Epidemic Models

Based on recent work with **Edouard Strickler** (ArXiv, 2017)

Lajmanovich and Yorke SIS Model, 1976

- d groups
- In each group each individual can be infected
- $0 \leq x_i \leq 1 =$ proportion of infected individuals in group i.
- $C_{ij} =$ rate of infection from group i to group j.
- D_i cure rate in group i

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$$\frac{dx_i}{dt} = (1 - x_i)(\sum_j C_{ij} x_j) - D_i x_i$$

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Theorem (Lajmanovich and Yorke 1976)

If $\lambda(A) \leq 0$, *the* disease free equilibrium 0 *is a global attractor*
If $\lambda(A) > 0$ *there exists another equilibrium* $x^* >> 0$ *and every non zero trajectory converges to* x^*

- What if the environment fluctuates ?

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- Example: Two environments

$$C^1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix},\ D^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

and

$$C^2 = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix},\ D^2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

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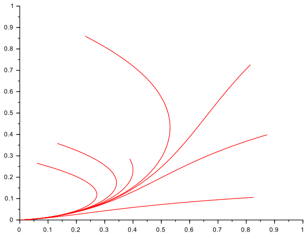
$$\lambda(A^1) = \lambda(A^2) = -1 < 0$$

$$\Rightarrow$$

The disease free equilibrium is a global attractor in each environment



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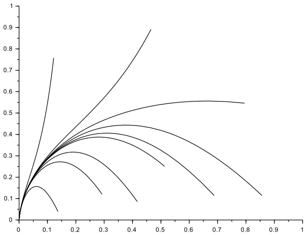

Figure: Phase portraits of F^1 and F^2

Constant switching

$$Q(x) = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, \beta >> 1.$$

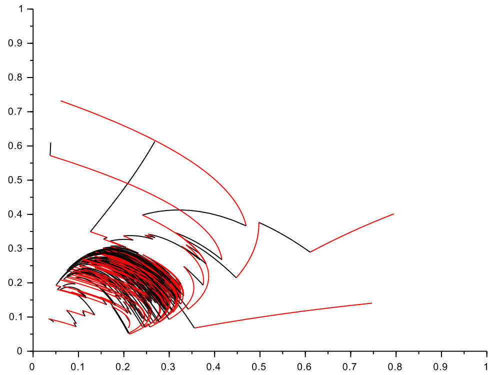

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Figure: Random Switching may reverses the trend

- **More surprising !**

$$A^0 = \begin{pmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{pmatrix},\ A^1 = \begin{pmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{pmatrix}.$$

$$D^0 = \begin{pmatrix} 11 \\ 11 \\ 20 \end{pmatrix},\ D^1 = \begin{pmatrix} 20 \\ 20 \\ 11 \end{pmatrix}$$

$$C^i = A^i + D^i.$$

$$F^{0,1} = \text{ the LY vector field on } [0,1]^3 \text{ induced by } (C^{0,1}, D^{0,1}).$$

$$F^t = (1-t)F^0 + tF^1$$

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• Still, a random switching between the dynamics leads to the persistence of the disease.

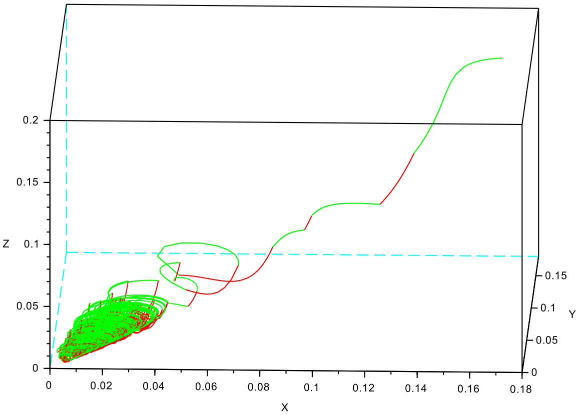

Figure: Simulation of X_t for $\beta = 10$.

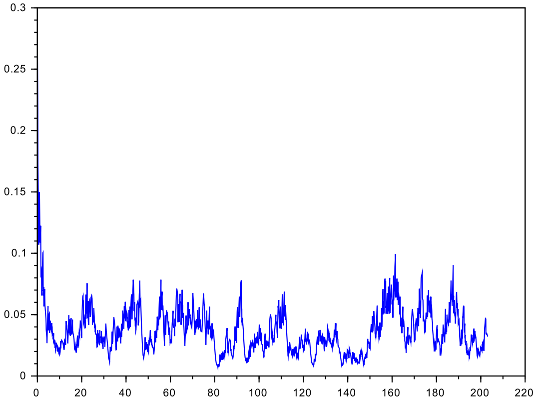

Figure: Simulation of $\|X_t\|$ for $\beta = 10$.

Analysis

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- $E = \{1, \ldots, m\}$,
- $F^1, \ldots, F^m$ smooth vector fields on $\mathbb{R}^d$,

$$F^1(0) = \ldots = F^m(0) = 0.$$

- $0 \in M \subset \mathbb{R}^d$ = compact positively invariant set under each Φ^i,

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$$\dot{X} = F^{I_t}(X),$$

(I_t) jump process controled by X

$$\mathbf{P}(I_{t+s} = j | I_t = i, X_t = x, \mathcal{F}_t) = Q_{ij}(x)t + o(s).$$

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$$\textbf{Replace } \{0_{\mathbb{R}^d}\} \textbf{ by } \{0\} \times S^{n-1}$$

$$\theta_t = \frac{X_t}{\|X_t\|}, \rho_t = \|X_t\|$$

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$$\Rightarrow$$

$$\frac{d\theta}{dt} = \hat{F}^{I_t}(\rho,\theta) - \langle \theta, \hat{F}^{I_t}(\rho,\theta)\rangle\theta.$$

$$\frac{d\rho}{dt} = \rho\langle \hat{F}^{I_t}(\rho,\theta), \theta\rangle$$

where

$$\hat{F}^j(\rho,\theta) = \begin{cases} \dfrac{F^j(\rho\theta)}{\rho} & \text{if } \rho > 0 \\ DF^j(0)\theta & \text{if } \rho = 0 \end{cases}$$

- New state space:

$$M = S^{d-1} \times \mathbb{R}_+ \times E$$

$$M_0 = S^{d-1} \times \{0\} \times E \approx S^{d-1} \times E$$

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- It also induces a PDMP on $\mathbb{P}^{d-1} \times E$, where

$$\mathbb{P}^{d-1} = S^{d-1}/x \sim -x$$

is the projective space

Growth rates

$$V(\theta, \rho, i) = -\log(\rho)$$
$$\Rightarrow H(\theta, \rho, \theta, i) = -\langle \hat{F}^i(\rho, \theta), \theta \rangle$$

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$$H(\theta, 0, i) = -\langle A^i \theta, \theta \rangle$$

Growth rate : For each μ ergodic for (Θ, J)

$$\Lambda(\mu) = \sup \int \langle A^i \theta, \theta \rangle \mu(d\theta di) = -\mu H$$

Maximal growth rates :

$$\Lambda^- = \min_\mu \Lambda(\mu) \leq \Lambda^+ = \sup_\mu \Lambda(\mu).$$

Link with Lyapunov expoents

Proposition (BS 16)

Λ^+ coincides with the top-Lyapounov exponent given by the Multiplicative Ergodic Theorem of the skew product system

$$\frac{dY}{dt} = A^{J_t} Y$$

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Remark

If there exists for (Θ, J) an accessible point (on $\mathbb{P}^{d-1}$) at which the weak bracket condition holds. Then

$$\Lambda^+ = \Lambda^-.$$

Example in dimension 2

Corollary (BS 16)

Suppose $d = 2$ and that either

(a) *One matrix A^i has no real eigenvalues; or*

(b) *Two matrices A^i, A^j have no common eigenvectors*

Then $\Lambda^+ = \Lambda^-$.

Metzler matrices

Proposition (BS, 16)

Assume the matrices are Metzler and at least one convex combination is irreducible. Then $\Lambda^+ = \Lambda^-$

Main results

Extinction : $\Lambda^+ < 0$

Assume $\Lambda^+ < 0$.

Theorem (BS, 16)

There exists a neighborhood $\mathcal{U}$ of 0 and $\eta > 0$ such that for all $x \in \mathcal{U}$ and $i \in E$

$$\mathbb{P}_{x,i}(\limsup_{t\to\infty} \frac{1}{t} \log(\|X_t\|) \leq \Lambda^+) \geq \eta.$$

If furthermore 0 is accessible then for all $x \in M$ and $i \in E$

$$\mathbb{P}_{x,i}(\limsup_{t\to\infty} \frac{1}{t} \log(\|X_t\|) \leq \Lambda^+) = 1.$$

Persistence : $\Lambda^- > 0$.

Assume $\Lambda^- > 0$.

Theorem (BS, 16)

$\forall x \in M^* = M \setminus \{0\}$, $\mathbb{P}_{x,i}$ almost surely, every limit point Π of (Π_t) belongs to $\mathcal{P}_{inv}$ and

$$\Pi(\{0\} \times E) = 0.$$

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More precisely, $\exists \theta, K > 0$ (independent of Π) such that

$$\int \frac{1}{\|x\|^\theta} d\Pi \leq K.$$

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Assume $\Lambda^- > 0$.

$$\tau = \inf\{t \geq 0 : \ \|X_t\| \geq \epsilon\}.$$

Theorem (BS, 16)

$\exists b > 1, c > 0$ *such that* $\forall x \in M^*$

$$\mathbb{E}_x(b^\tau) \leq c(1 + \|x\|^{-\theta}).$$

Persistence : $\Lambda^- > 0$.

Assume $\Lambda^- > 0$. In addition assume $\exists p \in M^*$ **accessible from** M^*

Theorem (BS, 16)

Weak Bracket condition at $p \Rightarrow$

$$\mathcal{P}_{inv} \cap \mathcal{P}(M^* \times E) = \{\Pi\}$$

and $\forall x \in M^*$

$$\lim_{t \to \infty} \Pi_t = \Pi$$

$\mathbb{P}_{x,i}$ *almost surely.*

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Theorem (BS, 16)

Strong Bracket condition at $p \Rightarrow$

$$\forall x \in M^* \parallel \mathbb{P}_{x,i}(Z_t \in \cdot) - \Pi(\cdot) \parallel \leq const(1 + \parallel x \parallel^{-\theta})e^{-\kappa t}$$

for some $\kappa, \theta > 0$.

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- $DF(x)$ is Metzler (off diagonal entries ≥ 0) irreducible
- F is (strongly) sub homogeneous on $]0,1[^d$: $F(tx) << tF(x)$ for $t \geq 1$.

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LY vector field is epidemic

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Theorem (BS, 16)

Suppose the $F^i, i = 1 \ldots m$ are epidemic. Then

(a) $\Lambda^+ = \Lambda^- = \Lambda$

(b) $\Lambda < 0 \Rightarrow$ *almost sure Extinction*

(c) $\Lambda > 0 \Rightarrow$ *Persistence*

Back to the "surprising" example

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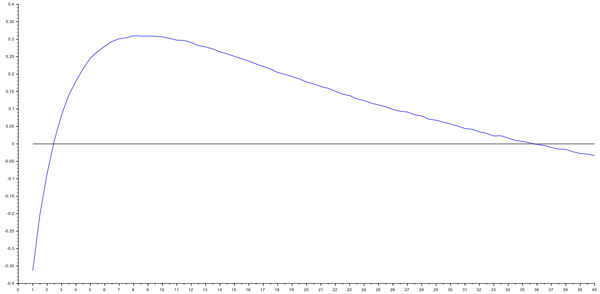
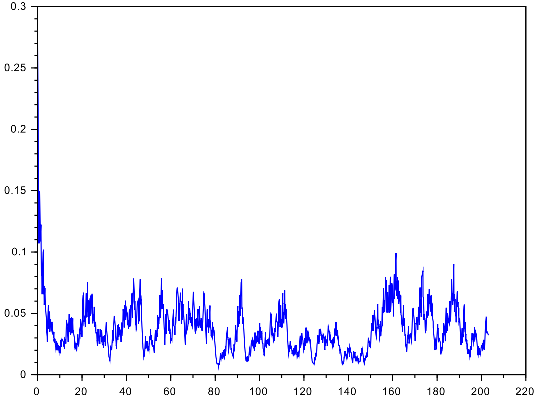


Figure: $\beta \mapsto \Lambda(\beta)$.



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