Spine constructions for branching populations with interactions. Application to a Yule process

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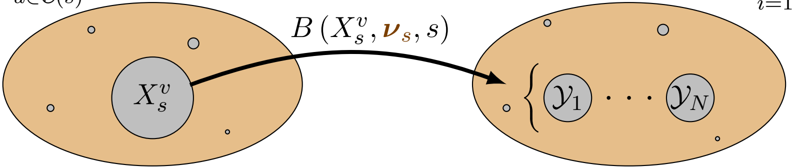
Sommaire

Definition of the process $(\boldsymbol{\nu}_s)_{s\geq 0}$

Type = (size,position,gender,...) $\in \mathcal{X} \subset \mathbb{R}^d$.

$$\boldsymbol{\nu}_s = \sum_{u\in\mathbb{G}(s)} \delta_{X_s^u} \qquad \boldsymbol{\nu}_{s+} = \boldsymbol{\nu}_s - \delta_{X_s^v} + \sum_{i=1}^{N} \delta_{\mathcal{Y}_i}$$



Notations

- Offspring size : $N \sim (p_n(X_s^v, \boldsymbol{\nu}_s, s))_{n\geq 0}$.
- Type distribution at birth :

$$(\mathcal{Y}_1, \cdots, \mathcal{Y}_n) \sim K_n(X_s^v, \boldsymbol{\nu}_s, s, \cdot).$$

- Deterministic dynamics between jumps : $\dot{x}_s = \mu(x_s, \boldsymbol{\nu}_s, s)$.

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Size-biased trees and Many-to-One formula

Size-biased tree [CR88], [LPP95]
We assume the following parameters to be constant :

- Branching rate ; $B(x, \nu, t) = B$,
- Reproduction law ; for all $k \geq 0$, $p_k(x, \nu, t) = p_k$

$$\mathbb{P}(\text{"having } k \text{ children"}) = p_k$$

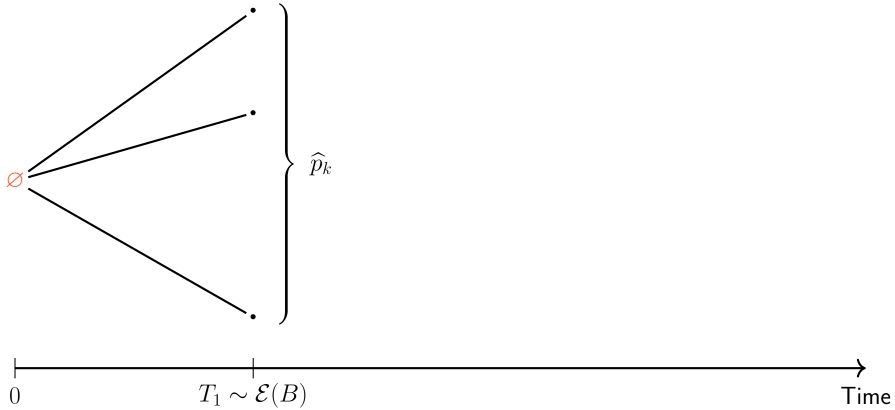
$$\widehat{p}_k = \widehat{\mathbb{P}}(\text{"having } k \text{ children"}) := \frac{kp_k}{m}$$

where $m := \sum_{k \geq 0} kp_k$ is the mean number of children.

- **Spine** $\rightarrow \widehat{p}_k$
- **Other individuals** $\rightarrow p_k$

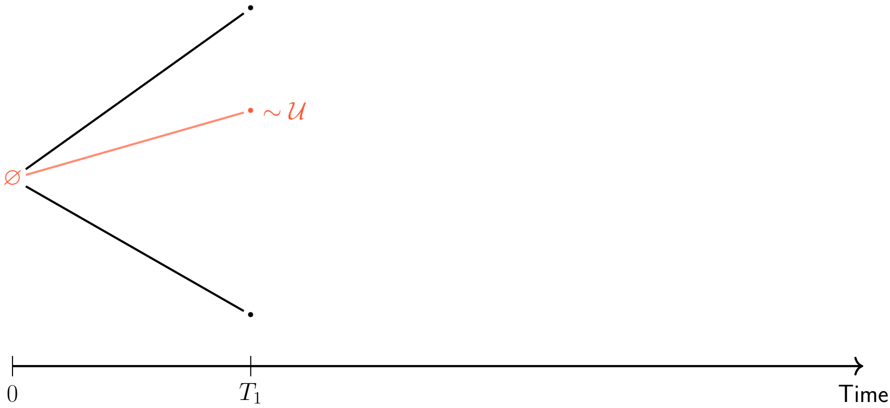
Type at birth of the k children $\sim K_k(x_t, \boldsymbol{\nu}_t, t, \cdot)$
Deterministic dynamics between jumps : $\dot{x}_t = \mu\left(x_t, \boldsymbol{\nu}_t, t\right)$

Spinal process



Size-biased trees and Many-to-One formula

Spinal process



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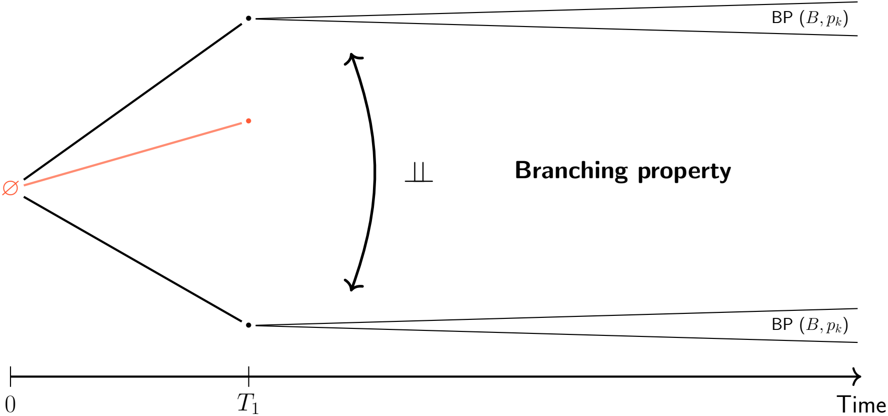
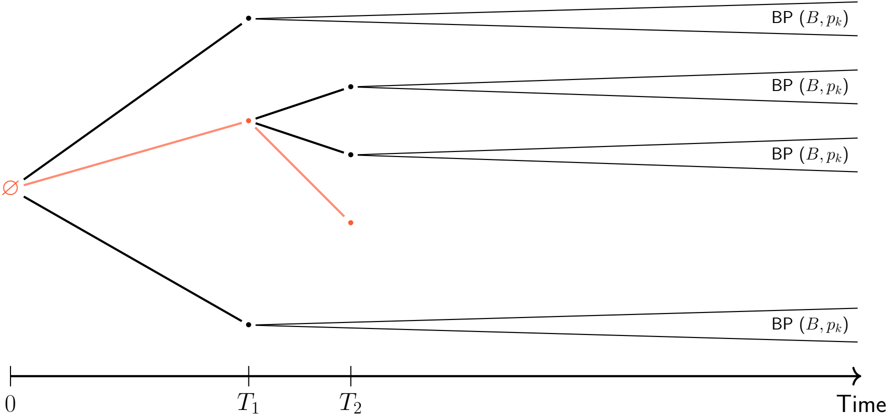
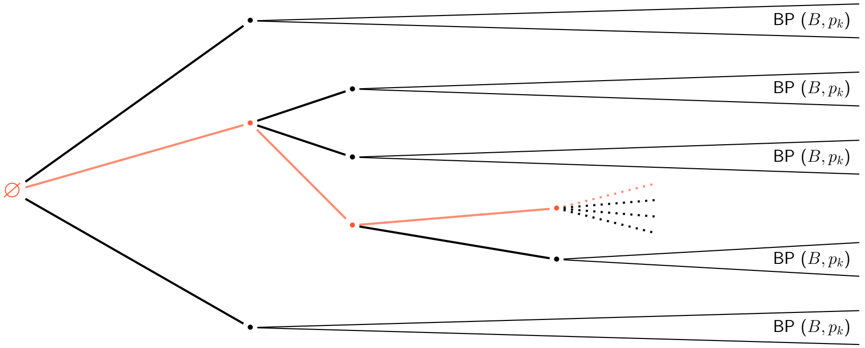


Spinal process



Size-biased trees and Many-to-One formula

Spinal process



Many-to-One formula without interactions, [BDMT11]

$$\mathbb{E}\left[\sum_{u\in\mathbb{G}(t)} F\left(\left(X_s^u\right)_{0\le s\le t}\right)\right] = \mathbb{E}\left[e^{B(m-1)t} F\left(\left(Y_s\right)_{0\le s\le t}\right)\right]$$

Spinal trees and Many-to-One formula

- Type space $\mathcal{X}$ finite or countable
- $\mu(x, \nu, t) = 0$
- $B(x, \nu, t) = B(x, \nu)$
- For all $n \geq 0$, $p_n(x, \nu, t) = p_n(x, \nu)$ and $K_n(x, \nu, t, \cdot) = K_n(x, \nu, \cdot)$

Many-to-One formula with interactions, [Ban21]

$$\mathbb{E}\left[\sum_{u \in \mathbb{G}(t)} F\left((X_s^u)_{0 \leq s \leq t}\right)\right] =$$

$$C\mathbb{E}\left[\frac{1}{\psi\left(Y_t, \chi_t\right)} \exp\left(\int_0^t \frac{\mathcal{G}\psi\left(Y_s, \chi_s\right)}{\psi\left(Y_s, \chi_s\right)} \mathrm{d}s\right) F\left((Y_s)_{0 \leq s \leq t}\right)\right].$$

Spinal trees and Many-to-One formula

- Type space $\mathcal{X}$ finite or countable
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Many-to-One formula with interactions, [Ban21]

With $\psi \equiv 1$

$$\mathbb{E}\left[\sum_{u \in \mathbb{G}(t)} F\left(\left(X_s^u\right)_{s \leq t}\right)\right] = \mathbb{E}\left[e^{\int_0^t B(Y_s, \chi_s)(m(Y_s, \chi_s)-1)\mathrm{d}s} F\left(\left(Y_s\right)_{s \leq t}\right)\right].$$

Spinal process $(Y_t, \chi_t)_{t\geq 0}$

We denote χ_+ the new population after a branching event

Biased distributions using $\psi_\phi(x,\chi,t) := \psi(x, \langle \chi, \phi(\cdot,t)\rangle, t)$

Spine

$$\widehat{p}_k^*\left(x_e,\chi\right) \propto \frac{\sum_{i=1}^k \psi_\phi\left(y_i,\chi_+\right)}{\psi_\phi\left(x_e,\chi\right)} p_k\left(x_e,\chi\right).$$

The j-th child becomes the new spine with probability

$$\frac{\psi_\phi\left(y_j,\chi_+\right)}{\sum_{i=1}^k \psi_\phi\left(y_i,\chi_+\right)}.$$

Outside the Spine

$$\widehat{p}_k\left(x,x_e,\chi\right) \propto \frac{\psi_\phi\left(x_e,\chi_+\right)}{\psi_\phi\left(x_e,\chi\right)} p_k\left(x,\chi\right).$$

Spinal process $(Y_t, \chi_t)_{t\geq 0}$

We denote χ_+ the new population after a branching event

Biased distributions using $\psi_\phi(x,\chi,t) := \psi(x, \langle \chi, \phi(\cdot,t)\rangle, t)$

Spine

$$\widehat{p}_k^* (x_e, \chi) \propto \frac{\sum_{i=1}^k \psi_\phi (y_i, \chi_+)}{\psi_\phi (x_e, \chi)} p_k (x_e, \chi) .$$

Outside the Spine

$$\widehat{p}_k (x, x_e, \chi) \propto \frac{\psi_\phi (x_e, \chi_+)}{\psi_\phi (x_e, \chi)} p_k (x, \chi) .$$

If $\psi_\phi^t \equiv 1$,

$$\widehat{p}_k (x, x_e, \chi) = p_k (x, \chi) \quad \text{et} \quad \widehat{p}_k^* (x_e, \chi) = \frac{k p_k (x_e, \chi)}{m (x_e, \chi)}$$

Many-to-one in general case

For a given function ψ_ϕ, we denote $(Y_t, \chi_t)_{t\geq 0}$ the associated spinal process.

Theorem

Under non-explosion assumptions for both processes, for all $t \geq 0$, for every initial population z and every positive function F,

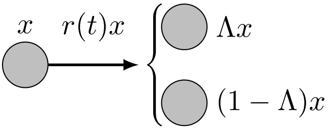
$$\mathbb{E}_z\left[\sum_{u\in\mathbb{G}(t)} F\left((X_s^u)_{0\leq s\leq t}\right)\right] =$$

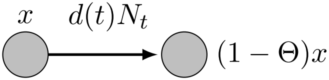
$$C(z)\mathbb{E}_z\left[\frac{1}{\psi_\phi\left(Y_t, \chi_t, t\right)}\exp\left(\int_0^t \frac{\boldsymbol{\mathcal{G}}\boldsymbol{\psi_\phi}\left(Y_s, \chi_s, s\right)}{\boldsymbol{\psi_\phi}\left(Y_s, \chi_s, s\right)}\mathrm{d}s\right) F\left((Y_s)_{0\leq s\leq t}\right)\right].$$

Sommaire

Yule process with growth and loss events

Type = mass $x \in \mathbb{R}_+^*$ of each individual, growing as $\dot{x} = \mu(t)x$.
Branching events :





- Λ is a random variable in $[0,1]$
- Θ is a random variable in $(0,1)$

Yule process with growth and loss events

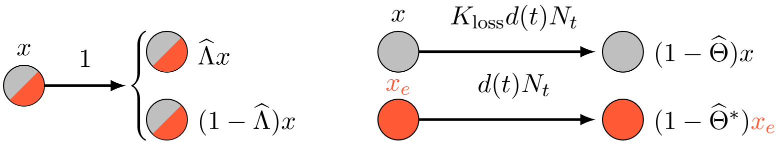
We assume

$$K_{\mathrm{div}} := \mathbb{E}\left[\frac{1}{\Lambda(1-\Lambda)}\right] < +\infty, \qquad K_{\mathrm{loss}} := \mathbb{E}\left[\frac{1}{\Theta}\right] < +\infty.$$

We take $\psi(x,y,t) := xe^{-y}$ and $\phi(x,t) := \ln\left(xK_{\mathrm{div}}r(t)\right)$. We thus get

$$\psi_\phi(x,\nu_t,t) = \frac{x}{\prod_{u\in\mathbb{G}(t)} X_t^u K_{\mathrm{div}} r(t)}$$

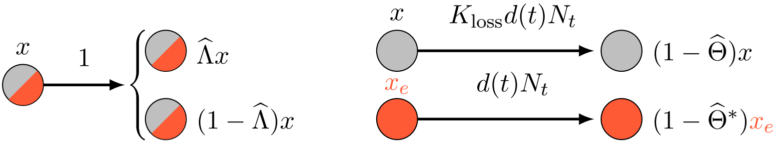
Branching rates of the associated spinal process



Yule process with growth and loss events

$$K_{\text{div}} := \mathbb{E}\left[\frac{1}{\Lambda(1-\Lambda)}\right], \quad K_{\text{loss}} := \mathbb{E}\left[\frac{1}{\Theta}\right], \quad \psi_\phi(x, \nu_t, t) = \frac{x}{\prod_{u\in\mathbb{G}(t)} X_t^u K_{\text{div}} r(t)}.$$

Branching rates of the spinal process

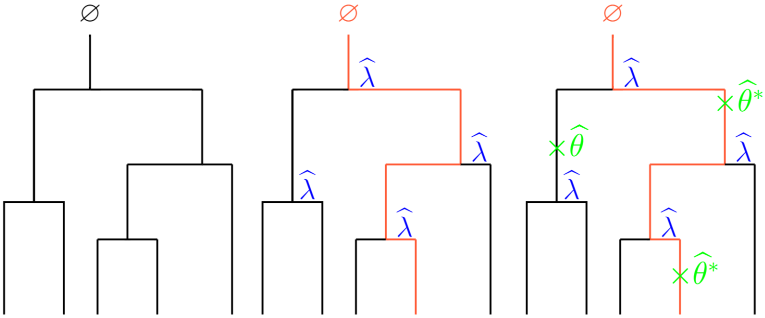


$$\frac{\mathcal{G}\psi_\phi(x,\nu_t,t)}{\psi_\phi(x,\nu_t,t)} = \mu(t) + \left[1 - \frac{\dot{r}(t)}{r(t)} - \mu(t) + d(t)(N_t - 1)\left(K_{\text{loss}} - 1\right)\right] N_t - r(t)B_t$$

Yule process with growth and loss events

$$K_{\text{div}} := \mathbb{E}\left[\frac{1}{\Lambda(1-\Lambda)}\right], \quad K_{\text{loss}} := \mathbb{E}\left[\frac{1}{\Theta}\right], \quad \psi_\phi(x, \nu_t, t) = \frac{x}{\prod_{u \in \mathbb{G}(t)} X_t^u K_{\text{div}} r(t)}.$$

Algorithm's sketch



(a) Step 1 (b) Step 2 (c) Step 3

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