Multitype linear population models in random environment

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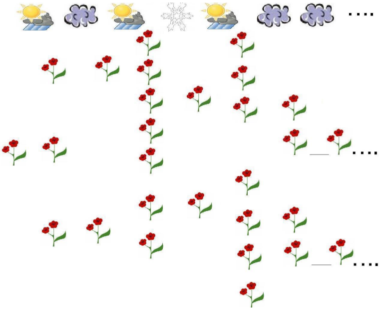
Problem : How does the fluctuation of environmental parameters affect the dynamics of a (structured) population ?



- Discrete time $n \in \mathbb{N} = \{0, 1 \cdots\}$.
- Set of types/traits $\mathbb{X}$ e.g. :
  - Size $\mathbb{X} = \mathbb{R}^+$
  - Phenotypical/Genotypical groups $\mathbb{X} = \{1, \ldots, p\}$ or $[0, 1]$
- Set of possible environments $\mathcal{E}$, e.g. :
  - $\mathcal{E} = [-50, 50]$ for temperatures
  - $\mathcal{E} = \{\text{"Normal"}, \text{"Disaster"}\}$
- Random sequence of environments $(\xi_n)_{n \in \mathbb{N}}$
  - i.i.d environments
  - Stationnary and ergodic sequence
  - Markov chain started on a invariant distribution.

- If $\mathbb{X} = \{1, \dots, d\}$, the population at time n is represented by

$$\mu_n = (\mu_n(1)), \dots \mu_n(d)) \in \mathbb{R}_+^d.$$

Then, if $\xi_n = e$,

$$\mu_{n+1}(j) = \mu_n(1) m_e(1, j) + \cdots + \mu_n(d) m_e(d, j)$$

$$\mu_{n+1} = \mu_n M_e$$

where $M_e = (m_e(i, j))_{i,j \in \mathbb{X}}$ is a $d \times d$ matrix.

- $\mu_0 \in \mathbb{R}_+^d$,

$$\mu_{n+1} = \mu_n M_{\xi_n}, \text{ thus } \mu_n = \mu_0 M_{\xi_0} \cdots M_{\xi_{n-1}}$$

Note

$$M_{0,n} = M_{\xi_0} \cdots M_{\xi_{n-1}}$$

Questions :

- Asymptotic behavior of the total mass $\langle \mu_n, \mathbf{1} \rangle$, extinction, explosion ?
- Asymptotic behavior of the distribution of traits $\frac{\mu_n}{\langle \mu_n, \mathbf{1} \rangle}$

Example 1 : Almost sure extinction and explosion in average

- Monotype example, $\mathbb{X} = \{1\}$, (ξ_n) i.i.d
- $\mu_n = \mu_0 M_{\xi_0} \cdots M_{\xi_{n-1}} = \mu_0 \exp\left(\log(M_{\xi_0}) + \cdots \log(M_{\xi_{n-1}})\right).$

Proposition

Suppose $\mathbb{E}\left[(\log M_\xi)^+\right] < \infty$, *Then,*

- *If* $\mathbb{E}[M_\xi] < 1$ *and* $\mathbb{E}[\log M_\xi] < 0$, *then* $\mathbb{E}[\mu_n] \longrightarrow 0$ *and* $\mathbb{P}\left(\mu_n \to 0\right) = 1$,
- *If* $\mathbb{E}[M_\xi] > 1$ *and* $\mathbb{E}[\log M_\xi] < 0$, *then* $\mathbb{E}[\mu_n] \longrightarrow \infty$ *and* $\mathbb{P}\left(\mu_n \to 0\right) = 1$,
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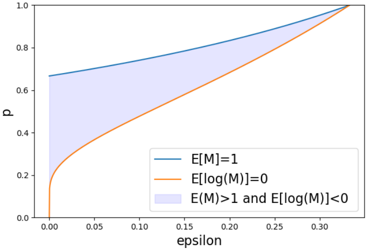
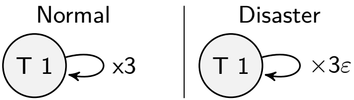


Example :

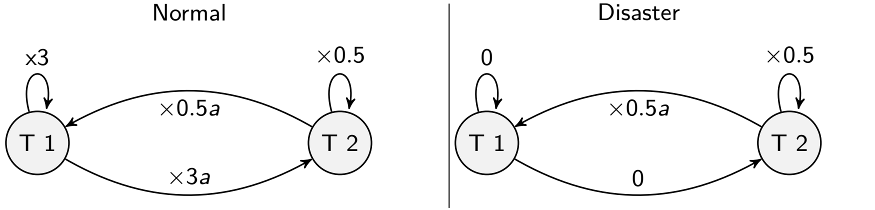
- $\mathcal{E} = \{\text{"Normal"},\text{"Disaster"}\}$
- $\mathbb{P}(N) = 1 - p, \mathbb{P}(D) = p > 0$
- $M_N = 3,\ M_D = 3\varepsilon,\ 0 \leq \varepsilon < \frac{1}{3}$



- If $\varepsilon = 0$, $\mathbb{E}[M_\xi] = 3(1 - p)$.

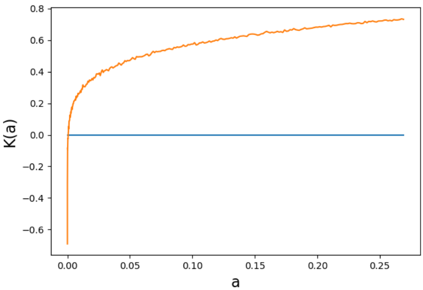
Example 2 : Connexion of types allowing survival [Jansen, Yoshimura 98']

Take $\mathcal{E} = \{\text{"Normal"}, \text{"Disaster"}\}$, $\mathbb{X} = \{1, 2\}$, $a \in [0, 1]$.



If $a = 0$, and $p > 0$, then $\mu_n \longrightarrow 0$.

Ex : with $p = 0.1$



Proposition

Assume $\mathbb{E}\left[\log(|M_\xi|)^+\right] < \infty$, *then there exists a deterministic number* $K(a) \in [-\infty, +\infty)$ *such that, with probability 1*

$$\frac{1}{n}\log\left(\|M_{0,n}\|\right) \underset{n\to\infty}{\rightarrow} K(a) = \inf_N \frac{\mathbb{E}\left[\log\|M_{0,N}\|\right]}{N}.$$

If $K(a) < 0$, $\mathbb{P}\left[\mu_n \to 0\right] = 1$.
If $K(a) > 0$, *and* $\mu_0 > 0$ $\mathbb{P}\left[\mu_n \to +\infty\right] = 1$.

Towards the infinite dimensional case

Theorem (L.)

Consider an ergodic sequence $(\xi_n)_n$ and suppose the $(M_{\xi_n})_n$ are bounded linear operators on the set of positive measures on $\mathbb{X}$. Under suitable positivity, boundedness, and moments assumptions, among which a ***Doeblin minoration*** *condition,*

i) *There exists $\tilde{\eta} \in (0,1)$, and almost surely, there exists a random function h on $\mathbb{X}$, such that, for any $\eta \in (0, \tilde{\eta})$, for n large enough,*

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n} \right\|_{TV} \leq \eta^n \langle \mu_1 M_{0,n}, \mathbf{1} \rangle.$$

ii) *Setting $\lambda := \inf_N \frac{1}{N} \mathbb{E}\left[\log \|M_{0,n}\|\right] \in [-\infty, \infty)$, it holds, almost surely, for any μ*

$$\frac{1}{n} \log \langle \mu M_{0,n}, \mathbf{1} \rangle \to \lambda.$$

iii) *There exists a random probability π on $\mathbb{X}$ such that for any μ*

$$\left(\frac{\mu M_{0,n}}{\langle \mu M_{0,n}, \mathbf{1} \rangle} \right)_{n \geq 0} \xrightarrow{d} \pi,$$

Proof elements

1. Control of the difference $\left\| \frac{\mu_1 M_{0,n}}{\langle \mu_1 M_{0,n}, \mathbf{1} \rangle} - \frac{\mu_2 M_{0,n}}{\langle \mu_2 M_{0,n}, \mathbf{1} \rangle} \right\|_{TV}$ with a Doeblin contraction (method used by Bansaye, Champagnat, Cloez, Del Moral, Gabriel, Horton, Jasra, Miclos, Villemonais)
2. Geometric decay of the Doeblin error terms with ergodic theorems
3. Existence of a reproductive value h such that $\lim_{n \to +\infty} \frac{\langle \mu_1 M_{0,n}, \mathbf{1} \rangle}{\langle \mu_2 M_{0,n}, \mathbf{1} \rangle} = \frac{\mu_1(h)}{\mu_2(h)}$.
4. Deducing the approximation for large n : $\mu_1 M_{0,n} \approx \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n}$.
5. Proving the existence of a deterministic Malthusian growth rate $\lambda = \lim \frac{1}{n} \log \langle \mu M_{0,n}, \mathbf{1} \rangle$ with subadditive ergodic theorem
6. Establishing the convergence (in distribution) of the type distribution $\frac{\mu_1 M_{0,n}}{\langle \mu_1 M_{0,n}, \mathbf{1} \rangle}$ with a time reversal trick

Thank you for your attention !