Spatial Epidemic Models

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Contents

The models

- We consider an infectious disease which spreads in a human population ;
- $D \subset \mathbb{R}^d$ (d=1, 2 or 3) ;
- Grid $D_\varepsilon := [0,1]^d \cap \varepsilon\mathbb{Z}^d$;

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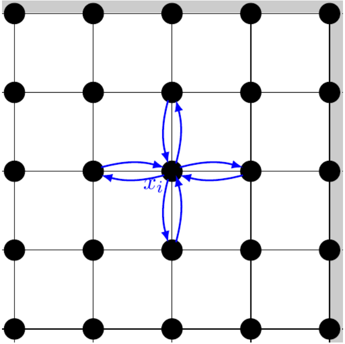

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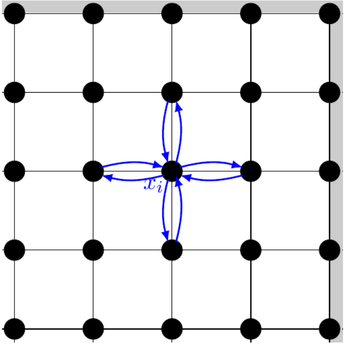
- $D \subset \mathbb{R}^d$ (d=1, 2 or 3) ;
- Grid $D_\varepsilon := [0,1]^d \cap \varepsilon\mathbb{Z}^d$;
- Total population size : $N\varepsilon^{-d}$.



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- $S_\varepsilon(t, x_i)$: susceptible, $I_\varepsilon(t, x_i)$: infectious, $R_\varepsilon(t, x_i)$: removed

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- $S_\varepsilon(t,x_i)$: susceptible, $I_\varepsilon(t,x_i)$: infectious, $R_\varepsilon(t,x_i)$: removed

(1)

$$\begin{cases} \dfrac{d\,S_\varepsilon}{dt}(t,x_i) = \mu_S\,\Delta_\varepsilon S_\varepsilon(t,x_i) - \dfrac{\beta(x_i)\,S_\varepsilon(t,x_i)I_\varepsilon(t,x_i)}{S_\varepsilon(t,x_i)+I_\varepsilon(t,x_i)+R_\varepsilon(t,x_i)} \\ \dfrac{d\,I_\varepsilon}{dt}(t,x_i) = \mu_I\,\Delta_\varepsilon I_\varepsilon(t,x_i) + \dfrac{\beta(x_i)\,S_\varepsilon(t,x_i)I_\varepsilon(t,x_i)}{S_\varepsilon(t,x_i)+I_\varepsilon(t,x_i)+R_\varepsilon(t,x_i)} - \alpha(x_i)\,I_\varepsilon(t,x_i) \\ \dfrac{d\,R_\varepsilon}{dt}(t,x_i) = \mu_R\,\Delta_\varepsilon R_\varepsilon(t,x_i) + \alpha(x_i)\,I_\varepsilon(t,x_i), \quad (t,x_i)\in(0,T)\times D_\varepsilon \\ \left.\begin{array}{r} S_\varepsilon(t,x_i) = S_\varepsilon(t,y_i) \\ I_\varepsilon(t,x_i) = I_\varepsilon(t,y_i) \\ R_\varepsilon(t,x_i) = R_\varepsilon(t,y_i) \end{array}\right\} \text{for } x_i\in\partial D_\varepsilon,\ x_i\sim y_i \text{ and } y_i\in\partial_{\vec{n}.out}D_\varepsilon \\ S_\varepsilon(0), I_\varepsilon(0), R_\varepsilon(0)\geq 0,\ 0 < S_\varepsilon(0)+I_\varepsilon(0)+R_\varepsilon(0)\leq M \end{cases}$$

The limiting system

$$
(2) \quad \begin{cases}
\dfrac{\partial \mathbf{s}}{\partial t}(t,x) = \mu_S \, \Delta \mathbf{s}(t,x) - \dfrac{\beta(x)\, \mathbf{s}(t,x)\mathbf{i}(t,x)}{\mathbf{s}(t,x)+\mathbf{i}(t,x)+\mathbf{r}(t,x)} \\
\dfrac{\partial \mathbf{i}}{\partial t}(t,x) = \mu_I \, \Delta \mathbf{i}(t,x) + \dfrac{\beta(x)\, \mathbf{s}(t,x)\mathbf{i}(t,x)}{\mathbf{s}(t,x)+\mathbf{i}(t,x)+\mathbf{r}(t,x)} - \alpha(x)\, \mathbf{i}(t,x) \\
\dfrac{\partial \mathbf{r}}{\partial t}(t,x) = \mu_S \, \Delta \mathbf{r}(t,x) + \alpha(x)\, \mathbf{i}(t,x), \quad (t,x) \in (0,T) \times D \\
\dfrac{\partial \mathbf{s}}{\partial n_{\text{out}}}(t,x) = \dfrac{\partial \mathbf{i}}{\partial n_{\text{out}}}(t,x) = \dfrac{\partial \mathbf{r}}{\partial n_{\text{out}}}(t,x) = 0, \quad \text{for } x \in \partial D \\
\mathbf{s}(0), \mathbf{i}(0), \mathbf{r}(0) \geq 0,\ 0 < \mathbf{s}(0)+\mathbf{i}(0)+\mathbf{r}(0) \leq M
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\end{cases}
$$

$$X = (\mathbf{s}, \mathbf{i}, \mathbf{r})^T$$

Theorem

$$\mathcal{S}_\varepsilon(t,x) = \sum_{i=1}^{\varepsilon^{-d}} S_\varepsilon(t,x_i)\mathbf{1}_{V_i}(x), \quad \mathcal{I}_\varepsilon(t,x) = \sum_{i=1}^{\varepsilon^{-d}} I_\varepsilon(t,x_i)\mathbf{1}_{V_i}(x),$$

$$\mathcal{R}_\varepsilon(t,x) = \sum_{i=1}^{\varepsilon^{-d}} R_\varepsilon(t,x_i)\mathbf{1}_{V_i}(x), \quad X_\varepsilon = \left(\mathcal{S}_\varepsilon, \mathcal{I}_\varepsilon, \mathcal{R}_\varepsilon\right)^T.$$

Theorem

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Theorem 1.1

Let us consider an initial condition $X(0) \in \left(L^\infty(D)\right)^3$. For all $T > 0$,

$$\sup_{t\in[0,T]} \left\| X_\varepsilon(t) - X(t) \right\|_\infty \longrightarrow 0, \text{ as } \varepsilon \to 0.$$

The stochastic model

(3)

$$
\begin{cases}
S_{N,\varepsilon}(t,x_i) = S_{N,\varepsilon}(0,x_i) - \frac{1}{N}\mathbf{P}^{inf}_{x_i}\left(\int_0^t N\frac{\beta(x_i)S_{N,\varepsilon}(r,x_i)I_{N,\varepsilon}(r,x_i)}{S_{N,\varepsilon}(r,x_i)+I_{N,\varepsilon}(r,x_i)+R_{N,\varepsilon}(r,x_i)}dr\right) \\
\quad - \sum_{y_i\sim x_i}\frac{1}{N}\mathbf{P}^{mig}_{S,x_i,y_i}\left(\int_0^t N\frac{\mu_S}{\varepsilon^2}S_{N,\varepsilon}(r,x_i)dr\right) + \sum_{y_i\sim x_i}\frac{1}{N}\mathbf{P}^{mig}_{S,y_i,x_i}\left(\int_0^t N\frac{\mu_S}{\varepsilon^2}S_{N,\varepsilon}(r,y_i)dr\right) \\
I_{N,\varepsilon}(t,x_i) = I_{N,\varepsilon}(0,x_i) + \frac{1}{N}\mathbf{P}^{inf}_{x_i}\left(\int_0^t N\frac{\beta(x_i)S_{N,\varepsilon}(r,x_i)I_{N,\varepsilon}(r,x_i)}{S_{N,\varepsilon}(r,x_i)+I_{N,\varepsilon}(r,x_i)+R_{N,\varepsilon}(r,x_i)}dr\right) \\
\quad - \frac{1}{N}\mathbf{P}^{rec}_{x_i}\left(\int_0^t N\alpha(x_i)I_{N,\varepsilon}(r,x_i)dr\right) - \sum_{y_i\sim x_i}\frac{1}{N}\mathbf{P}^{mig}_{I,x_i,y_i}\left(\int_0^t N\frac{\mu_I}{\varepsilon^2}I_{N,\varepsilon}(r,x_i)dr\right) \\
\quad + \sum_{y_i\sim x_i}\frac{1}{N}\mathbf{P}^{mig}_{I,y_i,x_i}\left(\int_0^t N\frac{\mu_I}{\varepsilon^2}I_{N,\varepsilon}(r,y_i)dr\right), \\
(t,x_i)\in[0,T]\times D_\varepsilon.
\end{cases}
$$

Law of Large Numbers ($N \to \infty$, ε being fixed)

Theorem 2.1

Let $Z_{N,\varepsilon}$ denote the solution of the SDE and Z_ε the solution of the ODE

$$\frac{dZ_\varepsilon(t)}{dt} = b_\varepsilon(t, Z_\varepsilon(t)).$$

Let us fix an arbitrary $T > 0$ and assume that $\left\| Z_{N,\varepsilon}(0) - Z_\varepsilon(0) \right\| \longrightarrow 0$, as $N \to +\infty$.

Then $\sup_{0 \le t \le T} \left\| Z_{N,\varepsilon}(t) - Z_\varepsilon(t) \right\| \longrightarrow 0$ *a.s. , as* $N \to +\infty$.

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- $\{ \, k/N, \;\; 0 \le k \le N \, \}$

LLN in sup-norm : $N \to \infty$ & $\varepsilon \to 0$

LLN in sup-norm : $N \to \infty$ & $\varepsilon \to 0$

$$\mathcal{S}_{N,\varepsilon}(t,x) = \sum_{i=1}^{\varepsilon^{-d}} S_{N,\varepsilon}(t,x_i)\mathbf{1}_{V_i}(x),$$

$$\mathcal{I}_{N,\varepsilon}(t,x) = \sum_{i=1}^{\varepsilon^{-d}} I_{N,\varepsilon}(t,x_i)\mathbf{1}_{V_i}(x),$$

$$\mathcal{R}_{N,\varepsilon}(t,x) = \sum_{i=1}^{\varepsilon^{-d}} R_{N,\varepsilon}(t,x_i)\mathbf{1}_{V_i}(x),$$

$$X_{N,\varepsilon} = (\mathcal{S}_{N,\varepsilon}, \mathcal{I}_{N,\varepsilon}, \mathcal{R}_{N,\varepsilon})^T.$$

- Recall that $X_\varepsilon = (\mathcal{S}_\varepsilon, \mathcal{I}_\varepsilon, \mathcal{R}_\varepsilon)^T$ and $X = (\mathbf{s}, \mathbf{i}, \mathbf{r})^T$.

Theorem 3.1 (Law of Large Numbers in Sup-norm)

Let us assume that $N \to \infty$ *and* $\varepsilon \to 0$, *in such way that*

(i) $\dfrac{N}{\log(1/\varepsilon)} \longrightarrow \infty$;

(ii) $\left\| X_{N,\varepsilon}(0) - X(0) \right\|_\infty \longrightarrow 0$ *in probability.*

Then for all $T > 0$, $\sup_{t \in [0,T]} \left\| X_{N,\varepsilon}(t) - X(t) \right\|_\infty \longrightarrow 0$ *in probability .*

Law of Large Numbers : $N \to \infty$ & $\varepsilon \to 0$

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(i) : D. Blount (1992).

Central Limit Theorem

$$\Psi_{N,\varepsilon}(t) = \begin{pmatrix} U_{N,\varepsilon}(t) \\ V_{N,\varepsilon}(t) \\ W_{N,\varepsilon}(t) \end{pmatrix},$$

where

$$(*) \quad U_{N,\varepsilon}(t) = \begin{pmatrix} \sqrt{N}\Big(S_{N,\varepsilon}(t,x_1) - S_\varepsilon(t,x_1)\Big) \\ \vdots \\ \sqrt{N}\Big(S_{N,\varepsilon}(t,x_\ell) - S_\varepsilon(t,x_\ell)\Big) \end{pmatrix}$$

CLT : $N \to \infty$, ε being fixed

Theorem 4.1

Assume that

(i) $\left\|Z_{N,\varepsilon}(0) - Z_\varepsilon(0)\right\| \longrightarrow 0$, *as* $N \to +\infty$ *and*

(ii) $\sqrt{N}\Big(Z_{N,\varepsilon}(0) - Z_\varepsilon(0)\Big) \longrightarrow 0$, *as* $N \to \infty$.

Then, as $N \to +\infty$, $\{\ \Psi_{N,\varepsilon}(t),\ t \geq 0\ \} \Longrightarrow \{\ \Psi_\varepsilon(t),\ t \geq 0\ \}$, *for the topology of locally uniform convergence, where the limit process*

$\Psi_\varepsilon(t) = \begin{pmatrix} U_\varepsilon(t) \\ V_\varepsilon(t) \\ W_\varepsilon(t) \end{pmatrix}$ *satisfies,* $t \geq 0$

$$(4) \quad \Psi_\varepsilon(t) = \int_0^t \nabla_z b_\varepsilon\Big(r, Z_\varepsilon(r)\Big).\Psi_\varepsilon(r)dr + \sum_{j=1}^{k_\varepsilon} \int_0^t \sqrt{\beta_j\Big(r, Z_\varepsilon(r)\Big)}dB_j(r).$$

Limit as $\varepsilon \to 0$

$$\mathbb{U}_\varepsilon(t,x) = \frac{1}{\varepsilon^{d/2}} \sum_{i=1}^{\varepsilon^{-d}} U_\varepsilon(t,x_i)\mathbf{1}_{V_i}(x),$$

$$\mathbb{V}_\varepsilon(t,x) = \frac{1}{\varepsilon^{d/2}} \sum_{i=1}^{\varepsilon^{-d}} V_\varepsilon(t,x_i)\mathbf{1}_{V_i}(x),$$

$$\mathbb{W}_\varepsilon(t,x) = \frac{1}{\varepsilon^{d/2}} \sum_{i=1}^{\varepsilon^{-d}} W_\varepsilon(t,x_i)\mathbf{1}_{V_i}(x),$$

and $\widetilde{\Psi}_\varepsilon(t) = \begin{pmatrix} \mathbb{U}_\varepsilon(t) \\ \mathbb{V}_\varepsilon(t) \\ \mathbb{W}_\varepsilon(t) \end{pmatrix}$.

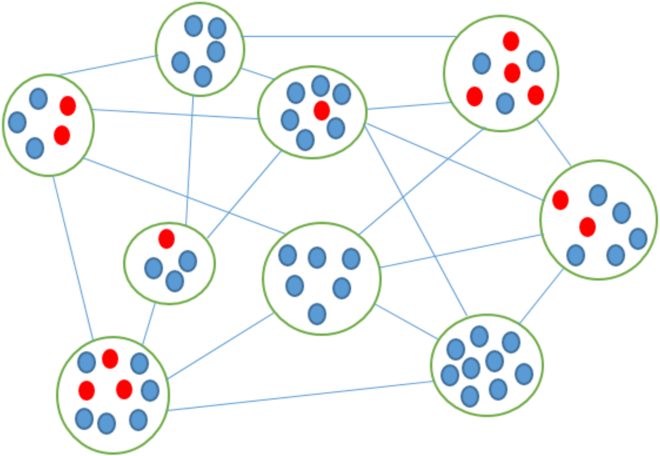
Limit as $\varepsilon \to 0$?

$$\alpha > \frac{d}{2} + 1, \ \{ \widetilde{\Psi}_\varepsilon(t), t \geq 0 \} \Longrightarrow \{ \widetilde{\Psi}(t), t \geq 0 \} \text{ in } C\Big([0, T] \,;\, (H_{-\alpha})^3\Big),$$

where the limit $\widetilde{\Psi}$ is solution of the stochastic differential equation

$$(5) \qquad \frac{\partial \widetilde{\Psi}}{\partial t}(t,x) = \nabla_x b\Big(t, X(t,x)\Big).\widetilde{\Psi}(t) + d\mathcal{M}(t,x)$$

SIS Epidemic Patch Model



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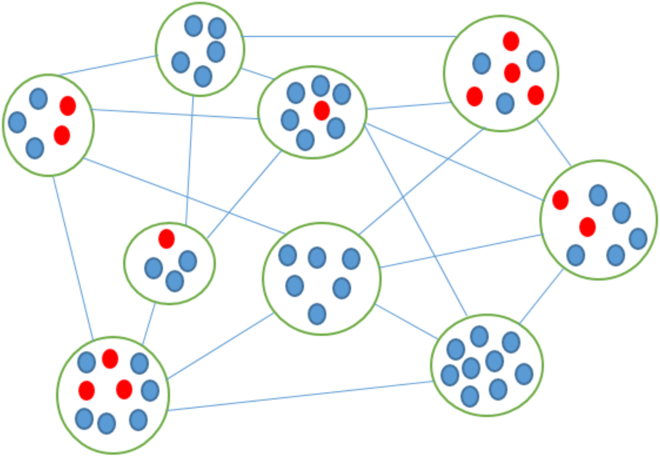

- Total population size : N

The deterministic model

$$(6) \quad \begin{cases} \dfrac{dS}{dt}(t,x_j) = -\lambda_j \dfrac{S(t,x_j)I(t,x_j)}{S(t,x_j)+I(t,x_j)} + \gamma_j\, I(t,x_j) \\ \qquad\qquad + \nu_S \displaystyle\sum_{k=1}^{\ell} \Big(a_{kj} S(t,x_k) - a_{jk} S(t,x_j) \Big) \\ \dfrac{dI}{dt}(t,x_j) = \lambda_j \dfrac{S(t,x_j)I(t,x_j)}{S(t,x_j)+I(t,x_j)} - \gamma_j\, I(t,x_j) \\ \qquad\qquad + \nu_I \displaystyle\sum_{k=1}^{\ell} \Big(a_{kj} I(t,x_k) - a_{jk} I(t,x_j) \Big) \\ \quad j = 1, \cdots, \ell. \end{cases}$$

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The deterministic model

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(6) \qquad \begin{cases}
\dfrac{dS}{dt}(t, x_j) = -\lambda_j \dfrac{S(t, x_j) I(t, x_j)}{S(t, x_j) + I(t, x_j)} + \gamma_j\, I(t, x_j) \\
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\qquad\qquad + \nu_I \displaystyle\sum_{k=1}^{\ell} \Big(a_{kj} I(t, x_k) - a_{jk} I(t, x_j) \Big) \\
\quad j = 1, \cdots, \ell.
\end{cases}
$$

- Allen et al. (2007)

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The stochastic model

(7)

$$
\begin{cases}
S_N(t,x_j) = S_N(0,x_j) - \frac{1}{N}\mathbf{P}^{inf}_{x_j}\left(\int_0^t N\lambda_j \frac{S_N(t,x_j)I_N(t,x_j)}{S_N(t,x_j)+I_N(t,x_j)}dr\right) \\
\quad + \frac{1}{N}\mathbf{P}^{rec}_{x_j}\left(\int_0^t N\gamma_j I_N(r,x_j)dr\right) - \sum_{k=1}^{\ell}\frac{1}{N}\mathbf{P}^{mig}_{S,x_j,x_k}\left(\int_0^t N\nu_S a_{jk} S_N(r,x_j)dr\right) \\
\quad + \sum_{k=1}^{\ell}\frac{1}{N}\mathbf{P}^{mig}_{S,x_k,x_j}\left(\int_0^t N\nu_S a_{kj} S_N(r,x_k)dr\right) \\
I_N(t,x_j) = I_N(0,x_j) + \frac{1}{N}\mathbf{P}^{inf}_{x_j}\left(\int_0^t N\lambda_j \frac{S_N(t,x_j)I_N(t,x_j)}{S_N(t,x_j)+I_N(t,x_j)}dr\right) \\
\quad - \frac{1}{N}\mathbf{P}^{rec}_{x_j}\left(\int_0^t N\gamma_j I_N(r,x_j)dr\right) - \sum_{k=1}^{\ell}\frac{1}{N}\mathbf{P}^{mig}_{I,x_j,x_k}\left(\int_0^t N\nu_I a_{jk} I_N(r,x_j)dr\right) \\
\quad + \sum_{k=1}^{\ell}\frac{1}{N}\mathbf{P}^{mig}_{I,x_k,x_j}\left(\int_0^t N\nu_I a_{kj} I_N(r,x_k)dr\right), \quad j = 1,\cdots,\ell.
\end{cases}
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LLN

Theorem 5.1

Let Z_N *denote the solution of the SDE (7) and* Z *the solution of the ODE*
$$\frac{dZ(t)}{dt} = b(t, Z(t)).$$
Let us fix an arbitrary $T > 0$ *and assume that* $\Big\|Z_N(0) - Z(0)\Big\| \longrightarrow 0$*, as* $N \to +\infty$.
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Then $\sup_{0 \le t \le T} \left\| Z_N(t) - Z(t) \right\| \longrightarrow 0$ *a.s. , as* $N \to +\infty$.

- $a_{kj} = a$, $\nu_S = \nu_I$, $\lambda_j = \lambda$, $\gamma_j = \gamma$, for all k, j

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LLN

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- $a_{kj} = a$, $\nu_S = \nu_I$, $\lambda_j = \lambda$, $\gamma_j = \gamma$, for all k, j

$$(**) \qquad Var\left(\sum_{i=1}^{\ell} I_N(t, x_i) \right) \approx \frac{1}{N} \frac{\gamma}{\lambda} \left(1 - e^{-2t(\lambda - \gamma)}\right).$$

Set-up

- For $\phi \in \mathcal{AC}_{T,2\ell}$, $\mathcal{A}_{2\ell}(\phi)$ denotes the set of functions $c \in L^1(0, T\,;\, \mathbb{R}_+^k)$ such that

$$(8)\quad c_t^j = 0 \text{ on the set } \Big\{\, t,\ \beta_j(\phi) = 0 \Big\} \text{ and } \frac{d\phi}{dt} = \sum_{j=1}^{k} c_t^j(t) h_j.$$

We define the rate function

$$(9)\qquad I_T(\phi) := \begin{cases} \inf\limits_{c \in \mathcal{A}_k(\phi)} I_T(\phi \mid c), & \text{if } \phi \in \mathcal{AC}_{T,2\ell} \\ \infty, & \text{otherwise,} \end{cases}$$

where

$$(10)\qquad I_T(\phi \mid c) = \int_0^T \sum_{j=1}^{k} g(c_t^j, \beta_j(\phi_t))\, dt,$$

$$(11)\qquad g(x, y) = x \log(x/y) - x + y.$$

Large Deviations Principle

Theorem 5.2 (Lower bound)

Let set $Z_N(0) = z_{_N}$ *and* $Z(0) = z$
For every open set $O \subset \mathbb{D}([0,T]\,;\,\mathbb{R}^{2\ell})$, *if* $z_N \longrightarrow z$ *as* $N \to \infty$, *then*

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(Z_N^{z_N} \in O) \geq - \inf_{\phi \in O, \phi_0 = z} I_T(\phi). \tag{12}$$

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Theorem 5.3 (Upper bound)

For every closed set $F \subset \mathbb{D}\Big([0, T]\,;\,\mathbb{R}^{2\ell}\Big)$, *if* $z_N \longrightarrow z$ *as* $N \longrightarrow \infty$, *then*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(Z_N^{z_N} \in F) \leq - \inf_{\phi\in F, \phi_0 = z} I_T(\phi). \tag{13}$$

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Time of extinction

(14)
$$A = \left\{ (u_i)_{1\leq i\leq 2\ell} : u_i \geq 0,\ \sum_{i=1}^{2\ell} u_i \leq 1 \right\},$$

(15)
$$\overline{V} = \inf_{z\in(0)_\ell\times[0,1]^\ell} V(z^*, z),$$

where

(16)
$$V(z, z', T) = \inf_{\phi\in D_{T,A},\phi_0=z,\phi_T=z'} I_T(\phi),$$

(17)
$$V(z, z') = \inf_{T>0} V(z, z', T).$$

Time of extinction in the SIS model

Theorem 5.4

Let $T^N_{Ext} := \inf\{t \geq 0, I_N = 0\}$ *be the extinction time in the SIS model. Given* $\eta > 0$*, for all* $z \in A$ *,*

$$(18) \quad \lim_{N \to \infty} \mathbb{P}\Big(\exp\{N(\overline{V} - \eta)\} < T^{N,z}_{Ext} < \exp\{N(\overline{V} + \eta)\} \Big) = 1,$$

and for N large enough,

$$(19) \qquad \exp\{N(\overline{V} - \eta)\} \leq \mathbb{E}\Big(T^{N,z}_{Ext} \Big) \leq \exp\{N(\overline{V} + \eta)\}.$$

Time of extinction in the SIS model

Theorem 5.4

Let $T_{Ext}^{N} := \inf\{t \geq 0, I_N = 0\}$ *be the extinction time in the SIS model. Given* $\eta > 0$, *for all* $z \in A$,

$$(18) \quad \lim_{N\to\infty} \mathbb{P}\Big(\exp\{N(\overline{V} - \eta)\} < T_{Ext}^{N,z} < \exp\{N(\overline{V} + \eta)\}\Big) = 1,$$

and for N large enough,

$$(19) \qquad \exp\{N(\overline{V} - \eta)\} \leq \mathbb{E}\Big(T_{Ext}^{N,z}\Big) \leq \exp\{N(\overline{V} + \eta)\}.$$

- Homogeneous case : $\overline{V} = \log(\lambda/\gamma) + \dfrac{\gamma}{\lambda} - 1.$

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Numerical methods

- $\boxed{\lambda = 1.5,\ \gamma = 1 : \overline{V} = 0.07213177477}$

"Bocop"

λ_1	λ_2	γ_1	γ_2	ν_I	ν_S	$\overline{V}$
1.5	1.5	1	1	0.1	0.1	0.072137102821197
1.5	1.5	1	1	0.5	0.5	0.072137145028217
2	1	1	1	0.1	0.1	0.088623530930413
2	1	1	1	0.1	0.5	0.070604684417394
2	1	1	1	0.5	0.5	0.083004811687020
2	1	1	1	0.5	0.1	0.098608814914569
2.5	0.5	1	1	0.1	0.1	0.1287684362961
2.5	0.5	1	1	0.5	0.5	0.1066612071012

Thank you for your attention !