The field-road diffusion model : fundamental solution, asymptotic behavior

(and exclusion process...)

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Content

- Biological motivations
- The field-road diffusion model
- Fundamental solution and asymptotic behavior
- Perspectives : microscopic scale and exclusion process

Alberta, Canada

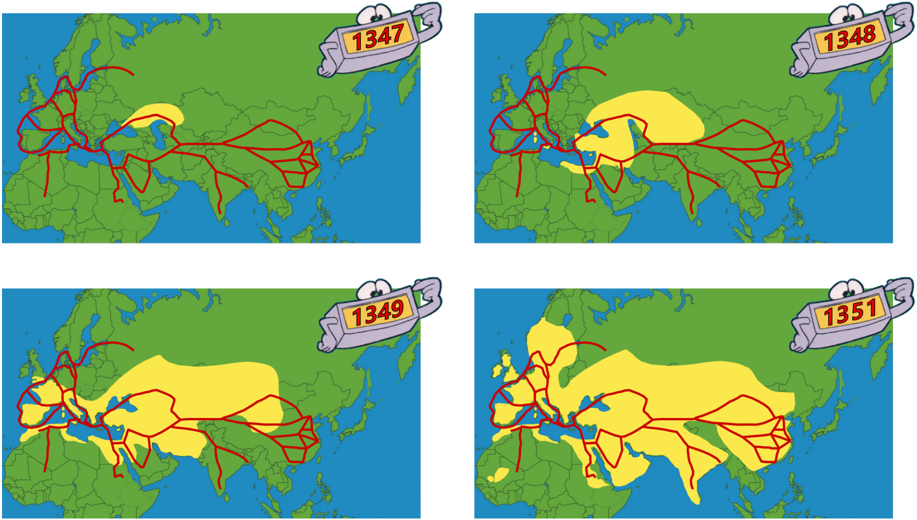
Wolfs travel faster along these lines

This increases their chances of encountering prey...

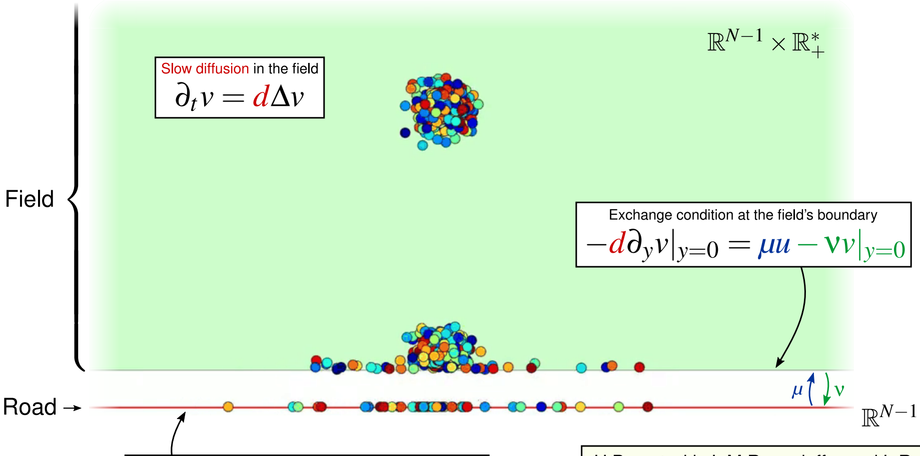
Biological motivations

The Silk Road accelerated the spread of the Black Plague during the 14th century...

Route de la soie — Peste noire



...for fast diffusion channels in population dynamics and ecology.



Fast diffusion on the road

$$\partial_t u = D\partial_{xx} u - \mu u + \nu v|_{y=0}$$

$N \geq 2$
$\mu, \nu > 0$
$0 < d \lll D$

H.Berestycki, J.-M.Roquejoffre and L.Rossi
The influence of a line with fast diffusion on Fisher-KPP propagation
Journal of Mathematical Biology (2012)

How to find the solutions (explicitly)

Simplified model : Heat equation in half space
(but same ideas !)

$$\begin{cases} \partial_t v = d(\partial_{xx} v + \partial_{yy} v) & t>0,\ x\in\mathbb{R},\ y>0 & (i) \\ -d\partial_y v|_{y=0} = -\nu v|_{y=0} & t>0,\ x\in\mathbb{R},\ y=0 & (ii) \\ v|_{t=0} \equiv v_0 & x\in\mathbb{R},\ y>0 & (iii) \end{cases}$$

MAIN TOOL : Double integral transform : $\mathcal{F}$ourier on x / $\mathcal{L}$aplace on t

(i) $\quad \partial_t v = d(\partial_{xx} v + \partial_{yy} v) \xrightarrow{\ \mathcal{FL}\ } \widehat{\partial_t v} = \widehat{d(\partial_{xx} v + \partial_{yy} v)}$

$\mathcal{FL}$ linear and "breaks" ∂_{xx} and ∂_t

$$\underbrace{d\partial_{yy}\hat{v}(s,\xi,y) - (s+d\xi^2)\hat{v}(s,\xi,y) = -\widehat{v_0}(\xi,y)}_{\text{Linear 2nd order ODE in } y}$$

Solve using (ii) at $y=0$ and $\hat{v}=0$ as $y\to\infty$ $\longrightarrow$ $\hat{v}$ = something explicit

Looking for known forms of Fourier and Laplace transforms in the tables...

$$v(t,x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} \overbrace{H_\nu(t,x,y\,;z,\omega)}^{\text{Fondamental solution}} v_0(z,\omega)\,d\omega\,dz$$

The solutions to the field-road diffusion model

$$\begin{cases} \partial_t v = d\Delta v, & t>0,\ x\in\mathbb{R}^{N-1},\ y>0, \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t>0,\ x\in\mathbb{R}^{N-1}, \\ \partial_t u = D\Delta_x u - \mu u + \nu v|_{y=0}, & t>0,\ x\in\mathbb{R}^{N-1}, \end{cases}$$

$$\begin{cases} v|_{t=0} = v_0, & x\in\mathbb{R}^{N-1},\ y>0, \\ u|_{t=0} = u_0, & x\in\mathbb{R}^{N-1}. \end{cases}$$

Theorem (M.Alfaro, R.Ducasse, S.T.) (2022)

The solution to the latter Cauchy problem is

$$v(t,x,y) = V(t,x,y) + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}^{N-1}} \Lambda(t,z,y)u_0(x-z)dz + \frac{\mu\nu}{\sqrt{d}}\int_0^t\int_{\mathbb{R}^{N-1}} \Lambda(s,z,y)V|_{y=0}(t-s,x-z)dzds$$

$$u(t,x) = e^{-\mu t}U(t,x) + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}^{N-1}} G(t-s,x-z)v|_{y=0}(s,z)dzds,$$

where,

- $\begin{cases} \partial_t V = d\Delta V, \\ -d\partial_y V|_{y=0} = -\nu V|_{y=0}, \\ V|_{t=0} \equiv v_0, \end{cases}$

- $\begin{cases} \partial_t U = D\Delta_x U, \\ U|_{t=0} \equiv u_0, \end{cases}$

- $G(t,x) = \frac{1}{(4\pi Dt)^{N/2}} e^{-\frac{\|x\|^2}{4Dt}}$ is the $(N-1)$-dimensional D-diffusive Heat kernel,

- $\Lambda(t,x,y)$ has a quite complex expression and may be interpreted as an exchange kernel since $v \equiv \frac{\mu}{\sqrt{d}}\Lambda$ if $(v_0,u_0)\equiv(0,\delta_0)$.

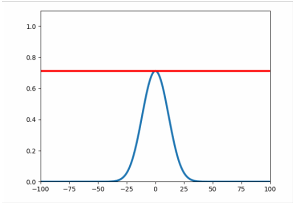
Asymptotic decay rate to tackle nonlinear issues

$$(\star)\ \ \partial_t u = \Delta u + u^{1+p}, \qquad t > 0, \ \ x \in \mathbb{R}^N$$

Diffusion part
$\partial_t u = \Delta u$

VS

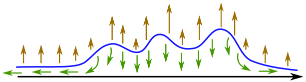
Reaction part
$U' = U^{1+p}$



$\|u(t,\cdot)\|_{L^\infty} \le c/t^{N/2}$

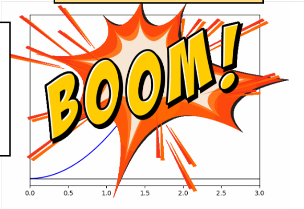
Vanish with algebraic rate $N/2$

Reaction tends to make the solution blow-up



Diffusion tends to crush the solution on zero

Global existence
or
blow-up in finite time ?



$U(t) = c/(T_{\text{boom}} - t)^{1/p}$

Blows-up with algebraic rate $1/p$

Theorem (Fujita) (1966)

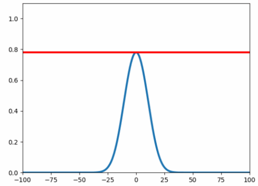
If $\frac{1}{p} > \frac{N}{2}$, then **reaction always prevails over diffusion**: any positive solution to $(\star)$ blows-up in finite time.

If $\frac{1}{p} < \frac{N}{2}$, then **diffusion may prevail over reaction**: $(\star)$ admits global positive solutions.

- For the Heat equation

$$\partial_t u = \Delta u, \qquad t > 0, \quad x \in \mathbb{R}^N,$$

$$\|u(t,\cdot)\|_{L^\infty} \leq c/t^{N/2}$$



- For the field-road diffusion model,

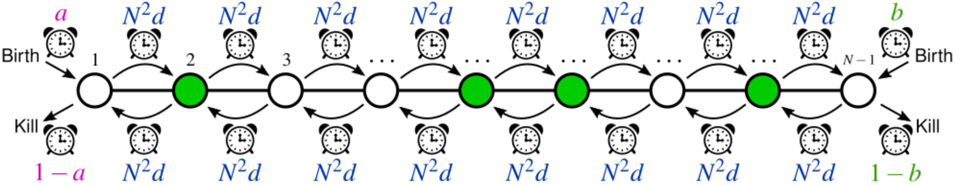
$$\begin{cases} \partial_t v = d\Delta v, & t > 0,\ x \in \mathbb{R}^{N-1},\ y > 0, \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t > 0,\ x \in \mathbb{R}^{N-1}, \\ \partial_t u = D\Delta_x u - \mu u + \nu v|_{y=0}, & t > 0,\ x \in \mathbb{R}^{N-1}, \end{cases}$$

Theorem (M.Alfaro, R.Ducasse, S.T.) (2022)

$$\|v(t,\cdot,\cdot)\|_{L^\infty} \leq \frac{c_1 \ln(1+t) + c_2}{t^{N/2}} \qquad \|u(t,\cdot)\|_{L^\infty} \leq \frac{c_1 \ln(1+t) + c_2}{t^{N/2}}$$

→ almost $c/t^{N/2}$...

- Next step: Fujita-type results on the field-road !...



- (clock τ) Rings with exponential law $\mathscr{E}(\tau)$ and the corresponding move is done.
- Exclusion process : **at most one particle per site** (illegal moves are inhibited)
- Let $(\mu_N)_{N\geq 1}$ be a sequence of probability measures on $\{0,1\}^{N-1}$ such that
  $\mu_N \xrightarrow[N\to\infty]{\text{in some sense}} u_0 : (0,1) \to (0,1)$
- Let $(\eta_N(t))_{t\geq 0}$ be the Markov process induced by μ_N and the dynamic described above.

Theorem (Baldasso *et al.*) (2017)

$$(\eta_N(t))_{t\geq 0} \xrightarrow[N\to\infty]{\text{in some sense}} u(t,\cdot) \quad \text{where} \quad \begin{cases} \partial_t u = d\Delta u, & t>0, \quad x\in(0,1), \\ \partial_n u|_{x=0} = \partial_n u|_{x=1} = 0, & t>0, \\ u|_{t=0} \equiv u_0, & \quad\quad\;\; x\in(0,1). \end{cases}$$

Our proposition for a particle field-road diffusion model...

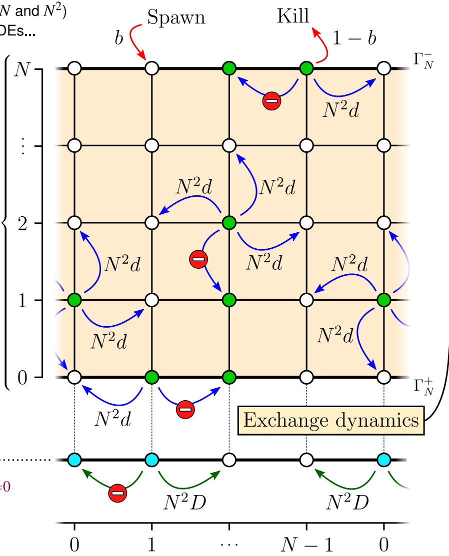
- Notice the different time scaling (1, N and N^2) to recover the good orders in the PDEs...

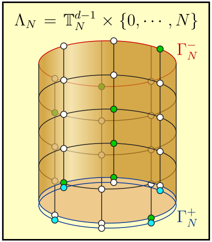
$$\partial_{n_-} v|_{y=1} = 0$$

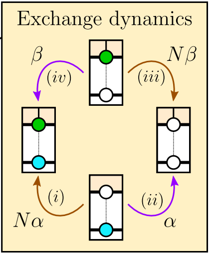
$$\partial_t v = d \Delta v$$

$$\partial_{n_+} v|_{y=0} = \alpha u - \beta v|_{y=0} - (\alpha - \beta) uv|_{y=0}$$

$$\partial_t u = D \partial_{xx} u - \alpha u + \beta v|_{y=0} + (\alpha - \beta) uv|_{y=0}$$



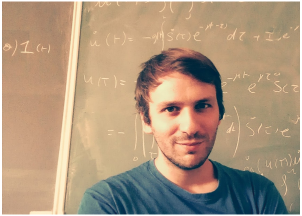




- Nonlinear terms appear due to the exclusion process...

Thanks for your attention !

Matthieu Alfaro



Romain Ducasse

Mustapha Mourragui

The original field-road model

H.Berestycki, J.-M.Roquejoffre and L.Rossi
The influence of a line with fast diffusion on Fisher-KPP propagation
Journal of Mathematical Biology (2012)

M.Alfaro, R.Ducasse, S.T.
The field-road diffusion model : fundamental solution and asymptotic behavior
Journal of Differential Equations (2023)

Fundamental solution

Heat equation with exclusion processes

R.Baldasso, O.Menezes, A.Neumann, R.Souza
Exclusion Process with Slow Boundary
Journal of Statistical Physics (2017)

H.W.McKenzie, E.H.Merrill, R.J.Spiteri, M.A.Lewis
How linear features alter predator movement and the functional response
Interface Focus (2012)

The wolfs travelling faster along the lines