov process which starts from $[Nx]/N$ at time $t = 0$, and evolves as follows $Y^{N,x}$ jumps from k/N to

$$\begin{cases} \frac{k+\ell-1}{N} \text{ at rate } \{(2cN + d_N)\nu_N(\ell) + \beta \mathbf{1}_{\{\ell=2\}}\}k, \text{ for all } \ell \geq 2, \\ \frac{k-1}{N} \quad \text{at rate } (2cN + d_N)\nu_N(0)k. \end{cases}$$

$Y^{N,x}$ is obtained from $Z^{N,x}$ by replacing $f(z)$ by βz.

Idea of proof (Tightness)

We prove
$P1$) For all $T, \epsilon > 0$, there exists $k_\epsilon > 0$ such that

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} Y_t^{N,x} > k_\epsilon\right) \leq \frac{\epsilon}{2}.$$

-It is easy to see that $P1$ implies A, since
$\sup_{0\leq t\leq T} Z_t^{N,x} \leq \sup_{0\leq t\leq T} Y_t^{N,x}$, stochastically.
To prove $P1$, we will proceed in three steps :
First step : : It is not hard to prove : For all $T > 0$, $x \geq 0$, for all $\lambda \geq 0$,

$$\mathbb{E}\left(e^{-\lambda Y_T^{N,x}}\right) \to \mathbb{E}\left(e^{-\lambda Y_T^{x}}\right),$$

as $N \to \infty$, where Y^x is the unique solution of the SDE (1).

Idea of proof (Tightness)

Combining this with the Portmanteau theorem, we have

$$\forall\ M_\epsilon > 0, \quad \limsup_{N\to\infty} \mathbb{P}\left(Y_T^{N,x} \geq M_\epsilon\right) \leq \mathbb{P}\left(Y_T^x \geq M_\epsilon\right). \tag{4}$$

Since from **(H3)** $Y_T^x < \infty$ *a.s*, we can choose M_ϵ such that

$$\mathbb{P}\left(Y_T^x \geq M_\epsilon\right) \leq \frac{\epsilon}{4}. \tag{5}$$

Second step : : We next define the process $\bar{Y}^{N,x}$ in the same way as $Y^{N,x}$ with the measure μ replaced by $\bar{\mu} = \mu\mathbf{1}_{[0,1]}$.
It is not hard to establish :
For all $T > 0$, $x \geq 0$, there exists a constant $C_1 > 0$ such that for all $N \geq 1$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \bar{Y}_t^{N,x}\right) \leq C_1.$$

Idea of proof (Tightness)

Third step : : Now, switching from $Y^{N,x}$ to $\bar{Y}^{N,x}$ consists in removing some of the positive jumps of $Y^{N,x}$. So, the time reversed process $\bar{K}_t^{N,x} = \bar{Y}_{T-t}^{N,x}$ behaves as $K_t^{N,x} = Y_{T-t}^{N,x}$, with some negative jumps deleted. Consequently

$$\mathbb{P}\left(\sup_{0\leq t\leq T} Y_t^{N,x} > k_\epsilon \middle| \; Y_T^{N,x} < M_\epsilon\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T} \bar{Y}_t^{N,x} > k_\epsilon \middle| \; \bar{Y}_T^{N,x} < M_\epsilon\right).$$

Combining this with the two first steps, it is easy to deduce $P1$.

Idea of proof (Convergence)

- By A and B, we may as well assume that $\{Z_t^{N,x},\ t \geq 0\}$ converges weakly to a process $\{Z_t^x,\ t \geq 0\}$ for the Skorohod topology of $\mathcal{D}([0,\infty);\mathbb{R}_+)$.
- Since the solution of the martingale problem of (2) is unique, it suffices to prove that the weak limit point $\{Z_t^x,\ t \geq 0\}$ of the sequence $\{Z_t^{N,x},\ t \geq 0\}$ is the solution of the martingale problem.

Thank you for your kind attention !