Back to Lotka Volterra

based on Benaim & Lobry, AAP 2016

$$(\dot{X}, \dot{Y}) = F_{\mathcal{E}_{u(t)}}(X, Y)$$

$$F_{\mathcal{E}_u}(x, y) = \begin{cases} x\alpha_u(1 - a_u x - b_u y) \\ y\beta_u(1 - c_u x - d_u y) \end{cases}$$

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- $u(t) \in \{0, 1\}$ jump process with rates

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- Environments $\mathcal{E}_0, \mathcal{E}_1$ are both *favorable to species* **x** :

$$a_u < c_u \text{ and } b_u < d_u.$$

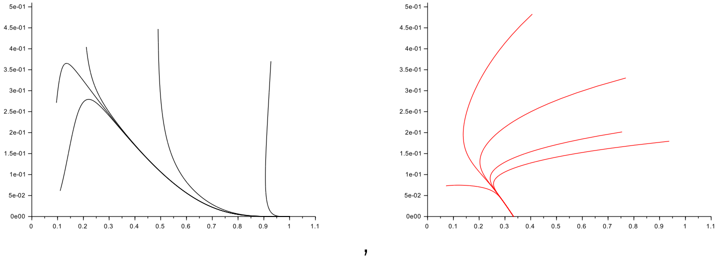

Figure: Phase portraits of $F_{\mathcal{E}_0}$ and $F_{\mathcal{E}_1}$

- State space $M = \{x, y \geq 0 : \; x + y \geq \eta\} \times \{0, 1\}$

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- The process $Z_t = (X_t, Y_t, u(t))$ is a *Piecewise Deterministic Markov Process* on M.

Extinction set

- Extinction set $M_0 = M_0^{\mathbf{x}} \cup M_0^{\mathbf{y}}$

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Easily implies $\rightsquigarrow$

Proposition

The process $\{Z_t\}$ restricted to $M_0^{\mathbf{y}}$ has a unique invariant probability μ given as :

$$\mu(dx,1) = h_1(x)\mathbf{1}_{[p_1,p_0]}(dx)dx,$$

$$\mu(dx,0) = h_0(x)\mathbf{1}_{[p_1,p_0]}(dx)dx$$

where

$$h_1(x) = C\frac{p_1|x-p_1|^{\gamma_1-1}|p_0-x|^{\gamma_0}}{\alpha_1 x^{1+\gamma_0+\gamma_1}},$$

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Where

$$p_i = \frac{1}{a_i}, \gamma_i = \frac{\lambda_i}{\alpha_i}.$$

Invasion rate of species **y**

The invasion rate of **y** is the **growth rate of y averaged over** μ :

Invasion rate of species $\mathbf{y}$

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$$= C[\int_{p_1}^{p_0} \beta_0(1-c_0x)\frac{p_0|x-p_1|^{\gamma_1}|p_0-x|^{\gamma_0-1}}{\alpha_0 x^{1+\gamma_0+\gamma_1}}dx$$

$$+ \int_{p_1}^{p_0} \beta_1(1-c_1x)\frac{p_1|x-p_1|^{\gamma_1-1}|p_0-x|^{\gamma_0}}{\alpha_1 x^{1+\gamma_0+\gamma_1}}dx]$$

Properties of Λ_y

For all $0 \le s \le 1$, let $\mathcal{E}_s = (\alpha_s, a_s, b_s, \beta_s, c_s, d_s)$ be the environment defined by

$$sF_{\mathcal{E}_1} + (1-s)F_{\mathcal{E}_0} = F_{\mathcal{E}_s}$$

Set

$$I = \{0 < s < 1 : \; a_s > c_s\}, J = \{0 < s < 1 : \; b_s > d_s\}.$$

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$\bullet I$ (respectively J) is either empty or is an open interval which closure is contained in $]0,1[$.

$$\bullet s \in I^c \cap J^c \Rightarrow \mathcal{E}_s \in \mathrm{Env}_{\mathbf{x}}$$

$$\bullet s \in I \cap J \Leftrightarrow \mathcal{E}_s \in \mathrm{Env}_{\mathbf{y}}$$

Conjecture

If $I =]s_0, s_1[\neq \emptyset$ *the set*

$$\{(p, \lambda) \in]0,1[\times \mathbb{R}_+^* \ : \Lambda_{\mathbf{y}}(p\lambda, (1-p)\lambda) = 0\}$$

is the graph of a **smooth** *function*

$$I \mapsto \mathbb{R}_+^*, p \mapsto \lambda(p)$$

with $\lim_{p \to s_0} \lambda(p) = \lim_{p \to s_1} \lambda(p) = \infty$.

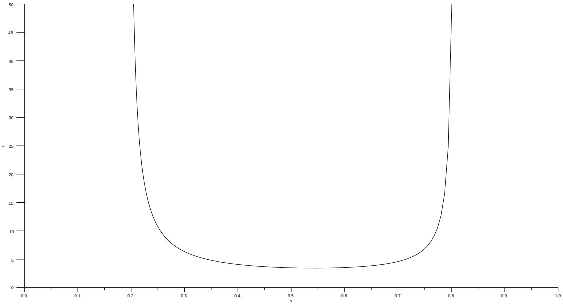

Figure: Zero set of $\Lambda_y(p\lambda, (1-p)\lambda)$

Proposition (Malrieu and Zitt, Arxiv February 2016)

If $I =]s_0, s_1[\neq \emptyset$ *the set*

$$\{(p,\lambda) \in]0,1[\times \mathbb{R}_+^* \ : \Lambda_{\mathbf{y}}(p\lambda, (1-p)\lambda) = 0\}$$

is the graph of a **continuous** *function*

$$I \mapsto \mathbb{R}_+^*, p \mapsto \lambda(p)$$

with $\lim_{p\to s_0} \lambda(p) = \lim_{p \to s_1} \lambda(p) = \infty$.

Good may be good

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Theorem

Assume that $\Lambda_{\mathbf{y}} < 0, \Lambda_{\mathbf{x}} > 0$ and $Z_0 = z \in M \setminus M_0$. Then, the following properties hold with probability one:

(a) $\limsup_{t\to\infty} \frac{\log(Y_t)}{t} \leq \Lambda_{\mathbf{y}}$,

(b) *The limit set of $\{X_t, Y_t\}$ equals $[p_0, p_1] \times \{0\}$,*

(c) *$\{\Pi_t\}$ converges weakly to μ, where μ is the invariant probability of Z on $M_0^{\mathbf{y}}$*

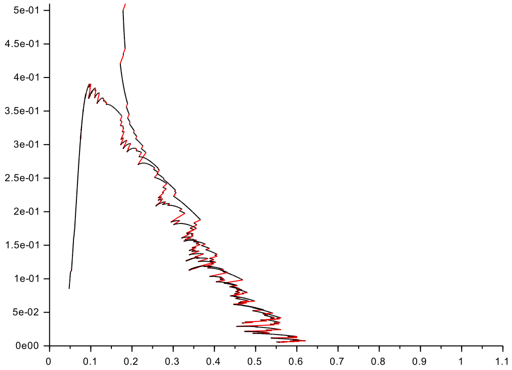

Figure: extinction of 2

Good may be bad

Theorem

Assume that $\Lambda_{\mathbf{x}} < 0, \Lambda_{\mathbf{y}} > 0$*, and* $Z_0 = z \in M \setminus M_0$*. Then, the following properties hold with probability one:*

(a) $\limsup_{t\to\infty} \frac{\log(X_t)}{t} \leq \Lambda_{\mathbf{x}}$,

(b) *The limit set of* $\{X_t, Y_t\}$ *equals* $\{0\} \times [\hat{p}_0, \hat{p}_1]$,

(c) $\{\Pi_t\}$ *converges weakly to* $\hat{\mu}$*, where* $\hat{p}_i = \frac{1}{d_i}$ *and* $\hat{\mu}$ *is the probability on* $M_0^{\mathbf{x}}$ *defined analogously to* μ *(by permuting* α_i *and* β_i*, and replacing* (a_i, c_i) *by* (d_i, b_i)*).*

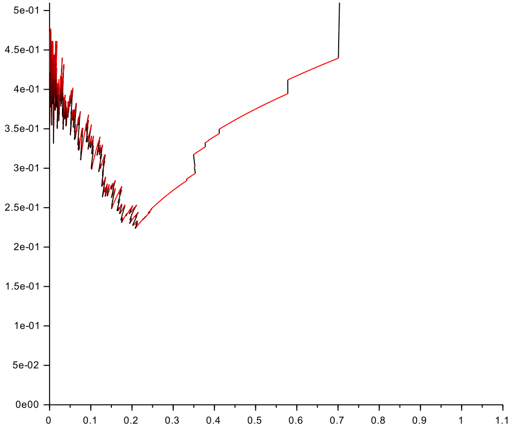

Figure: Extinction of 1

Good may be fair (Persistence)

Theorem

Suppose that $\Lambda_{\mathbf{x}} > 0, \Lambda_{\mathbf{y}} > 0$ Then, there exists a unique invariant probability (for the process $\{Z_t\}$) Π on $M \setminus M_0$ i.e $\Pi(M \setminus M_0) = 1$. Furthermore,

(i) *Π is absolutely continuous with respect to the Lebesgue measure $dxdy \otimes (\delta_0 + \delta_1)$;*

(ii) *There exists $\theta > 0$ such that*

$$\int (\frac{1}{x^\theta} + \frac{1}{y^\theta}) d\Pi < \infty;$$

(iii) *For every initial condition $z = (x, y, i) \in M \setminus M_0$*

$$\lim_{t \to \infty} \Pi_t = \Pi$$

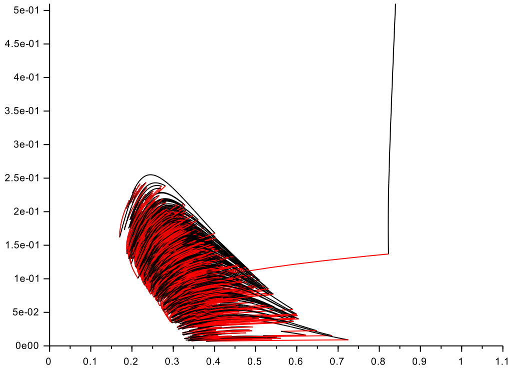

Figure: Persistence

Good may be fair : Exponential convergence

Theorem

Suppose that $\Lambda_{\mathbf{x}} > 0, \Lambda_{\mathbf{y}} > 0$. Then for all (but possibly a positive codimension set of environments) there are constants $C, \gamma, \theta > 0$ such that for every Borel set $A \subset M \setminus M_0$ and every $z = (x, y, i) \in M \setminus M_0$

$$|\mathbf{P}(Z_t \in A | Z_0 = z) - \Pi(A)| \leq C(1 + \frac{1}{x^\theta} + \frac{1}{y^\theta})e^{-\gamma t}.$$

Good may be fair : Properties of the support

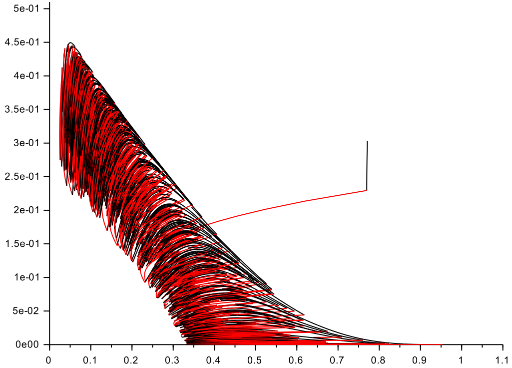

Figure: Extinction of 1 or 2

Good may be fair : Properties of the support

Let $\Psi = \{\Psi_t\}$ the *set valued dynamical system* induced by

$$\dot{\eta}(t) \in conv(F_{\mathcal{E}_0}, F_{\mathcal{E}_1})(\eta(t)) \tag{1}$$

$$\Psi_t(x, y) = \{\eta(t) : \eta \text{ is solution to (1) with initial condition } \eta(0) = (x, y)\}$$

$$\omega_\Psi(x, y) = \bigcap_{t \geq 0} \overline{\Psi_{[t,\infty[}(x, y)}$$

Good may be fair : Properties of the support

Theorem

Under the previous assumptions the topological support of Π *writes* $supp(\Pi) = \Gamma \times \{0,1\}$ *where*

(i) $\Gamma = \omega_\Psi(x,y)$ *for all* $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. *In particular,* Γ *is compact connected strongly positively invariant and invariant under* Ψ;

(ii) Γ *equates the closure of its interior;*

(iii) $\Gamma \cap \mathbb{R}_+ \times \{0\} = [p_0, p_1] \times \{0\}$;

(iv) *If* $I \cap J \neq \emptyset$ *then* $\Gamma \cap \{0\} \times \mathbb{R}_+ = \{0\} \times [\hat{p}_0, \hat{p}_1]$.

(v) $\Gamma \setminus \{0\} \times [\hat{p}_0, \hat{p}_1]$ *is contractible (hence simply connected).*