# General stochastic epidemic models: non-Markovian SIRS model with varying infectivity and gradual loss immunity

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# Kermack Mckendrick Model

$$\frac{\partial \overline{S}}{\partial t}(t,\tau) + \frac{\partial \overline{S}}{\partial \tau}(t,\tau) = -\overline{S}(t,\tau)\int_0^{+\infty} \sigma(r)\overline{I}(t,r)dr$$

$$\frac{\partial \overline{I}}{\partial t}(t,\tau) + \frac{\partial \overline{I}}{\partial \tau}(t,\tau) = -\mu(\tau)\overline{I}(t,\tau)$$

$$\frac{\partial \overline{R}}{\partial t}(t,\tau) + \frac{\partial \overline{R}}{\partial \tau}(t,\tau) = -\overline{R}(t,\tau)\beta(\tau)\int_0^{+\infty} \sigma(r)\overline{I}(t,r)dr$$

$$\overline{S}(t,0) = 0$$

$$\overline{I}(t,0) = \int_0^{+\infty} \left(\overline{S}(t,\tau) + \beta(\tau)\overline{R}(t,\tau)\right) d\tau \int_0^{+\infty} \sigma(\tau)\overline{I}(t,\tau)d\tau$$

$$\overline{R}(t,0) = \int_0^{+\infty} \mu(\tau)\overline{I}(t,\tau)d\tau$$

# Kermack Mckendrick Model

where

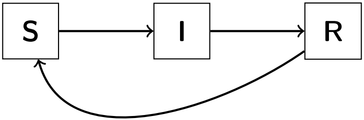
- $\overline{S}(t,\tau)$ is the density of susceptible population without infection at time $t$ and which has been susceptible since the time $t-\tau$ ;
- $\overline{I}(t,\tau)$ is the density of infected and infectious population at time $t$ and which has been infected since the time $t-\tau$ ;
- $\overline{R}(t,\tau)$ is the density of recovered at time $t$ and which has been recovered since the time $t-\tau$ ;
- $\sigma(\tau)$ is the infectivity of infected individuals at age $\tau$ ;
- $\beta(\tau)$ is the susceptibility of recovered individuals at age $\tau$ and assume that $\beta(\tau) \to 1$ when $\tau$ tends to $+\infty$ ;
- $\mu(\tau)$ is the recovery rate at class age $\tau$ ;

# Kermack Mckendrick Model

Here

- The infectivity of the individual is represented by an arbitrary function ;
- Initially all naïve individual have the same susceptibility ;
- the susceptibility of individual after the recovery is assumed to be a function of the time elapsed since his complete recovery.

# Markovian SIRS




- The infectivity rate $\sigma$ is a constant
- The duration of infection and immunity are exponential.

This model are not realist.

# Illustration

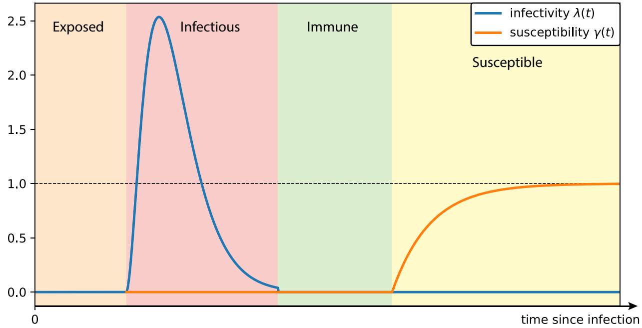



Figure – Evolution of infectivity and susceptibility in the time.

# Gradual loss of immunity model

- Let $(\lambda_{k,i}, \gamma_{k,i})_{k\geq 1, i\geq 1} \subset D^2$ be an i.i.d collection of r.v.
- Let $(\lambda_{k,0}, \gamma_{k,0})_{k\geq 1} \subset D^2$ be an i.i.d collection of r.v. independent from the previous one.
- $\lambda_{k,i}(t)$ is the infectivity of the $k-th$ individual $t$ units time after its $i-th$ infection and $\gamma_{k,i}(t)$ is its susceptibility $t$ units time after its $i-th$ infection.

## Model description

Moreover,

- each individual has infectious contacts at a rate equal to its current infectivity.
- At each infectious contact, an individual is chosen uniformly in the population and this individual becomes infected with probability given by its current susceptibility.

Hence if we denote $A_k^N(t)$ the number of times that the $k-th$ individual has been infected between time 0 and $t$,

- the infectivity of the $k-th$ individual at time $t$ is given by $\lambda_{k,A_k^N(t)}(\mathcal{T}_k^N(t))$ amd its susceptibility is $\gamma_{k,A_k^N(t)}(\mathcal{T}_k^N(t))$,

where

$$\mathcal{T}_k^N(t) := t - \sup\{s \in [0,t] : A_k^N(s) = A_k^N(s^-) + 1\} \vee 0$$

is the time elapsed since the last time where it was infected or since the start of the epidemic if it has not been infected yet.

## Model description

More precisely,
the $k-th$ individual becomes infected at time $t$ at the rate

$$\Upsilon_k^N(t) = \gamma_{k,A_k^N(t)}(\mathcal{T}_k^N(t))\overline{\mathfrak{I}}^N(t)$$

where

$$\overline{\mathfrak{I}}^N(t) = \frac{1}{N}\sum_{k=1}^{N} \lambda_{k,A_k^N(t)}(\mathcal{T}_k^N(t))$$

is the total force of infection at time $t$, ie the sum of the infectivities of all the (infected) individuals at time $t$. We also define the average of susceptibility by

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^{N} \gamma_{k,A_k^N(t)}(\mathcal{T}_k^N(t)).$$

# Assumptions

- For some deterministic $\lambda^*$, $\sup_{i,j} \sup_{t\in[0,T]} \lambda_{i,j}(t) \leq \lambda^*$;
- For $k \geq 1$, $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$, $\lambda_{k,i}(t) \geq 0$ and $\gamma_{k,i}(t) \in [0,1]$;
- $\sup\{t \geq 0,\ \lambda_{k,i}(t) > 0\} \leq \inf\{t \geq 0,\ \gamma_{k,i}(t) > 0\}$.

Let us define : $\eta_k^0 = \inf\{t > 0,\ \lambda_{k,0}(t+s) = 0,\ \forall s \geq 0\}$
$\overline{\lambda}^0(t) = \mathbb{E}\left[\lambda_{1,0}(t) \middle| \eta^0 > 0\right]$, $\overline{\lambda}(t) = \mathbb{E}\left[\lambda_{1,1}(t)\right]$.

# Functional law of large numbers

**Theorem 1**

*Under the assumptions above,*

$$(\overline{\mathfrak{S}}^N, \overline{\mathfrak{I}}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{\mathfrak{S}}, \overline{\mathfrak{I}}) \text{ in } D^2$$

*where* $(\overline{\mathfrak{S}}, \overline{\mathfrak{I}})$ *solves the set of equation* (1)–(2).

# Functional law of large numbers

$$\overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma^0(t)\exp\left(-\int_0^t \gamma^0(r)\overline{\mathfrak{I}}(r)dr\right)\right] + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{I}}(r)dr\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{I}}(s)ds, \quad (1)$$

where $\gamma_0$ and $\gamma$ are r.v. which have respectively the same law as $\gamma_{1,0}$ and $\gamma_{1,1}$ and

$$\overline{\mathfrak{I}}(t) = \bar{I}(0)\overline{\lambda}^0(t) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{I}}(s)ds. \quad (2)$$

# Illustration

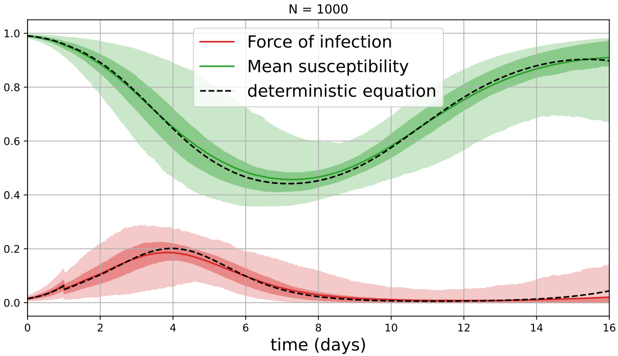



Figure – Stochastic model vs deterministic model.

# Schema of proof

We use the same technical of Chevallier in us article on the propagation of chao to prove this theorem.
The difference between the model of Chevallier and our model is that, Chevallier suppose that, each individual have the same infectivity function after each infection and the same deterministic function of susceptibility for all susceptible.

# Example of Kermack-Mckendrick model

Let us define $X$ be a r.v such that

$$X = \overline{S}(0)\mathbb{1}_{X=1} + \overline{I}(0)\mathbb{1}_{X=2} + \overline{R}(0)\mathbb{1}_{X=3}$$

Let $\tau$ be a non-negative r.v such that

$$\mathbb{P}(\tau > x) = \frac{1}{\overline{I}(0)} \int_x^{+\infty} \overline{I}(0, r)dr$$

and let $\xi$ be a non-negative r.v such that

$$\mathbb{P}(\xi > x) = \frac{1}{\overline{R}(0)} \int_x^{+\infty} \overline{R}(0, r)dr$$

where $\tau$, $\xi$ and $X$ are independent.
Let $\eta_0$ be a non-negative r.v such that

$$\mathbb{P}(\eta_0 > t \big| \tau) = \exp\left(-\int_\tau^{\tau+t} \mu(r)dr\right) \mathbb{1}_{X=2}.$$

Then if we take

# Example of Kermack-Mckendrick model

$$\lambda_0(t) = \sigma(t+\tau)\mathbb{1}_{t<\eta_0, X=2}$$
$$\gamma_0(t) = \mathbb{1}_{X=1} + \beta(t-\eta_0)\mathbb{1}_{t>\eta_0, X=2} + \beta(t+\xi)\mathbb{1}_{X=3},$$
$$\lambda(t) = \sigma(t)\mathbb{1}_{t<\eta}$$
$$\gamma(t) = \beta(t-\eta)\mathbb{1}_{t>\eta}$$

we have particular case of Kermack-Mckendrick.

## Remark

The case where $\gamma = 0$ and

$$\gamma^0 = \begin{cases} 0 & \text{if the individual is initially infectious} \\ 1 & \text{if the individual is initially susceptible} \end{cases}$$

is done in the previous article by *Forien – Pang – Pardoux*.

# Perspective

- Endemic equilibrium (in progress)
- Central limits theorem
- Large deviation principle.

Thank you for your attention !