The Multi-Type bisexual Galton-Watson branching process

Nicolás Zalduendo Vidal

Joint work with Coralie Fritsch and Denis Villemonais

École de recherche de la Chaire MMB

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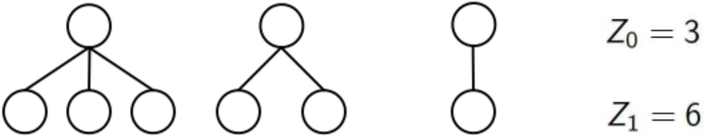
The Galton-Watson process

$Z_0 = 3$

The Galton-Watson process



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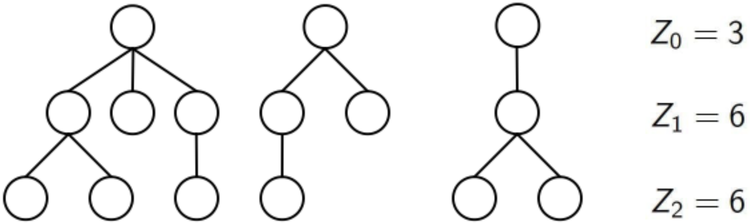

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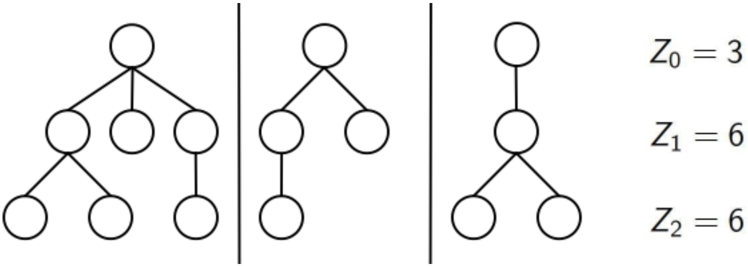
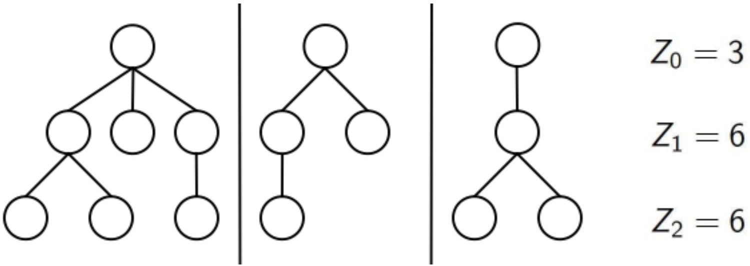


Very important property: INDEPENDENCE!

The Galton-Watson process



Very important property: INDEPENDENCE!

$$m := \mathbb{E}(Z_1 | Z_0 = 1) \leq 1 \iff Z_n \to 0 \; a.s.$$

The Multi-Type Galton Watson process

We now consider a process with types:

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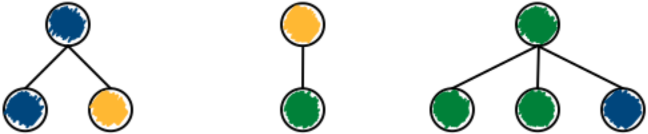

$$Z_0 = (1, 1, 1)$$

$$Z_1 = (2, 1, 3)$$

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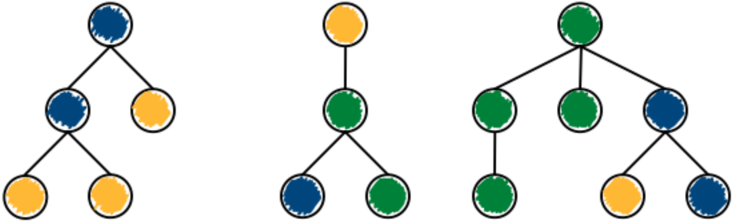

$$Z_0 = (1,1,1)$$

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$$Z_2 = (2,3,2)$$

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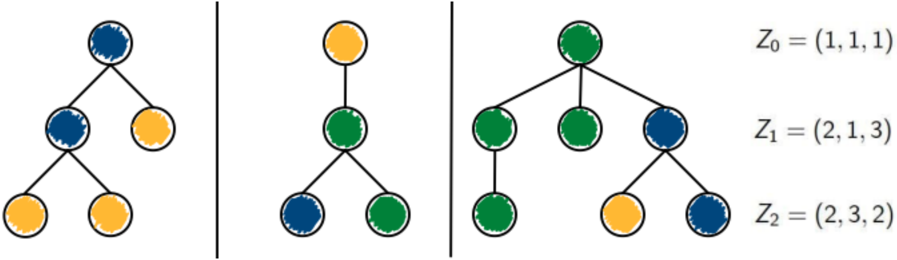

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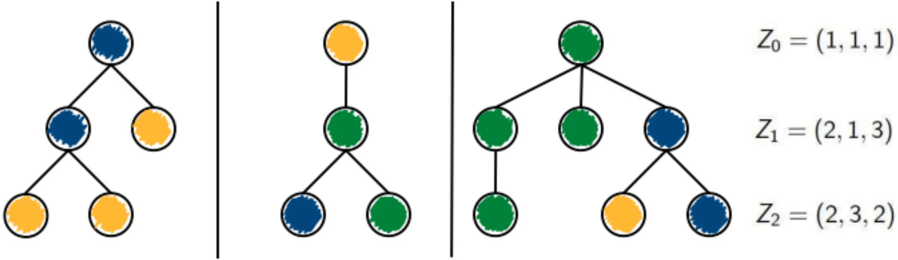

If we define $\mathbb{A}_{i,j} = \mathbb{E}(Z_1^j | Z_0 = e_i)$, then

$$\lambda^* \leq 1 \iff Z_n \to 0, \ a.s.$$

with λ^* the greatest eigenvalue of $\mathbb{A}$.

The bisexual GW branching process [Daley, '68]

Consider the function $\xi(x, y) = x \min\{y, 1\}$.

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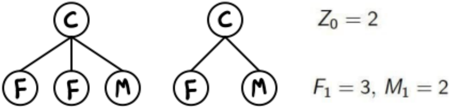

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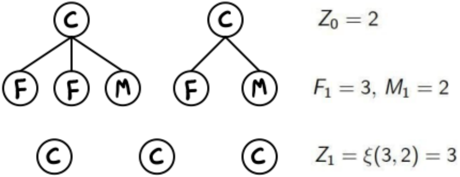

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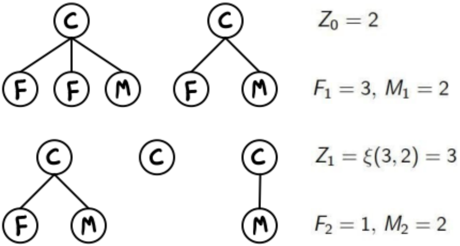

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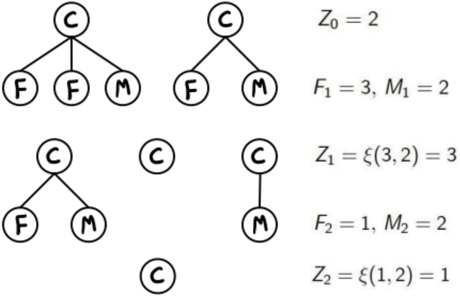

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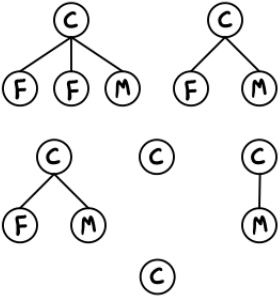

$Z_0 = 2$

$F_1 = 3,\ M_1 = 2$

$Z_1 = \xi(3, 2) = 3$

$F_2 = 1,\ M_2 = 2$

$Z_2 = \xi(1, 2) = 1$

Difficulty: We lose the independence property!

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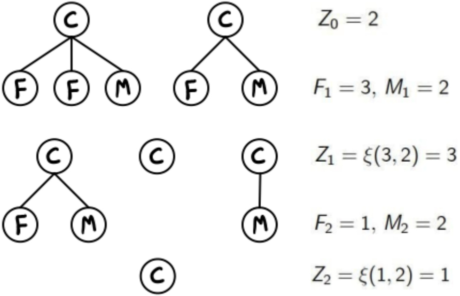

Difficulty: We lose the independence property!

Superadditive model [Hull, 1982]:

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2),\ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}_+$$

What about Multi-Type?

Some Multi-Type models that have been studied:

- Mode, 1972: *A 3-type bisexual model where inherits the type of the male.*
- Karlin - Kaplan, 1973: *A Multi-Type version of the Cows and Bulls model, where the couple inherits the type of the female.*
- Hull, 1998: *A 2-type bisexual model where the couple inherits the type of the male.*

But not as deeply as the previous processes!

The Multi-Type bGWbp

We consider a multi-dimensional model

In the general case,

$$Z_n = (Z_n^1, \ldots, Z_n^p)$$

The Multi-Type bGWbp

We consider a multi-dimensional model

$Z_0 = (1,2)$

$F_1 = (1,1,2)$, $M_1 = (3,2)$

In the general case,

$$Z_n = (Z_n^1, \ldots, Z_n^p)$$

$$F_{n+1} = (F_{n+1}^1, \ldots, F_{n+1}^{n_f}),\ M_{n+1} = (M_{n+1}^1, \ldots, M_{n+1}^{n_m})$$

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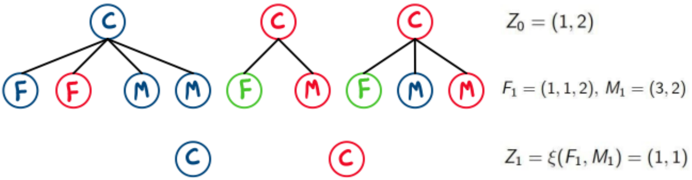

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$$Z_{n+1} = \xi((F_{n+1}^1, \ldots, F_{n+1}^{n_f}), (M_{n+1}^1, \ldots, M_{n+1}^{n_m}))$$

The Multi-Type bGWbp

Assumptions:

- Superadditivity:

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2).$$

- Integrability: The matrices

$$\mathbb{F}_{i,j} = \mathbb{E}(F_1^j | Z_0 = e_i),\ \mathbb{M}_{i,j} = \mathbb{E}(M_1^j | Z_0 = e_i)$$

are well defined.

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  are well defined.

This implies:

Proposition

The function $R : \mathbb{N}_+^p \longrightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$ given by

$$R(z) = \lim_{k \to +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = kz)}{k}.$$

is well defined.

1. Motivation
2. The Multi-Type bGWbp
3. Results
    - Law of Large Numbers
    - Condition for certain extinction

Law of Large Numbers

What is the role of R?

$$R(z) = \lim_{k \to +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = kz)}{k}.$$

Theorem [Fritsch - Villemonais - Z.]

We denote $(Z_{n,k})_{n \in \mathbb{N}}$ the process with $Z_{0,k} = kz$. Then

$$\frac{Z_{n,k}}{k} \xrightarrow[a.s.,\, L^1]{k \to +\infty} R^n(z).$$

Law of Large Numbers

The proof relies on additional properties of R:

Lemma

For any $z \in \mathbb{N}^p$,

$$R(z) = \lim_{k \to +\infty} \frac{\xi(kz\mathbb{F}, kz\mathbb{M})}{k}$$

$$\mathbb{F}_{i,j} = \mathbb{E}(F_1^j | Z_0 = e_i), \ \mathbb{M}_{i,j} = \mathbb{E}(M_1^j | Z_0 = e_i)$$

Fact: The function R is **concave**.

Condition for certain extinction

Extra assumptions:

- Transcience:

$$\mathbb{P}(Z_n \to 0 \mid Z_0 = z) + \mathbb{P}(Z_n \to +\infty \mid Z_0 = z) = 1, \quad \forall z \in \mathbb{N}^p \setminus \{0\}.$$

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Condition for certain extinction

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Theorem [Krause, '94]

The eigenvalue problem

$$R(z^*) = \lambda^* z^*$$

has a unique solution with $\lambda^* > 0$ and $z^* \in (\mathbb{R}_+)^p$, $z^* > 0$, $|z^*| = 1$.

Condition for certain extinction

Define

$$q_z = \mathbb{P}(\exists n \in \mathbb{N},\ Z_n = 0 | Z_0 = z)$$

Condition for certain extinction

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Theorem [Fritsch - Villemonais - Z.]

Assume R is finite. Then,

$$\lambda^* \leq 1 \iff q_z = 1, \forall z \in \mathbb{N}^p.$$

If $\lambda^* > 1$, then $\forall \varepsilon > 0, \exists v_0 \in \mathbb{N}^p$ such that if $Z_0 = v_0$

$$\mathbb{P}\left(Z_n > (\lambda^* - \varepsilon)^n v_0,\ \forall n \in \mathbb{N}\right) > 0.$$

If there exists $z \in (\mathbb{R}_+)^p$ such that $R(z)$ is not finite, then $q_v < 1$ for some $v \in \mathbb{N}^p$.

The Multi-Type bisexual Galton-Watson process

	Asexual Process	**Bisexual Process**
Single-Type	*Classic Galton-Watson process*	*Bisexual Galton-Watson process*
	$p = 1$, $\xi(x, y) = x$ $R(z) = mz$ Extinction condition: $R(1) \leq 1$	$p = 1$, ξ superadditive R linear Extinction condition: $R(1) \leq 1$
Multi-Type	*Multi-Type Galton-Watson process*	
	$p > 1$, $\xi(x, y) = x$ $R(z) = z\mathbb{A}$ Extinction condition: $\lambda^* \leq 1$	

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Multi-Type	*Multi-Type Galton-Watson process*	*Multi-Type bGWbp*
	$p > 1$, $\xi(x, y) = x$ $R(z) = z\mathbb{A}$ Extinction condition: $\lambda^* \leq 1$	$p > 1$, ξ superadditive R concave Extinction condition: $\lambda^* \leq 1$

Condition for certain extinction

Some examples:

1. Multi-Type perfect fidelity mating:
   - $n_f = n_m = p$.
   - $\xi(x, y) = \min\{x, y\}$.
   - $R(z) = \min\{z\mathbb{F}, z\mathbb{M}\}$.

Condition for certain extinction

Some examples:

1. Multi-Type perfect fidelity mating:
   - $n_f = n_m = p$.
   - $\xi(x, y) = \min\{x, y\}$.
   - $R(z) = \min\{z\mathbb{F}, z\mathbb{M}\}$.
   - A particular case: $\mathbb{F} = \alpha\mathbb{U}$, $\mathbb{M} = (1-\alpha)\mathbb{U}$.
     In this case $\lambda^* = \min\{\alpha, 1-\alpha\}\lambda^*_{\mathbb{U}}$, $z^* = z^*_{\mathbb{U}}$.

Condition for certain extinction

Some examples:

1. Multi-Type perfect fidelity mating:
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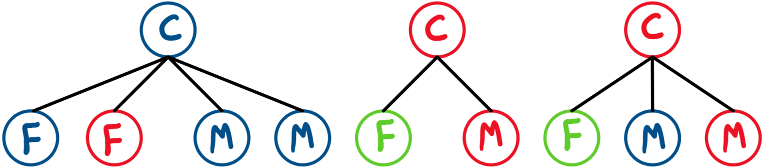
2. Multi-Type completely promiscuous mating [Karlin - Kaplan, 1973]:
   - $p = n_f$.
   - $\xi(x, y) = x \prod_{i=1}^{n_m} \mathbb{1}_{y_i > 0}$.
   - $R(z) = (z\mathbb{F})\mathbb{1}_{z\mathbb{M} > 0}$.
   - In this case $\lambda^* = \lambda^*_{\mathbb{F}}$.

The Multi-Type bisexual Galton-Watson process

Thank You!



Future Work

What can we say about the asymptotic behavior of the process?

Conjecture

There exists a real and positive random variable W such that

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[a.s.]{n\to\infty} Wz^*$$

If the conjecture is true, we want to find conditions for:

$$\{Z_n > 0, \forall n \in \mathbb{N}\} = \{W > 0\}.$$

Future Work

Other models to consider:

- bGWbp in varying or in random environment.
- bGWbp with immigration.
- Random mating of couples.
- Infinite numer of types.
- Scaling limits

Krause's Result

Consider a function $F : (\mathbb{R}_+)^p \longrightarrow (\mathbb{R}_+)^p$.

- We say F is primitive if $\exists n \in \mathbb{N}, \forall m \geq n, \forall z \neq 0, F^{(m)}(z) > 0$.
- We say F is positively homogeneous if $F(\lambda z) = \lambda F(z)$, $\forall \lambda \in \mathbb{R}, \forall z \in (\mathbb{R}_+)^p$.

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Theorem [Krause - 1994]

Consider $F : E \longrightarrow E$ a concave, primitive and positively homogeneous mapping. Then,

(i) The eigenvalue problem $F(x) = \lambda x$ has a unique solution $(\lambda^*, x^*) \in \mathbb{R}_+ \times (\mathbb{R}_+)^p$, with $z^* > 0$, $|z^*| = 1$. If $(\lambda, x) \in \mathbb{R} \times E \setminus \{0\}$ is another solution of the problem, then it must hold that $x = rx^*$ for some $r > 0$ and $\lambda = \lambda^*$.

(ii) $L(x) = \lim\limits_{k\to\infty} \frac{F^{(k)}(x)}{(\lambda^*)^k}$ exists on $(\mathbb{R}_+)^p$ and L is a concave and positively homogeneous mapping with $L(x) > 0$ for all $x \in (\mathbb{R}_+)^p \setminus \{0\}$.

A supermartingale

Krause's Theorem:

$$\lim_{k \to +\infty} \frac{R^k(z)}{(\lambda^*)^k} = C(z)z^*$$

with C a suitable function. We have that

$$C\left(\frac{Z_n}{(\lambda^*)^n}\right)_{n\in\mathbb{N}}$$

is a positive supermartingale.