The α-Ford algebraic measure trees

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June 17, 2021

The α-Ford model [Ford '05]

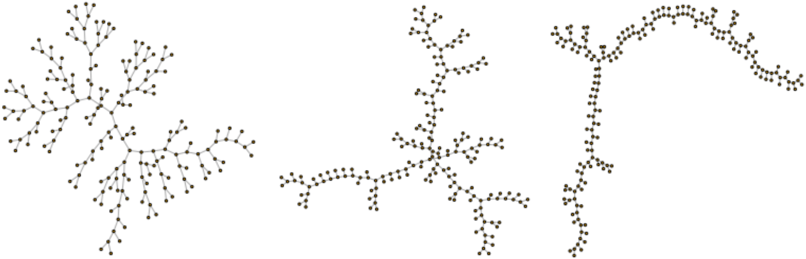

Figure: $\alpha = 0$ on the left; $\alpha = 0.5$ in the middle; $\alpha = 0.9$ on the right.

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Fix $\alpha \in [0, 1]$. The α-*Ford tree* is a random binary unrooted tree with *n* leaves:

- Start with one edge (yielding 2 leaves).
- Given the α-Ford tree with *k*-leaves, assign weight $1 - \alpha$ to each external and weight α to each internal edge.
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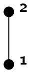

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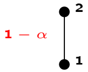

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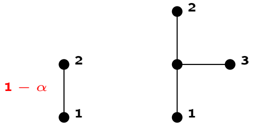

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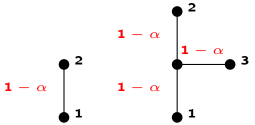

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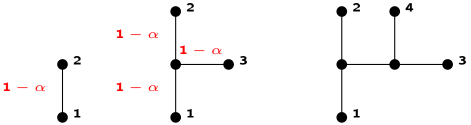

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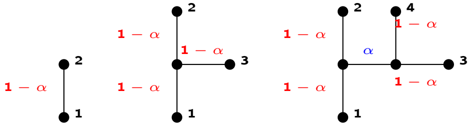

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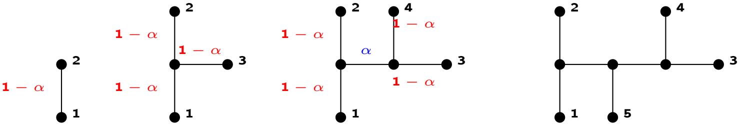

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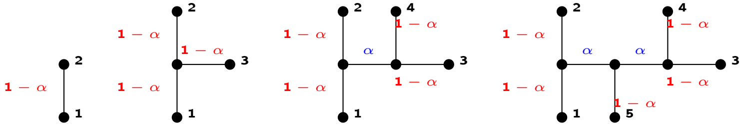

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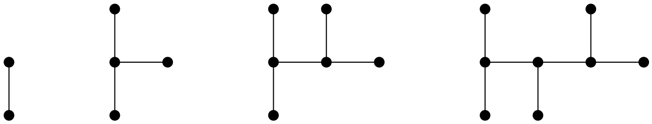

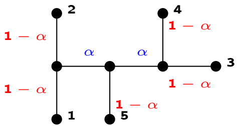

Special cases:

- $\alpha = 0$: *Kingman coalescent tree* whithout edge lengths or *Yule tree*.

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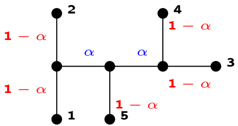

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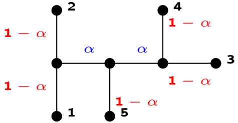

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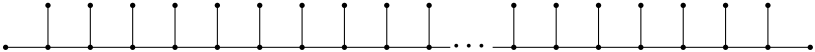

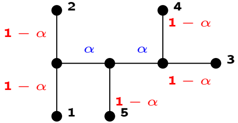

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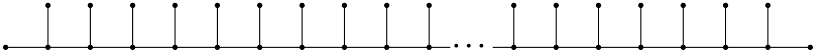

Goal

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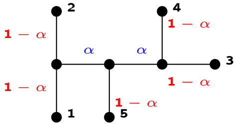

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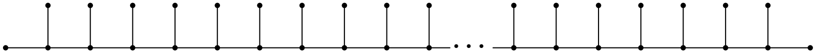
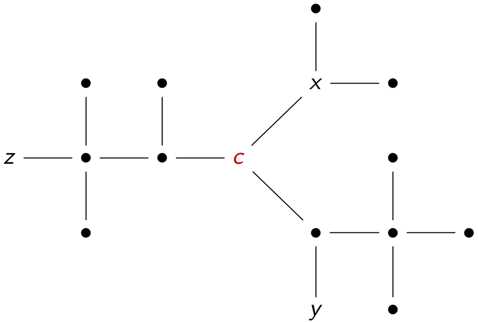


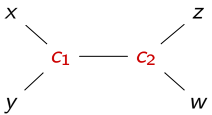
Goal

What if the tree gets large? Need a *state space* of *(infinite) trees* with a "nice" *topology*.



For $x, y, z \in T$ there exists a unique *branch point* $c = c(x, y, z) \in T$ with

$$[x, y] \cap [y, z] \cap [z, x] = \{c\}.$$



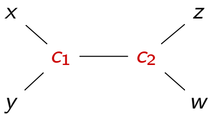
Definition: algebraic tree

An algebraic tree is a set T together with a symmetric map $c\colon T^3 \to T$ such that:

- (2-point condition) For $x, y \in T$, $c(x, y, y) = y$.
- (3-point condition) For $x, y, z \in T$, $c(x, y, c(x, y, z)) = c(x, y, z)$.
- (4-point condition) For $x, y, z, w \in T$,

$$c(x, y, z) \in \{c(x, y, w), c(x, z, w), c(y, z, w)\}.$$

The space of algebraic measure trees [Löhr, Winter '18]



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Definition

An algebraic measure tree (T, c, μ) consists of an order separable algebraic tree (T, c) together with a probability measure μ on $(T, \mathcal{B}(T, c))$.

We consider the subset of *binary algebraic measure trees*:

$$\mathbb{T}_2 := \{(T, c, \mu) : \deg(v) \leqslant 3 \ \forall v \in T, \ \operatorname{atoms}(\mu) \subseteq \operatorname{leaves}(T)\}.$$

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Idea: "Gromov-weak" topology

A sequence of trees converges to a limit tree if and only if all randomly sampled finite subtrees converge to the corresponding limit subtrees.

When we sample n points according to μ, we can look at the tree shape they span.

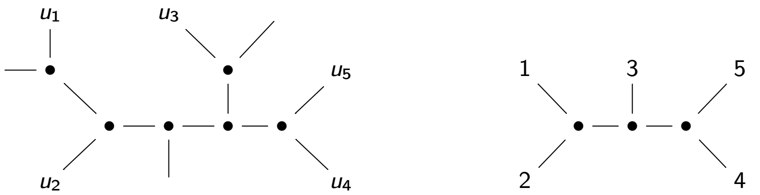

Figure: A tree T and the shape spanned by 5 points.

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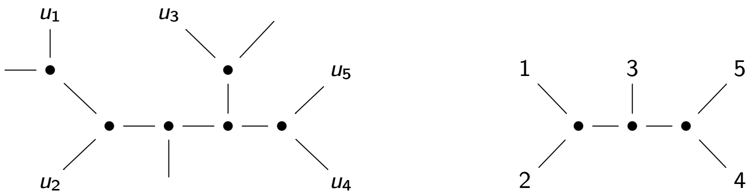

Figure: A tree T and the shape spanned by 5 points.

A sequence of algebraic measure trees is sample-shape convergent if for all n, the distributions of the n-tree shapes converge.

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Proposition [N., Winter '20]

As the number of leaves goes to infinity, the α-Ford model converges in distribution in $\mathbb{T}_2$ to a random *continuum* algebraic measure tree.

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We call the limit the α-Ford algebraic measure tree.

- $\alpha = 0$: the Kingman algebraic measure tree.
- $\alpha = \frac{1}{2}$: the algebraic measure Brownian Continuum Random Tree [Löhr, Mytnik, Winter '18].

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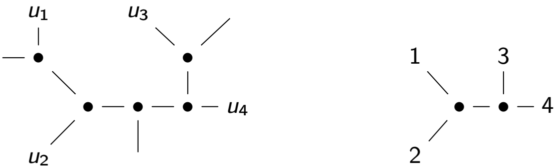
- **Sample shapes:** distribution of the shape spanned by m points sampled according to μ.



Statistics on the limit trees

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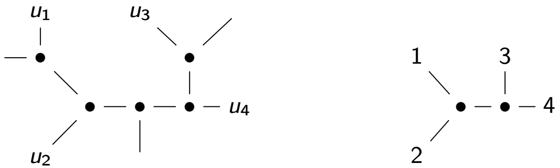
- **Sample shapes:** distribution of the shape spanned by m points sampled according to μ.



- **Subtree masses**: sample 3 points according to μ and look at how the mass is distributed around their branch point.

For $\underline{u} = u_1, u_2, u_3$ three leaves of T, we consider the vector of the three masses of the components connected to $c(\underline{u})$:

$$\underline{\eta}(u_1, u_2, u_3) = (\eta_1(\underline{u}), \eta_2(\underline{u}), \eta_3(\underline{u})).$$

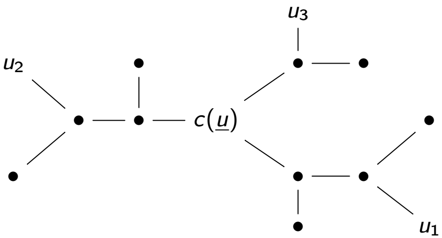


Figure: A tree with 8 leaves; $\underline{\eta}(\underline{u}) = \left(\frac{3}{8}, \frac{3}{8}, \frac{2}{8}\right)$.

We look at polynomials of the form

$$\Phi^f(\chi) := \int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u}) f(\underline{\eta}(\underline{u})).$$

Proposition [Aldous '94]

Let $\mathbb{P}_{\mathrm{CRT}}$ be the law of the Brownian algebraic CRT. Then for all $f\colon \Delta_2 \to \mathbb{R}$ continuous bounded,

$$\mathbb{E}_{\mathrm{CRT}}\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u}) f(\underline{\eta}(\underline{u}))\right] = \int_{\Delta_2} f(\underline{x}) \mathrm{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)(\mathrm{d}\underline{x}),$$

where $\mathrm{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the Dirichlet distribution.

Proposition [N., Winter '20]

Let $\mathbb{P}_{\mathrm{Kin}}$ be the law of the Kingman algebraic measure tree. Let $B_{1,2}$ and $B_{2,2}$ be two independent beta random variables, such that $B_{1,2}$ has law $\mathrm{Beta}(1,2)$ and $B_{2,2}$ has law $\mathrm{Beta}(2,2)$. Then for all $f\colon \Delta_2 \to \mathbb{R}$ continuous bounded,

$$\begin{aligned}\mathbb{E}_{\mathrm{Kin}}&\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u}) f(\underline{\eta}(\underline{u}))\right] \\ &= \frac{1}{6} \sum_{\pi \in \mathcal{S}_3} \mathbb{E}\big[f \circ \pi^*(B_{1,2}B_{2,2}, B_{1,2}(1-B_{2,2}), 1-B_{1,2})\big],\end{aligned}$$

where $\mathcal{S}_3$ is the set of permutations $\{1,2,3\}$, and for $\pi \in \mathcal{S}_3$, $\pi^*\colon \Delta_2 \to \Delta_2$ is the induced map $\pi^*(\underline{x}) = (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$.

Subtree masses

$\rightarrow$ What about the other $\alpha \in [0, 1]$?

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We can calculate all the moments through a recurrence relation.

$$\mathbb{E}_\alpha\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u})\eta_1(\underline{u})\right] = \frac{1}{3},$$

$$\mathbb{E}_\alpha\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u})(\eta_1(\underline{u}))^2\right] = \frac{1}{5},$$

$$\mathbb{E}_\alpha\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u})(\eta_1(\underline{u}))^3\right] = \frac{11-7\alpha}{15(5-3\alpha)},$$

$$\mathbb{E}_\alpha\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u})(\eta_1(\underline{u}))^4\right] = \frac{37-25\alpha}{63(5-3\alpha)},$$

$$\mathbb{E}_\alpha\left[\int_{T^3} \mu^{\otimes 3}(\mathrm{d}\underline{u})(\eta_1(\underline{u}))^5\right] = \frac{145-165\alpha+44\alpha^2}{42(5-3\alpha)(7-3\alpha)}.$$

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