# Optimal control of branching diffusion processes
## Through the modelling and their scaling limit

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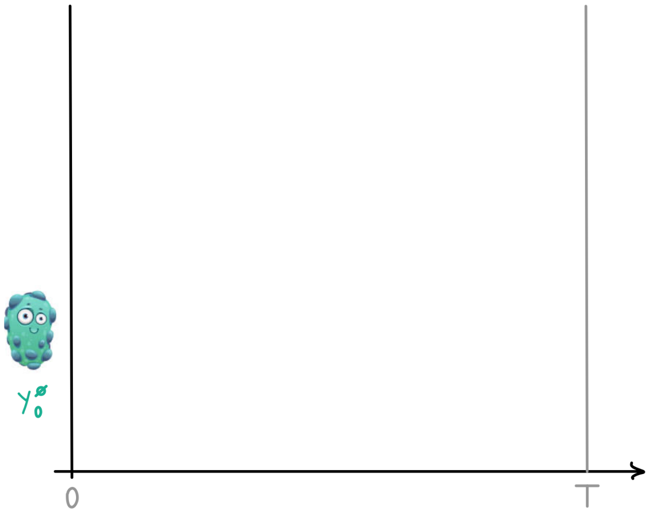
# Controlled branching diffusion processes

# Branching diffusion process



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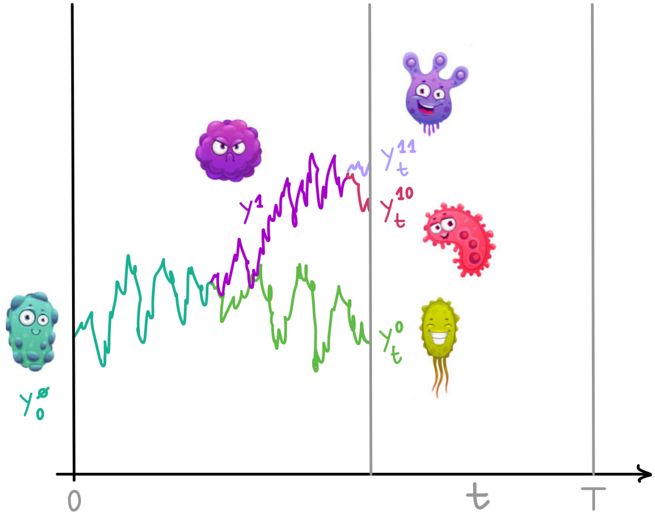

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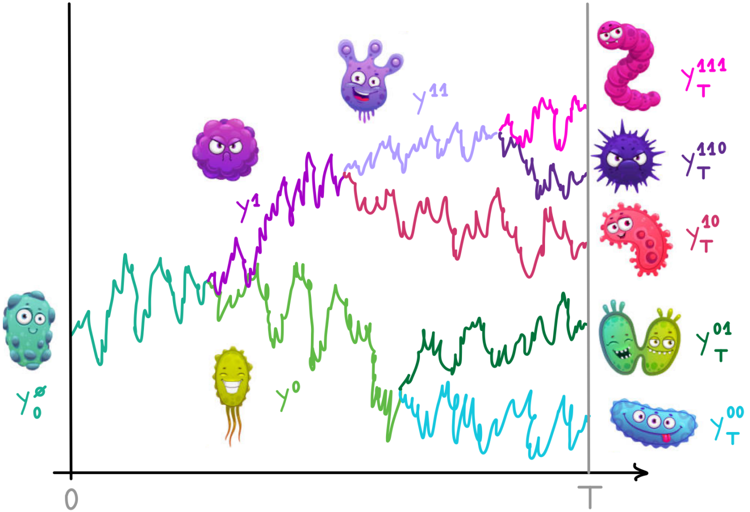

## Probabilistic setting

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- $(B_t^i)_{t\in[0,T]}$ is a standard Brownian motion in $\mathbb{R}^m$ for $i \in \mathcal{I}$;
- $Q^i(dt, dz)$ is a Poisson random measure on $[0, T] \times \mathbb{R}_+$ with intensity $dtdz$ for $i \in \mathcal{I}$;
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Consider $A$ a compact space subset of a Euclidean space.

**Definition (Standard strong control)**

We say that $\beta = (\beta_i)_{i\in\mathcal{I}}$ is a *standard strong control* if $\beta$ is an $\mathbb{F}$-predictable measurable $A^{\mathcal{I}}$-valued process.

# Controlled population

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Considering $V_s$ the set of alive particles at times $s$, the *controlled branching diffusion* is described by

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Considering $V_s$ the set of alive particles at times $s$, the *controlled branching diffusion* is described by

$$\xi_t^\beta = \sum_{i \in V_t} \delta_{Y_t^{i,\beta}} \ ,$$

such that

- Spatial motion: each $i$ moves according to the following stochastic differential equation

$$dY_s^{i,\beta} = b\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) ds + \sigma\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) dB_s^i \ .$$

  Therefore, the motion is associated with the generator $L$, with

$$L\varphi(x,\lambda,a) \quad = \quad b(x,\lambda,a)^\top D\varphi(x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,\lambda,a)D^2\varphi(x)\right) \ ,$$

- Branching rate: given that $i$ is alive at $s$, the probability that she dies in $[s, s+\delta s)$ is

$$\gamma\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) \delta s + o(\delta s) \ .$$

- Branching mechanism: When $i$ dies, she has an offspring with probability

$$\left(p_k\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right)\right)_{k\in\mathbb{N}} \ .$$

# Strong SDE

Let $\mathbf{D}^d = \mathbb{D}([0,T]; M(\mathbb{R}^d))$ be the set of càdlàg functions from $[0,T]$ to $M(\mathbb{R}^d)$. The controlled branching diffusion is associated with the following SDE on $\mathbf{D}^d$

$$
\begin{aligned}
\langle \varphi, \xi_s^\beta \rangle &= \langle \varphi, \xi_t^\beta \rangle + \int_t^s \sum_{i \in V_u} D\varphi(Y_u^{i,\beta})^\top \sigma\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) dB_u^i \\
&\quad + \int_t^s \sum_{i \in V_u} L\varphi\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) du \\
&\quad + \int_{(t,s]\times\mathbb{R}_+} \sum_{i \in V_{u-}} \sum_{k \geq 0} (k-1)\varphi(Y_u^{i,\beta}) \mathbb{1}_{I_k\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right)}(z) Q^i(dudz) \,,
\end{aligned}
$$

with $I_k(x,\lambda,a) = \left[\gamma(x,\lambda,a)\sum_{\ell=0}^{k-1} p_\ell(x,\lambda,a), \gamma(x,\lambda,a)\sum_{\ell=0}^{k} p_\ell(x,\lambda,a)\right)$ ,.

# Existence and uniqueness

## Proposition

*Assume the following conditions*

- $b$ *and* $\sigma$ *are Lipschitz continuous, i.e., there exists* $L > 0$ *such that*

$$\left|b(x,\lambda,a) - b(x',\lambda',a)\right| + \left|\sigma(x,\lambda,a) - \sigma(x',\lambda',a)\right| \leq L(|x-x'| + \mathbf{d}_{\mathbb{R}^d}(\lambda,\lambda'))\ ;$$

- $b$*,* $\sigma$ *and* $\gamma$ *are uniformly bounded;*
- *the first and second order moments related to* $(p_k)_k$ *are uniformly bounded, i.e., there exist a constant* $C > 0$ *such that*

$$\sum_{k\geq 1} kp_k(x,\lambda,a) \leq C, \qquad \sum_{k\geq 1} k(k-1)p_k(x,\lambda,a) \leq C\ .$$

*Let* $t \in [0,T]$*,* $\lambda \in M(\mathbb{R}^d)$ *with* $\lambda := \sum_{i\in V}\delta_{x_i}$ *and* $V$ *finite, and* $\beta$ *be a standard strong control. There exists a unique (up to indistinguishability) càdlàg and adapted process* $\left(\xi_s^\beta\right)_{s\geq t}$ *satisfying the previous SDE such that* $\xi_t^\beta = \lambda$.

# Martingale properties

## Proposition

*Fix $(t, \lambda) \in [0, T] \times M\left(\mathbb{R}^d\right)$, with $\lambda = \sum_{i \in V} \delta_{x^i}$, $V$ finite, and $\beta$ a standard strong control. Therefore,*

$$F_\varphi\left(\xi_s^\beta\right) - \int_t^s F'_\varphi\left(\xi_s^\beta\right) \sum_{i \in V_s} L\varphi\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) +$$

$$+\frac{1}{2} F''_\varphi\left(\xi_s^\beta\right) \sum_{i \in V_s} \left| D\varphi\left(Y_u^{i,\beta}\right) \sigma\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right)\right|^2$$

$$+\sum_{i \in V_s} \gamma\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right)$$

$$\left(\sum_{k \geq 0} F_\varphi\left(\sum_{j \in V_s} \delta_{Y_u^{j,\beta}} + (k-1)\delta_{Y_u^{i,\beta}}\right) p_k\left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i\right) - F_\varphi\left(\xi_u^\beta\right)\right) du\ ,$$

*where $F_\varphi$ denotes the the cylindrical function $F_\varphi = F(\langle \varphi, \cdot \rangle)$, for $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$*

## Control problem

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Reward function: Fix $\psi \in C_b(\mathbb{R}^d \times M(\mathbb{R}^d) \times A)$ and $\Psi \in C_b(M(\mathbb{R}^d))$. Consider the following reward function

$$J_1(t, \lambda; \beta) := \mathbb{E}\left[\int_t^T \sum_{i \in V_s} \psi\left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i\right) ds + \Psi\left(\xi_T^\beta\right) \middle| \xi_t^\beta = \lambda\right].$$

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*Control problem:*

$$v_1(t, \lambda) = \sup \left\{ J_1(t, \lambda, \beta) : \beta \in \mathcal{R}^1_{(t,\lambda)} \right\}.$$

# Verification theorem

## Proposition

*Let $w$ be a function in $C_b^0\left([0,T]\times M(\mathbb{R}^d)\right)$ and fix $(t,\bar\lambda)\in M(\mathbb{R}^d)$, and assume the following*

1. $w_T(\lambda)=\Psi(\lambda)$, *for* $\lambda\in M(\mathbb{R}^d)$;
2. $\left\{ w_s\left(\xi_s^{\beta}\right)+\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\beta},\xi_u^{\beta},\beta_u^i\right)du : s\in[t,T]\right\}$ *is a local submartingale, for any* $\beta\in\mathcal{R}^1_{(t,\bar\lambda)}$;
3. *there exists* $\hat\beta\in\mathcal{R}^1_{(t,\bar\lambda)}$ *such that* $\left\{ w_s\left(\xi_s^{\bar\beta}\right)+\int_t^s\sum_{i\in V_u}\psi\left(Y_u^{i,\bar\beta},\xi_u^{\bar\beta},\bar\beta_u^i\right)du : s\in[t,T]\right\}$ *is a local martingale.*

*Then,* $\bar\beta$ *is an optimal control for* $v(t,\bar\lambda)$*, i.e.,* $v_1(t,\bar\lambda)=J_1(t,\bar\lambda;\bar\beta)$*, and* $v_1(t,\bar\lambda)=w_t(\bar\lambda)$.

But ...

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what if the particles are too many?

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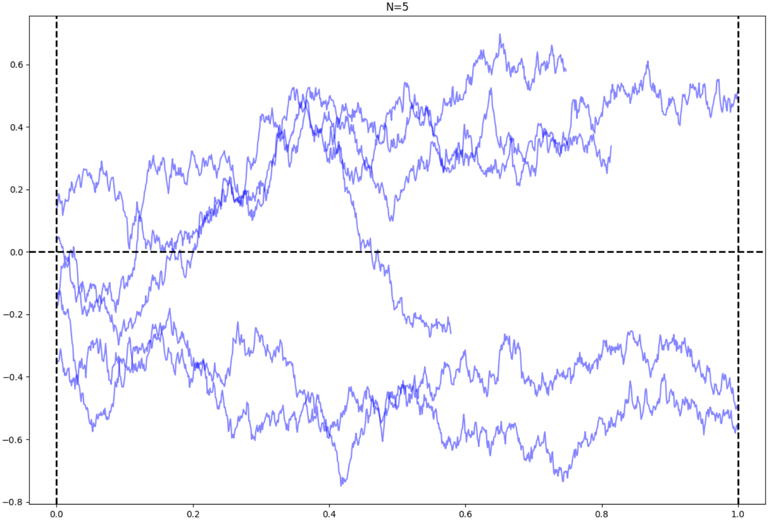

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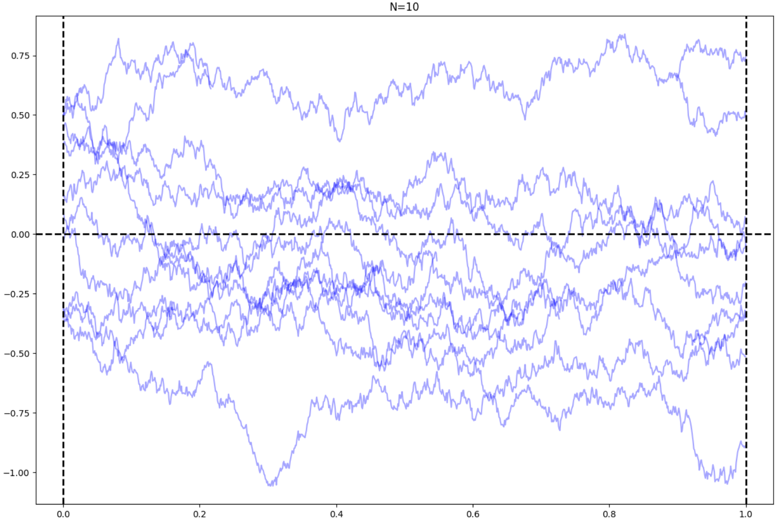

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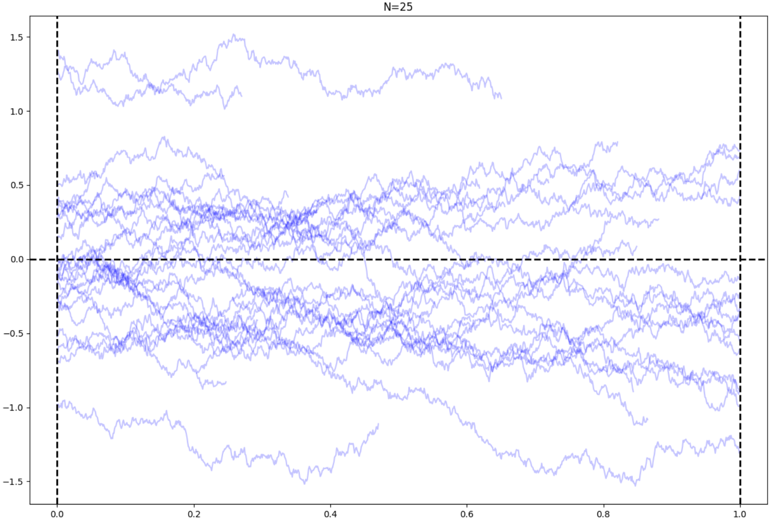

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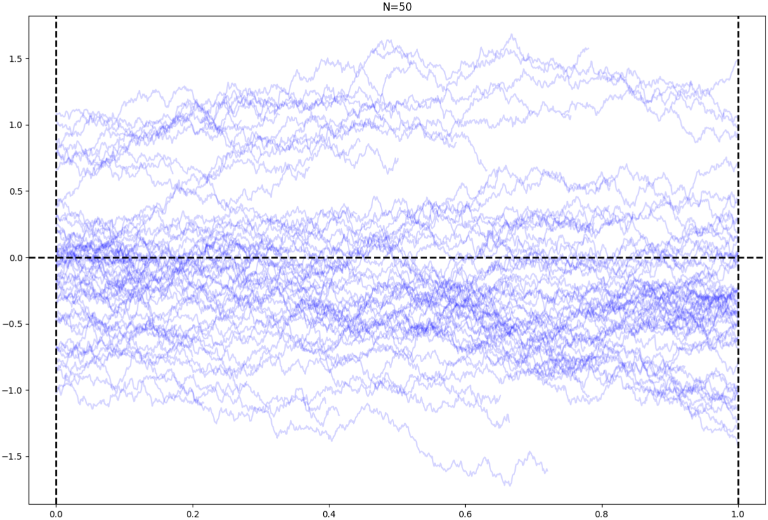

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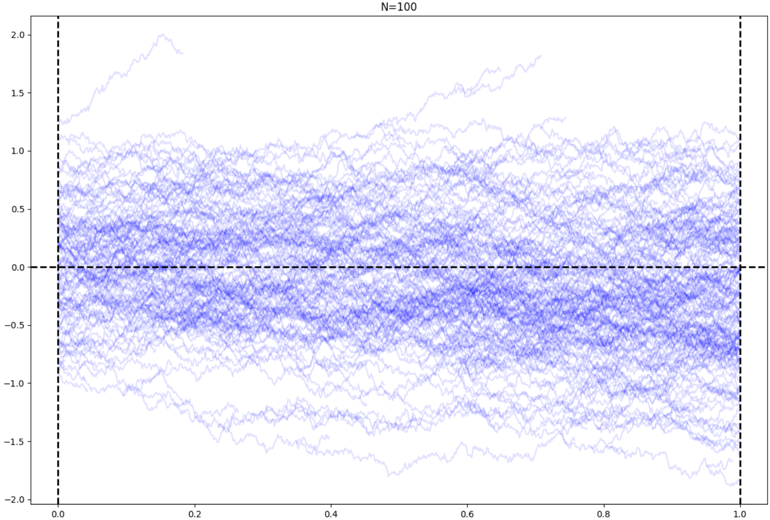

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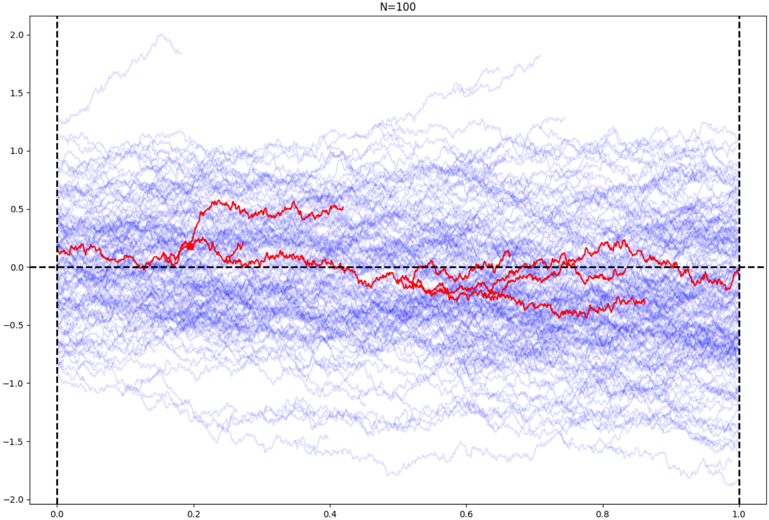

# Weak formulation

Let $\mathcal{A}$ be the set of $\left\{\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s\right\}_s$-predictable processes from $[0, T] \times \mathbb{R}^d$ to $A$.

# Weak formulation

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Let $\mathcal{L}^1$ be the generator

$$\begin{aligned}\mathcal{L}^1F_\varphi(x,\lambda,a) &= F'_\varphi(\lambda)L\varphi(x,\lambda,a)+\frac{1}{2}F''_\varphi(\lambda)\left|D\varphi(x)\sigma(x,\lambda,a)\right|^2\\ &\quad+\gamma(x,\lambda,a)\left(\sum_{k\geq 0}F_\varphi\left(\lambda+(k-1)\delta_x\right)p_k(x,\lambda,a)-F_\varphi\left(\lambda\right)\right).\end{aligned}$$

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### Definition

Fix $(t, \lambda) \in [0, T] \times M\left(\mathbb{R}^d\right)$ with $\lambda = \sum_{i \in V} \delta_{x^i}$ with $\lambda = \sum_{i \in V} \delta_{x^i}$ and $V$ finite. We say that $(\mathbb{P}, \alpha) \in \mathcal{P}(\mathbf{D}^d) \times \mathcal{A}$ is a *controlled branching diffusion process*, and we denote $(\mathbb{P}, \alpha) \in \mathcal{R}^1_{(t,\lambda)}$, if $\mathbb{P}(\mu_t = \lambda) = 1$ and the process

$$M_s^{F_\varphi} = F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}^1 F_\varphi(x, \mu_u, \alpha_u(x)) \mu_u(dx) du$$

is a $(\mathbb{P}, \mathbb{F})$-martingale for any $F \in C_b^2(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, and $s \geq t$.

## De-zooming

Consider the de-zooming of the population solution in $\mathbf{D}^d$ of the previously considered SDE

$$\xi_t^{(n)} = \frac{1}{n}\xi_{nt} = \frac{1}{n}\sum_{i\in V_{nt}} \delta_{Y^i_{nt}}.$$

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Therefore, the previous martingale problem translates into

$$F_\varphi(n\xi_s^{(n)}) - \int_t^s \int_{\mathbb{R}^d} \Bigg[\frac{1}{n}F'_\varphi(n\xi_u^{(n)})L\varphi\left(x, n\xi^{(n)}, \alpha_u(x)\right) + \\ + \frac{1}{2n^2}F''_\varphi\left(n\xi_u^{(n)}\right)\left|D\varphi(x)\sigma\left(x, n\xi^{(n)}, \alpha_u(x)\right)\right|^2 + \\ + n\gamma\left(x, n\xi^{(n)}, \alpha_u(x)\right)\left(\sum_{k\geq 0} F_\varphi\left(n\xi_u^{(n)} + (k-1)\delta_x\right)p_k - F_\varphi\left(n\xi_u^{(n)}\right)\right)\Bigg] n\xi_u^{(n)}(dx)du$$

is a $(\mathbb{P}, \mathbb{F})$-martingale for $s \geq t$, and for any $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$.

## Weak rescaled formulation

Renaming $\varphi/n$ with $\varphi$ and $\gamma$ for $n\gamma$ oin the previous computation, we obtain a new martingale problem. Let $\mathcal{L}^n$ be the generator

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\begin{aligned}
\mathcal{L}^n F_\varphi(x,\lambda,a) &= F'_\varphi(\lambda) L\varphi(x,\lambda,a) + \frac{1}{2n} F''_\varphi(\lambda) \left| D\varphi(x)\sigma(x,\lambda,a) \right|^2 \\
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## Controlled superprocesses

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# Scaling limit

## Theorem

*Fix $\alpha \in \mathcal{A}$, $t \in [0, T]$, and $\lambda, \lambda_n \in M(\mathbb{R}^d)$, for $n \geq 1$, such that $\lambda_n = \frac{1}{n} \sum_{i \in V_n} \delta_{x^{i,n}}$ and $\lambda_n \to \lambda$ for $n \to \infty$. Then,*

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- *$\mathbb{P}^n \to \mathbb{P}$ for $n \to \infty$;*
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Reward function:

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**Control problem:**

$$v_\infty(t, \lambda) = \sup \left\{ J_\infty(t, \lambda; \alpha) : \alpha \in \mathcal{A} \right\}.$$

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*And now, we can start optimizing ...*

# Optimization

# Dynamic programming principle

We recall

$$v_\infty(t,\lambda) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{(t,\lambda,\alpha)}}\left[\int_t^T \int_{\mathbb{R}^d} \psi\left(x,\mu_s,\alpha_s(x)\right)\mu_s(dx)ds + \Psi\left(\mu_T\right)\middle|\mu_t=\lambda\right] .$$

If we have an optimal control $\hat{\alpha}$, what is the behaviour of an optimally controlled trajectory $\left(\mu_s\right)_s$ under $\mathbb{P}^{(t,\lambda,\hat{\alpha})}$?
How $v_\infty\left(s,\mu_s\right)$ and $v_\infty\left(s+h,\mu_{s+h}\right)$ for $s, s+h \in [t,T]$ under $\mathbb{P}^{(t,\lambda,\hat{\alpha})}$?

**Theorem (Dynamic programming principle)**

*We have*

$$v_\infty(t,\lambda) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\left[\int_t^\tau \int_{\mathbb{R}^d} \psi(x,\mu_s,\alpha_s(x))\mu_s(dx)ds + v_\infty(\tau,\mu_\tau)\right],$$

*for any* $(t,\lambda)\in[0,T]\times M\left(\mathbb{R}^d\right)$, *and* $\tau$ *stopping time taking value in* $[t,T]$.

# Differential properties

## Definition

A continuous and bounded function $u : M(\mathbb{R}^d) \to \mathbb{R}$ has a **linear derivative $\delta_\lambda u$** if there exists a bounded function $\delta_\lambda u : M(\mathbb{R}^d) \times \mathbb{R}^d \ni (\lambda, x) \mapsto \delta_\lambda u(\lambda, x) \in \mathbb{R}$, continuous for the product topology, such that

$$u(\lambda) - u(\lambda') = \int_0^1 \int_{\mathbb{R}^d} \delta_\lambda u \left(t\lambda + (1-t)\lambda', x\right) (\lambda - \lambda')(dx)\, dt,$$

for $\lambda, \lambda' \in M(\mathbb{R}^d)$. We denote $C^1(M(\mathbb{R}^d))$ this class of functions.

We say $u$ has **intrinsic derivative $D_\lambda u$** if $u \in C^1(M(\mathbb{R}^d))$ and $\delta_\lambda u$ is of class $C^1$ with respect to the second variable, and

$$D_\lambda u(\lambda, x) = \partial_x \delta_\lambda u(\lambda, x).$$

We denote with $C^{1,1}(M(\mathbb{R}^d))$ this class of functions.

## Generalized martingale problem

We define the operator $\mathbf{L}$ on $u \in C_b^{2,2}(M(\mathbb{R}^d))$ by

$$\begin{aligned}\mathbf{L}u(\lambda, x, a) &= b(x,\lambda,a)^\top D_\lambda u(\lambda,x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,\lambda,a)\partial_x D_\lambda u(\lambda,x)\right)\\ &\quad + \frac{1}{2}\gamma(x,\lambda,a)\delta_\lambda^2 u(\mu,x,x)\end{aligned}$$

for $(x,\lambda,a) \in \mathbb{R}^d \times M(\mathbb{R}^d) \times A$.

### Proposition

*For $(t,\lambda) \in [0,T] \times M(\mathbb{R}^d)$ and $\alpha \in \mathcal{A}$, the following are equivalent:*

1. $(\mathbb{P}^{t,\lambda,\alpha}, \alpha) \in \mathcal{R}_{(t,\lambda)}$;
2. *the process*

$$M_s^u = u(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \boldsymbol{L}u(x,\mu_u,\alpha_u(x))\mu_u(dx)du$$

*is a $(\mathbb{P},\mathbb{F})$-martingale for any $u \in C_b^{2,2}(\mathbb{R}^d)$, and $s \geq t$.*

## HJB equation

$$H(x,\lambda,a,p,M,r) = b(x,\lambda,a)^\top p + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x,\lambda,a)M\right) + \frac{1}{2}\gamma(x,\lambda,a)r + \psi(x,\lambda,a).$$

**Theorem (Verification theorem)**

*Let* $V : [0,T] \times M(\mathbb{R}^d) \to \mathbb{R}$ *be a function living in* $C_b^{1,(2,2)}([0,T) \times M(\mathbb{R}^d)) \cap C^0([0,T] \times M(\mathbb{R}^d))$. *Suppose that* $V$ *satisfies*

$$\begin{cases} \partial_t V(t,\lambda) + \displaystyle\int_{\mathbb{R}^d} \inf_{a\in A} H\Big(x,\lambda,a,D_\lambda V(q),\partial_x D_\lambda V(q),\delta_\lambda^2 V(q,x)\Big)_{\big|q=(t,x,\lambda)} \lambda(dx) = 0 \\ V(T,\lambda) = \Psi(\lambda). \end{cases}$$

*and there exists a continuous function* $\hat{a}(t,x,\lambda)$ *valued in* $A$ *such that*

$$\hat{a}(t,x,\lambda) \in \arg\min_{a\in A} H\left(x,\lambda,a,D_\lambda v(t,\lambda,x),\partial_x D_\lambda v(t,\lambda,x),\delta_\lambda^2 v(t,\lambda,x,x)\right).$$

*Therefore, if* $\alpha^* = \{\alpha_s^*(x) := \hat{a}(s,x,\mu_s), s\in[t,T)\} \in \mathcal{A}$, *then* $V = v_\infty$ *and* $\alpha^*$ *is an optimal Markovian control.*

## Exercise

Assume that $b$, $\sigma$, and $\gamma$ do not depend on the measure. Fix $h \in C_b(\mathbb{R}^d)$ with $h \geq 0$ and define the following value function

$$v_\infty(t,\lambda) \quad = \quad \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^{t,\lambda,\alpha}}\Big[\exp(-\langle h,\mu_T\rangle)\Big] \ .$$

### Proposition

*Suppose there exists a function $w \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$, such that*

$$\begin{cases} -\partial_t w(t,x) - \sup_{a\in A}\Big\{ b(x,a)^\top Dw(t,x) + \frac{1}{2}Tr\left(\sigma\sigma^\top(x,a)D^2w(t,x)\right) \\ \qquad\qquad -\frac{1}{2}\gamma(x,a)w(t,x)^2 \Big\} = 0 \ , \\ w(T,x) = h(x) \ . \end{cases}$$

*Therefore, we have that*

$$v_\infty(t,\lambda) = \exp\left(\langle w(t,\cdot),\lambda\rangle\right) \ .$$

# Questions

# Questions

Modelling:

- Do these dynamics reflect reality?
- Which cost functions?
- What is important to control? Is it a direct control or an inverse control?

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Mathematics:

- Convergence rate for the scaling limit
- Characterization of regular solutions for the HJB equation
- Viscosity solutions for the HJB equation
- Simulations

Grazie per l'attenzione

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Domande ?