Large-scale dynamics of self-propelled particles moving through obstacles: How environment affects particle swarms

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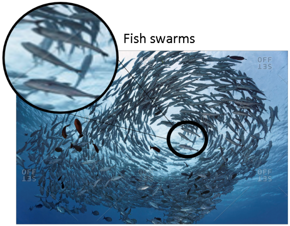
P. ACEVES-SANCHEZ - UCLA
E. KEAVENY - IMFT, Toulouse
L. SALA - INRAE

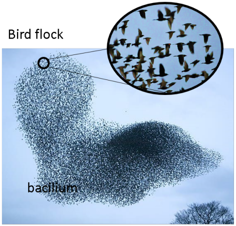
Collective motion: Examples in nature

Fish swarms



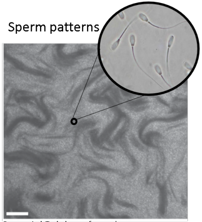
Swarm Bigeye trevallies (Caranx sexfasciatus) in the blue water, Pacific, French Polynesia, Oceania
Credit: Norbert Probst / ImageBROKER

Bird flock



Starlings flying to roost near Kendal in Cumbria, UK.
Credit: Ashley Cooper / The Image Bank / Getty

Sperm patterns

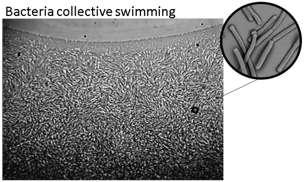


Creppy et al. Turbulence of swarming sperm. Phys. Rev. E. (2015)

Pedestrian crowds, lanes

Johansson et al, Crowd and environmental management during mass gatherings
The Lancet Infectious diseases (2012)

Bacteria collective swimming

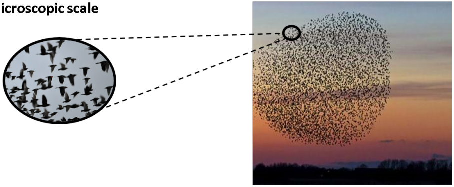


L. H. Cisneros, et al. Fluid dynamics of self-propelled microorganisms, from individuals to concentrated populations. Experiments in Fluids, 43:737–753, 2007.

- Self-organization: Emergence of large-scale ordered structures from local, small-scale interactions

Mathematical modelling: Choosing the right scale

Microscopic scale **Macroscopic scale**



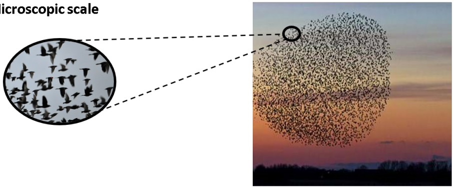
	Microscopic scale	Macroscopic scale
Variables	Agents positions, speed	Mean variables (density, orientation etc)
Model type	ODE systems (N equations)	PDE equations/systems
Advantages	- Modelling precision - Link with experimental data	- Theoretical analysis - Computational efficiency
Drawbacks	- Lack of theoretical results - Computationally challenging	- Loss of informations at the agent's scale
References	[Vicsek et al, (1995)],[Cucker and Smale. (2007)], [[S Bernardi, A Colombi, M Scianna (2018)] (swarms) [A. Kamal and E. E. Keaveny (2018)]	[J. A. Carrillo, Y.-P. Choi, S. Pérez (2017), J. A. Carrillo, D. Kalise, F. Rossi, E. Trélat (2022)] (flocks) [P. Degond, A. Manhart, and H. Yu (2018)] (alignment) [P. Degond, S. Merino-Aceituno, F. Vergnet, and H. Yu (2019)] [B. Maury, A. Roudneff-Chupin, and F. Santambrogio (2010)] (crowds) [M Burger, S Hittmeir, H Ranetbauer, MT Wolfram (2016)] (lane formation) [T. Hillen, K.J. Painter (2009)] (chemotaxis)

Mean-field: [Boley, Canizo, Carrillo (2011)], [J. A. Carrillo, Y.-P. Choi, M. Hauray (2014)]
Hydrodynamic limits: [Helbling (2001)], [Aw,Klar, Rascle, Materne (2002)], [Jiang, Xiong, Zhang (2016)], [Degond, Motsch (2008)], [Degond, Frouvelle, Liu (2012)], [Aceves-Sanchez, Bostan, Carrillo, Degond (2019)], [J.A Carrillo, M. Fornasier, J. Rosado, G. Toscani (2010)]
Diffusion limits: [HG Othmer, T Hillen (2000)]
Kinetic models:, [Calvez, V., Gosse, L. and Twarogowska, (2017)], [Peruani and Markus Bär 2013 New J. Phys. (2013)], [Carrillo, D'Orsogna, Panferov (2009)]

Mathematical modelling: Choosing the right scale

Macroscopic scale

Microscopic scale



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References	[Vicsek et al, (1995)],[Cucker and Smale. (2007)], [[S Bernardi, A Colombi, M Scianna (2018)] (swarms) [A. Kamal and E. E. Keaveny (2018)] **Swimmers in obstacles**	[J. A. Carrillo, Y.-P. Choi, S. Pérez (2017), J. A. Carrillo, D. Kalise, F. Rossi, E. Trélat (2022)] (flocks) [P. Degond, A. Manhart, and H. Yu (2018)] (alignment) **Swimmers in viscous fluid** [P. Degond, S. Merino-Aceituno, F. Vergnet, and H. Yu (2019)] [B. Maury, A. Roudneff-Chupin, and F. Santambrogio (2010)] (crowds) [M Burger, S Hittmeir, H Ranetbauer, MT Wolfram (2016)] (lane formation) [T. Hillen, K.J. Painter (2009)] (chemotaxis)

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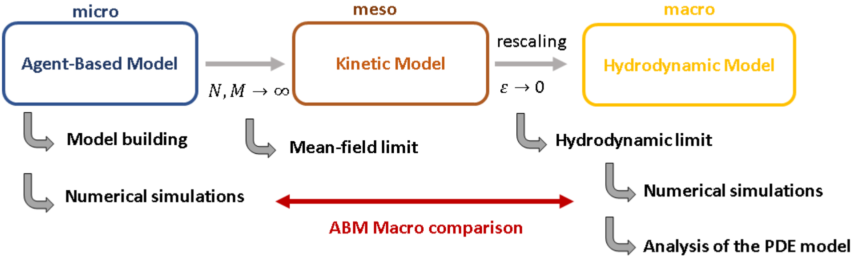
Plan of the talk

Objectives: Investigate pattern emergence using ABM and PDE



Plan of the talk

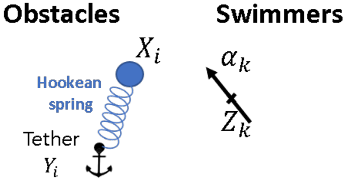
Objectives: Investigate pattern emergence using ABM and PDE

micro

Agent-Based Model

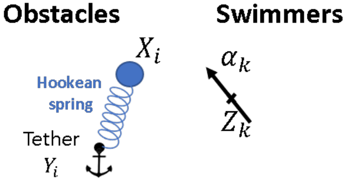
- **Model building**
- **Numerical simulations**

Agent-based model



- Obstacles: Centers $X_i(t) \in \mathbb{R}^2$, $i = 1 \dots N$ attached to anchor points $Y_i \in \mathbb{R}^2$ via hookean springs
- Swimmers: Centers $Z_k(t) \in \mathbb{R}^2$, self-propelled along orientation vectors $\alpha_k(t) \in \mathbb{S}^1$, $k = 1 \dots M$

Agent-based model



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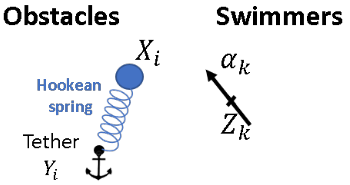
Equations of motion (Newton overdamped):

$$dX_i = -\frac{\kappa}{\eta}(X_i - Y_i)dt - \frac{1}{\eta}\frac{1}{M}\sum_{k=1}^{M} \nabla\phi\left(X_i - Z_k\right) dt + + \sqrt{2d_o}\, dB_t^i,$$

$$dZ_k = u_0 \alpha_k dt - \frac{1}{\zeta}\frac{1}{N}\sum_{i=1}^{N} \nabla\phi\left(Z_k - X_i\right) dt - \frac{1}{\zeta}\frac{1}{M}\sum_{l \neq k}^{M} \nabla\psi\left(Z_k - Z_l\right) dt,$$

$$d\alpha_k = P_{\alpha_k^\perp} \circ \left[\nu \bar{\alpha}_k dt + \sqrt{2d_s}\, d\tilde{B}_t^k\right]$$

Agent-based model



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$$dZ_k = u_0\alpha_k dt - \frac{1}{\zeta}\frac{1}{N}\sum_{i=1}^{N}\nabla\phi\,(Z_k - X_i)\,dt - \frac{1}{\zeta}\frac{1}{M}\sum_{l\neq k}\nabla\psi\,(Z_k - Z_l)\,dt,$$

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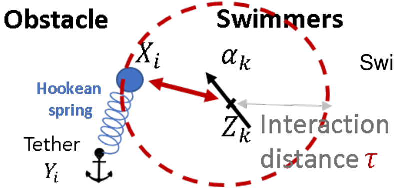
Agent-based model



Swimmer-Obstacle repulsion potential

$$\phi(x) = \frac{3C_\phi}{2\pi\tau}\left(1 - \frac{|x|}{\tau}\right)_+^2$$

Equations of motion (Newton overdamped):

$$dX_i = -\frac{\kappa}{\eta}(X_i - Y_i)dt - \underbrace{\frac{1}{\eta}\frac{1}{M}\sum_{k=1}^{M}\nabla\phi\,(X_i - Z_k)\,dt}_{\text{Sw-Ob repulsion}} + \sqrt{2d_o}\,dB_t^i,$$

$$dZ_k = u_0\alpha_k dt - \frac{1}{\zeta}\frac{1}{N}\sum_{i=1}^{N}\nabla\phi\,(Z_k - X_i)\,dt - \frac{1}{\zeta}\frac{1}{M}\sum_{l\neq k}\nabla\psi\,(Z_k - Z_l)\,dt,$$

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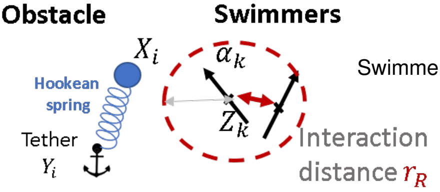
Agent-based model



Swimmer-Swimmer repulsion potential

$$\psi(x) = \frac{6\mu}{\pi {r_R}^2}\left(1 - \frac{|x|}{r_R}\right)_+^2$$

Equations of motion (Newton overdamped):

$$dX_i = -\frac{\kappa}{\eta}(X_i - Y_i)dt - \frac{1}{\eta}\frac{1}{M}\sum_{k=1}^{M}\nabla\phi\,(X_i - Z_k)\,dt + \sqrt{2d_o}\,dB_t^i,$$

$$dZ_k = u_0\alpha_k dt - \frac{1}{\zeta}\frac{1}{N}\sum_{i=1}^{N}\nabla\phi\,(Z_k - X_i)\,dt - \underbrace{\frac{1}{\zeta}\frac{1}{M}\sum_{l\neq k}^{M}\nabla\psi\,(Z_k - Z_l)\,dt}_{\text{Sw-Sw repulsion}},$$

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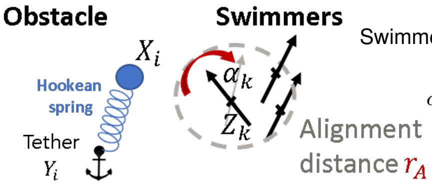
Agent-based model



Swimmer-Swimmer alignment

$$\bar{\alpha_k} = \frac{J_k}{|J_k|}, \quad J_k = \sum_{j=1, |Z_k - Z_j| \leq r_A}^{M} \alpha_j$$

Equations of motion (Newton overdamped):

$$dX_i = -\frac{\kappa}{\eta}(X_i - Y_i)dt - \frac{1}{\eta}\frac{1}{M}\sum_{k=1}^{M} \nabla\phi\,(X_i - Z_k)\,dt + \sqrt{2d_o}\,dB_t^i,$$

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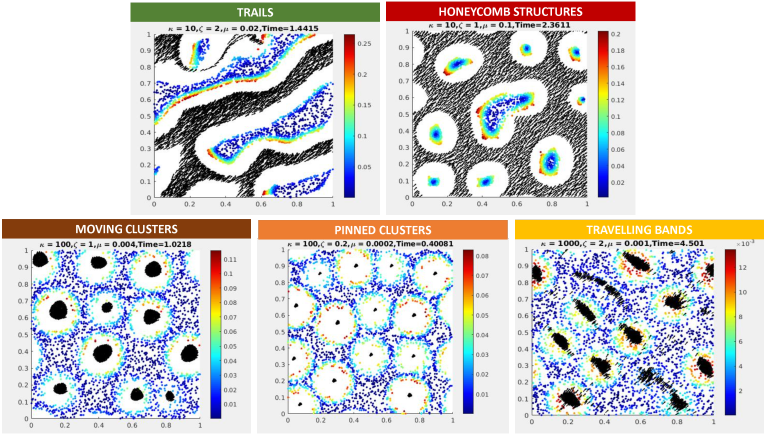
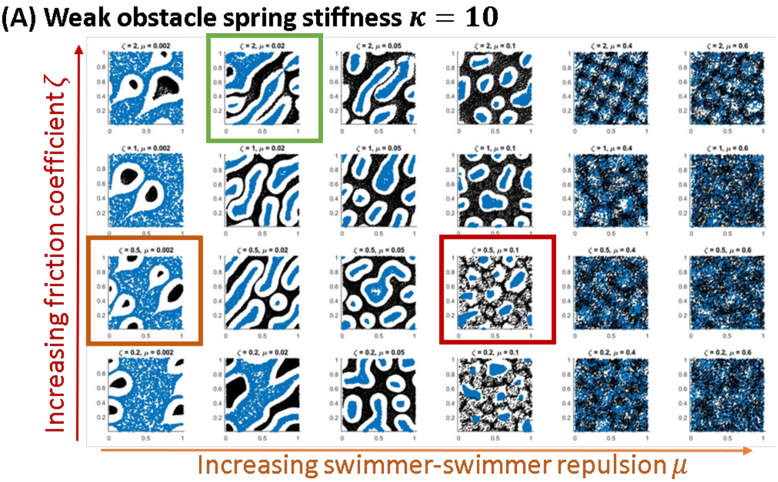
$$d\alpha_k = P_{\alpha_k^\perp} \circ \left[\overbrace{\nu\bar{\alpha}_k dt}^{\text{Sw-Sw alignment}} + \sqrt{2d_s}\,d\tilde{B}_t^k \right],$$

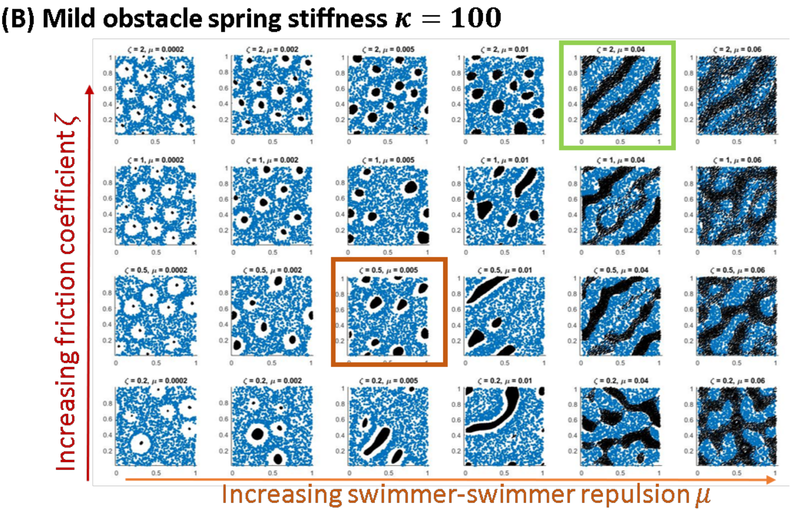
Numerical simulations - patterns



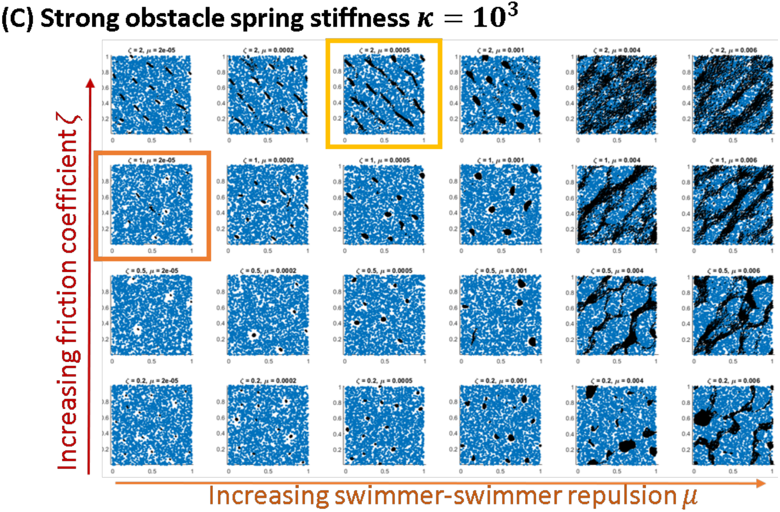
Numerical simulations - Weak obstacle spring stiffness



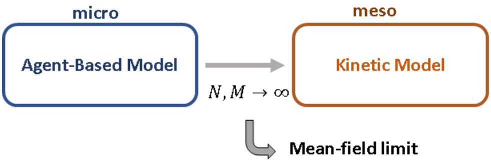
Numerical simulations - Mild obstacle spring stiffness



Numerical simulations - Strong obstacle spring stiffness



Derivation of a kinetic model in the limit $N, M \to \infty$



Distribution functions:

- $g^M(x, \alpha, t)$: density distribution of the M swimmers
- $f^N(x, y, t)$: density distribution of the N obstacles

Hydrodynamic model

Scaling assumptions:

- Scaling asumptions for the swimmers:
  - Small swimmer-swimmer alignment radius : $r_A = O(\epsilon)$
  - Small swimmer-swimmer repulsion distance: $r_R = O(\epsilon)$
  - Swimmer alignment rate and orientational noise intensity large: $d_S, \nu = O(\frac{1}{\epsilon})$, $\frac{d_S}{\nu} = O(1)$
- Scaling assumptions for the obstacles:
  - Uniform anchor density $\rho_A \equiv 1$
  - Stiff springs: $\gamma = \frac{\eta}{\kappa} \ll 1$
  - Low obstacle noise: $d = d_0 \gamma \ll 1$

Hydrodynamic model

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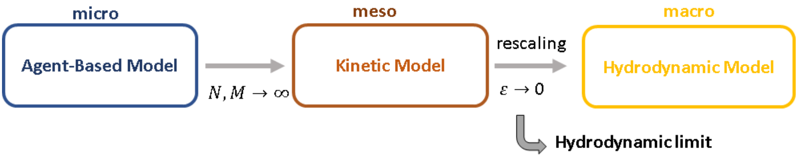
Hydrodynamic model



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  - Uniform anchor density $\rho_A \equiv 1$
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Observables:

- $\rho_g(x,t)$: local density of the swimmers
- $\Omega(x,t)$: local orientation of the swimmers
- $\rho_f(x,t)$: local density of the obstacles

Theorem (Macroscopic model)

The macroscopic swimmer density ρ_g and orientation Ω fulfil:

$$\partial_t \rho_g + \nabla \cdot (U \rho_g) = 0,$$
$$\rho_g \partial_t \Omega + \rho_g (V \cdot \nabla) \Omega + d_3 P_{\Omega^\perp} \nabla \rho_g = 0,$$

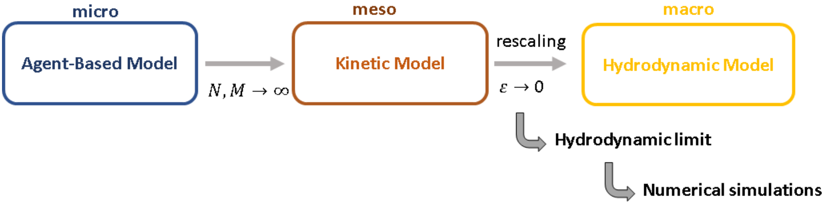
where

$$U = d_1 \Omega - \frac{1}{\zeta} \nabla \bar{\rho}_f - \frac{\mu}{\zeta} \nabla \rho_g,$$
$$V = d_2 \Omega - \frac{1}{\zeta} \nabla \bar{\rho}_f - \frac{\mu}{\zeta} \nabla \rho_g.$$

Under assumptions $\gamma \ll 1$ and $\delta \ll 1$, the obstacle density $\rho_f(x,t)$ is given by:

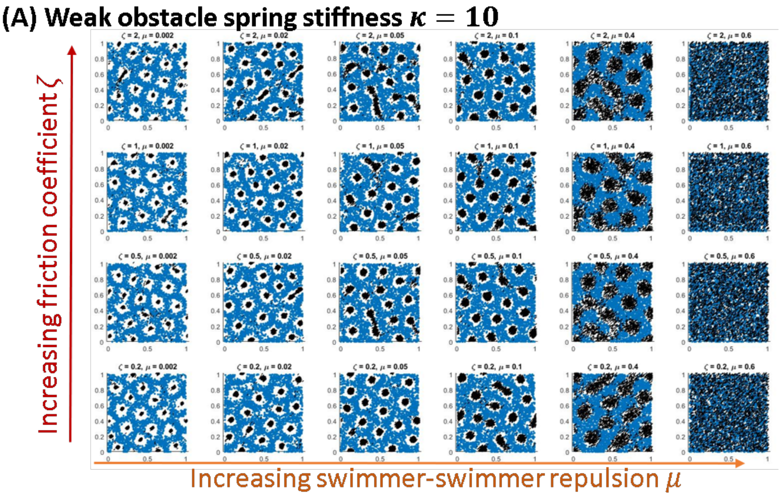
$$\rho_f = 1 + \frac{\gamma}{\eta} \Delta \bar{\rho}_g - \frac{\gamma^2}{\eta} \partial_t \Delta \bar{\rho}_g + \frac{\gamma^2}{\eta^2} \mathcal{N}(\bar{\rho}_g) + \mathcal{O}\left(\gamma^3\right), \quad \mathcal{N}(\bar{\rho}_g) := \det \mathbb{H}(\bar{\rho}_g),$$

Numerical simulations of the macroscopic model



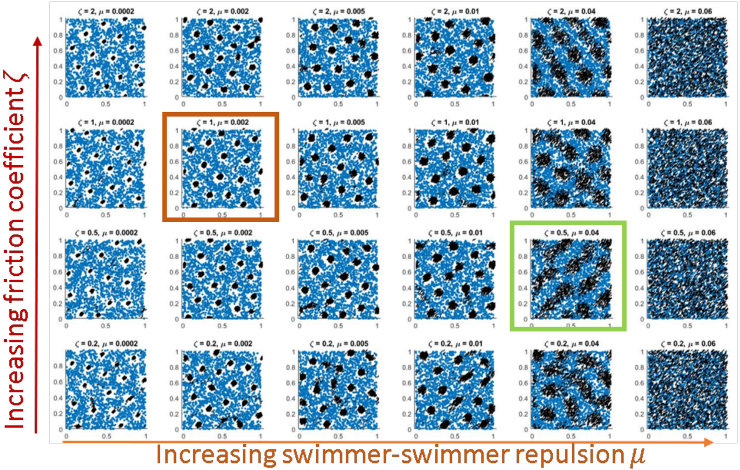
Numerical scheme adapted from [Motsch, Navoret, Mult. Mod. Simul., 2011], [Carrillo, Chertock, Huang. Comm. in Comp. Phys., 2015]

Numerical simulations - Weak obstacle spring stiffness



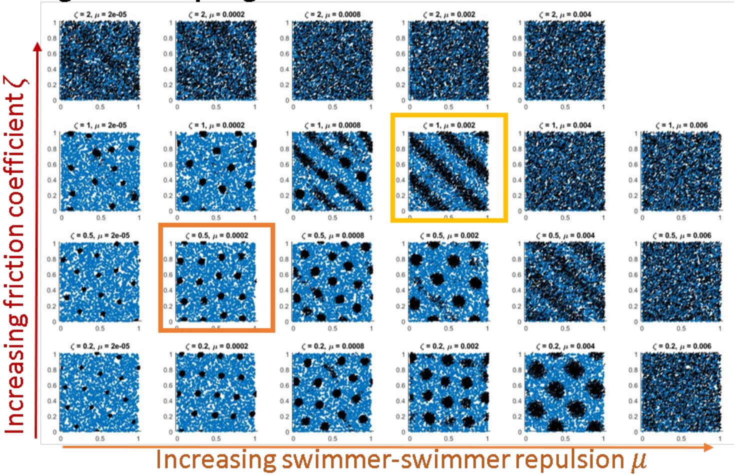
Numerical simulations - Mild obstacle spring stiffness

(B) Mild obstacle spring stiffness $\kappa = 100$

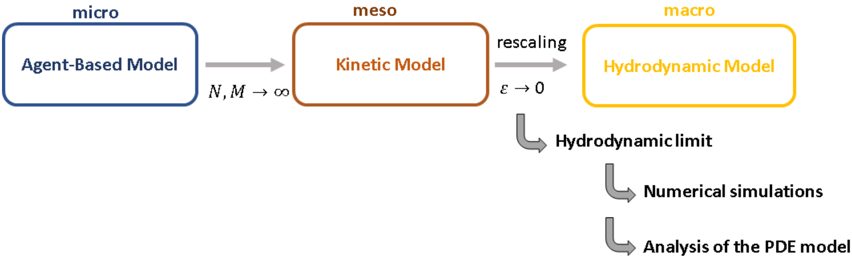


Numerical simulations - Strong obstacle spring stiffness

(C) Strong obstacle spring stiffness $\kappa = 10^3$



Numerical simulations - Mild obstacle spring stiffness



Linear stability analysis

- PDE system:

$$\partial_t \rho_g + \nabla \cdot \Big(\big(d_1 \Omega - \frac{1}{\zeta} \nabla \bar{\rho}_f - \frac{\mu}{\zeta} \nabla \rho_g \big) \rho_g \Big) = 0,$$

$$\rho_g \partial_t \Omega + \rho_g \Big(\big(d_2 \Omega - \frac{1}{\zeta} \nabla \bar{\rho}_f - \frac{\mu}{\zeta} \nabla \rho_g \big) \cdot \nabla \Big) \Omega + d_3 P_{\Omega^\perp} \nabla \rho_g = 0,$$

$$\rho_f = 1 + \frac{\gamma}{\eta} \Delta \bar{\rho}_g - \frac{\gamma^2}{\eta} \partial_t \Delta \bar{\rho}_g + \frac{\gamma^2}{\eta^2} \mathcal{N}(\bar{\rho}_g) + \mathcal{O}\left(\gamma^3\right),$$

- Linearize around (ρ_0, Ω_0): $\rho_g = \rho_0 + \epsilon \rho_1 + O(\epsilon^2)$, $\Omega = \Omega_0 + \epsilon \Omega_1 + O(\epsilon^2)$

$$\partial_t \rho_1 + d_1 \Omega_0 \cdot \nabla \rho_1 + d_1 \rho_0 \nabla \cdot \Omega_1 = \bar{\mu} \rho_0 \Delta \rho_1 + \rho_0 \bar{\lambda} (\Delta^2 \bar{\bar{\rho}}_1 - \gamma \Delta^2 \partial_t \bar{\bar{\rho}}_1),$$

$$\rho_0 \partial_t \Omega_1 + \rho_0 d_2 (\Omega_0 \cdot \nabla) \Omega_1 + d_3 P_{\Omega_0^\perp} \nabla \rho_1 = 0,$$

$$\Omega_0 \cdot \Omega_1 = 0, \ \bar{\lambda} = \frac{\rho_A}{\kappa \zeta}$$

- Consider plane wave perturbations

$$\rho_1(x,t) = \tilde{\rho} e^{ik \cdot x + \alpha t}, \qquad \Omega_1(x,t) = \tilde{\Omega} e^{ik \cdot x + \alpha t}$$

Linear stability analysis

- PDE system:

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$$\partial_t \rho_1 + d_1 \Omega_0 \cdot \nabla \rho_1 + d_1 \rho_0 \nabla \cdot \Omega_1 = \bar{\mu} \rho_0 \Delta \rho_1 + \rho_0 \bar{\lambda} \big(\Delta^2 \bar{\bar{\rho}}_1 - \gamma \Delta^2 \partial_t \bar{\bar{\rho}}_1 \big),$$

$$\rho_0 \partial_t \Omega_1 + \rho_0 d_2 \left(\Omega_0 \cdot \nabla \right) \Omega_1 + d_3 P_{\Omega_0^\perp} \nabla \rho_1 = 0,$$

$$\Omega_0 \cdot \Omega_1 = 0, \ \bar{\lambda} = \frac{\rho_A}{\kappa \zeta}$$

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Linear stability analysis

- PDE system:

$$\partial_t \rho_g + \nabla \cdot \left(\left(d_1 \Omega - \frac{1}{\zeta} \nabla \bar{\rho}_f - \frac{\mu}{\zeta} \nabla \rho_g \right) \rho_g \right) = 0,$$

$$\rho_g \partial_t \Omega + \rho_g \left(\left(d_2 \Omega - \frac{1}{\zeta} \nabla \bar{\rho}_f - \frac{\mu}{\zeta} \nabla \rho_g \right) \cdot \nabla \right) \Omega + d_3 P_{\Omega^\perp} \nabla \rho_g = 0,$$

$$\rho_f = 1 + \frac{\gamma}{\eta} \Delta \bar{\rho}_g - \frac{\gamma^2}{\eta} \partial_t \Delta \bar{\rho}_g + \frac{\gamma^2}{\eta^2} \mathcal{N}(\bar{\rho}_g) + \mathcal{O}\left(\gamma^3\right),$$

- Linearize around (ρ_0, Ω_0): $\rho_g = \rho_0 + \epsilon \rho_1 + O(\epsilon^2)$, $\Omega = \Omega_0 + \epsilon \Omega_1 + O(\epsilon^2)$

$$\partial_t \rho_1 + d_1 \Omega_0 \cdot \nabla \rho_1 + d_1 \rho_0 \nabla \cdot \Omega_1 = \bar{\mu} \rho_0 \Delta \rho_1 + \rho_0 \bar{\lambda} (\Delta^2 \bar{\bar{\rho}}_1 - \gamma \Delta^2 \partial_t \bar{\bar{\rho}}_1),$$

$$\rho_0 \partial_t \Omega_1 + \rho_0 d_2 \left(\Omega_0 \cdot \nabla \right) \Omega_1 + d_3 P_{\Omega_0^\perp} \nabla \rho_1 = 0,$$

$$\Omega_0 \cdot \Omega_1 = 0, \; \bar{\lambda} = \frac{\rho_A}{\kappa \zeta}$$

- Consider plane wave perturbations

$$\rho_1(x,t) = \tilde{\rho} e^{ik \cdot x + \alpha t}, \qquad \Omega_1(x,t) = \tilde{\Omega} e^{ik \cdot x + \alpha t}$$

Linear stability analysis

Theorem (Bifurcation parameter)

The linearized system around (ρ_0, Ω_0) is unstable iff

$$b_p = \frac{\mu\kappa}{C_\phi^2 c_0'} < 1.$$

Characterisation of the most unstable modes:
Maximally disturbed modes :

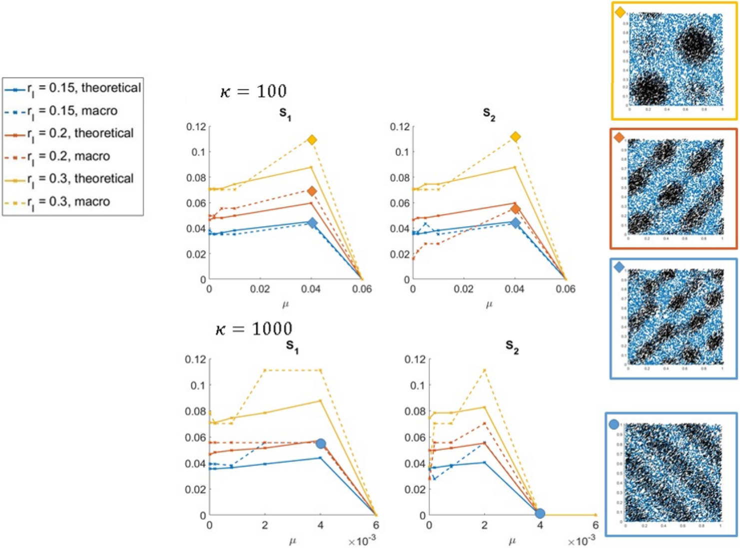
$$k_{\parallel}^{th} = \underset{k \parallel \Omega_0}{\operatorname{argmax}} \operatorname{Re}(\tilde{\alpha}(k)),$$

$$k_{\perp}^{th} = \underset{k \parallel \Omega_0^{\perp}}{\operatorname{argmax}} \operatorname{Re}(\alpha(k)),$$

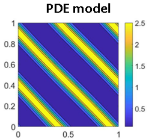
Related to size of perturbations

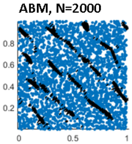
$$S_1^{th} = \frac{2\pi}{|k_{\parallel}^{th}|}, \quad S_2^{th} = \frac{2\pi}{|k_{\perp}^{th}|}.$$

Validation of the theoretical predictions (macro model)



Micro-macro comparison





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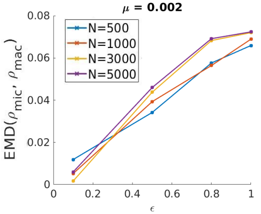
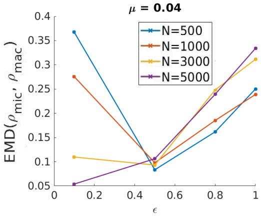
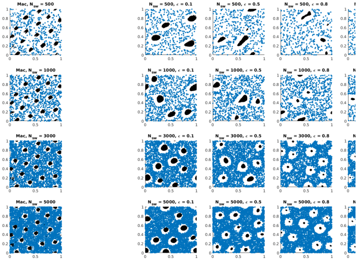
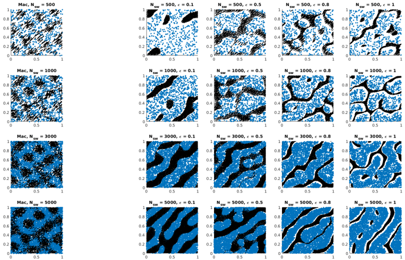








Conclusions/perspectives

Conclusions

- Model for collective motion of active particles interacting in a viscoelastic medium
  Variety of patterns depending on the interactions with the environment
- Attraction not required for swimmer aggregation
  Obstacles can induce aggregation irrespective of whether they repel or attract the particles => rethink cause of biological aggregation
- Linear stability of PDE
  Bifurcation parameter controling the appearance and shape of patterns
- Micro-macro comparison
  Quantitative agreement between ABM and PDE simulations

Perspectives

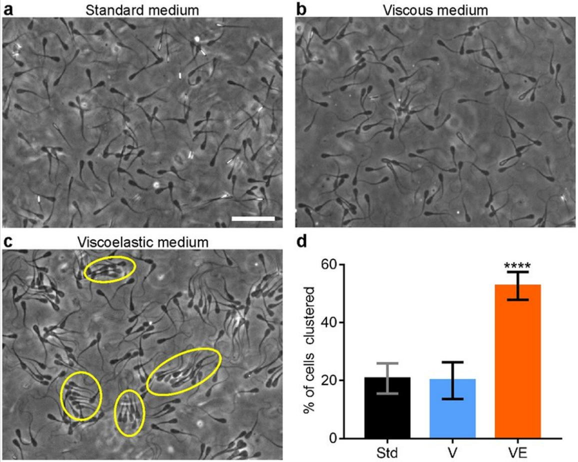
- Coupling with fluid model
- Different types of obstacles (elongated fibers)

[P. Aceves-Sanchez, P. Degond, E.E. Keaveny, A. Manhart, S. Merino-Aceituno, D. P., Large-scale dynamics of self-propelled particles moving through obstacles: model derivation and pattern formation. Bull. Math. Bio. (2020) 82(129)]

[P. Degond, A. Manhart, S. Merino-Aceituno, D. P., L. Sala, How environment affects active particle swarms: a case study, R. Soc. open sci (2022) 9:220791]

Thank you !

Collective motion: The role of the environment



Tung et al, Sci. Reports, 2017

Lemma 1 (Kinetic model)

Formally, as $N, M \to \infty$, $f^N \to f$ and $g^M \to g$ where $f(x, y, t)$ and $g(x, \alpha, t)$ fulfil:

$$\partial_t f + \nabla_x \cdot (\mathcal{W} f) = d_0 \Delta_x f$$

$$\partial_t g + \nabla_x \cdot (\mathcal{U} g) + \nabla_\alpha \cdot \left(P_{\alpha^\perp}[\nu \bar{\alpha}] g\right) = d_s \Delta_\alpha g,$$

where

$$\bar{\alpha} = \frac{J_g(x,t)}{|J_g(x,t)|}, \quad J_g(x,t) = \int_{|x-z| \le r_A} \alpha g(z, \alpha, t) dz d\alpha.$$

The velocities are given by

$$\mathcal{W} = -\frac{\kappa}{\eta}(x-y) - \frac{1}{\eta} \nabla_x \bar{\rho}_g(x,t)$$

$$\mathcal{U} = \alpha - \frac{1}{\zeta} \nabla_x \bar{\rho}_f(x,t) - \frac{1}{\zeta} \nabla_x \hat{\rho}_g(x,t),$$

with

$$\rho_g(x,t) = \int g(x, \alpha, t) d\alpha, \quad \rho_f(x,t) = \int f(x, y, t) dy$$

and

$$\bar{\rho}(x,t) = \phi * \rho(x,t), \quad \hat{\rho}(x,t) = \psi * \rho(x,t)$$

Elements of proof for the hydrodynamic limit:

- Swimmer equation: Self-Organized Hydrodynamics:
  Collision operator $Q(g) = -\nabla_\alpha \cdot (P_{\alpha^\perp}[\nu\bar{\alpha}]g) + d\Delta_\alpha g$
  [Degond, Motsch, 2008] (SOH)
  [Degond, Dimarco, Mac, Wang, 2015] (SOHR)
- Obstacle equation:

$$\partial_t f + \nabla_x \cdot (\mathcal{W} f) = d_0 \Delta_x f, \quad \mathcal{W} = -\frac{\kappa}{\eta}(x-y) - \frac{1}{\eta}\nabla\bar{\rho}_g(x,t)$$

- Rewrite & rescale: $\gamma = \eta/\kappa$, $\delta = d_0\gamma$

$$\partial_t f + \frac{1}{\eta}\nabla_x \cdot (-\nabla_x \bar{\rho}_g f) = \frac{1}{\gamma}\overbrace{[\nabla_x \cdot ((x-y)f + \delta\nabla_x f]}^{\text{Fokker-Planck Operator}}$$

- Expand $f(x,y,t)$

$$f(x,y,t) = f_0(x,y,t) + \gamma f_1(x,y,t) + \gamma^2 f_2(x,y,t) + O(\gamma^3)$$

- Strong spring limit $\gamma \to 0$, small noise limit $\delta \to 0$ => use Fokker-Planck Operator properties
- Gives $\rho_f(x,t)$ expanded in terms of γ and δ, $\rho_f(x,t) = \int f(x,y,t)dy$

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Substituting the plane wave ansatz into the equation yields

$$\tilde{\rho}\alpha + i\tilde{\rho}d_1\left(\Omega_0 \cdot k\right) + i\rho_0 d_1\left(\tilde{\Omega} \cdot k\right) = -|k|^2\bar{\mu}\rho_0\tilde{\rho} + |k|^4\bar{\lambda}\rho_0\tilde{\rho}(\hat{\phi}_k)^2\left(1 - \gamma\alpha\right), \tag{1a}$$

$$\rho_0\alpha\tilde{\Omega} + i\rho_0 d_2\tilde{\Omega}\left(\Omega_0 \cdot k\right) + i\tilde{\rho}d_3 P_{\Omega_0^\perp}k = 0, \tag{1b}$$

$$\Omega_0 \cdot \tilde{\Omega} = 0. \tag{1c}$$

or (if $\tilde{\Omega} = \omega\Omega_0^\perp$)

$$(G(|k|)\alpha - F(|k|) + id_1k_0)\tilde{\rho} + i\rho_0 d_1 k_1\omega = 0,$$
$$id_3k_1\tilde{\rho} + \rho_0(\alpha + id_2k_0)\omega = 0.$$

This is a homogeneous linear system in $(\tilde{\rho}, \omega)$ which has a non-trivial solution if and only if the determinant of the system is 0, *i.e.*:

$$\left(G(|k|)\alpha - F(|k|) + id_1k_0\right)\left(\alpha + id_2k_0\right) + d_1d_3k_1^2 = 0. \tag{2}$$

Linear stability analysis

Theorem (Bifurcation parameter)

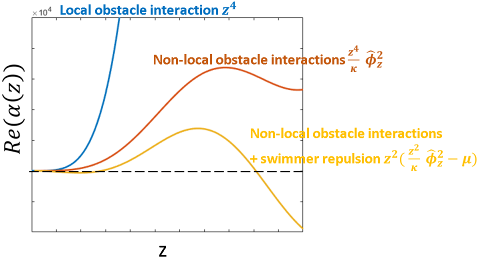
Consider fixed constant values $\rho_0 > 0$ and $\Omega_0 \in \mathbb{S}^1$. Then, the linearized system around (ρ_0, Ω_0) is unstable if and only if

$$\textit{there exists } z > 0 \textit{ such that } z^2(\hat{\phi})^2 > \mu\kappa.$$

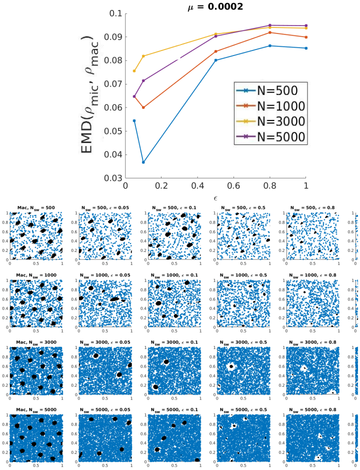
Obstacle interaction function:
$\phi(x) = \frac{3C_\phi}{2\pi\tau}\left(1 - \frac{|x|}{\tau}\right)_+^2$

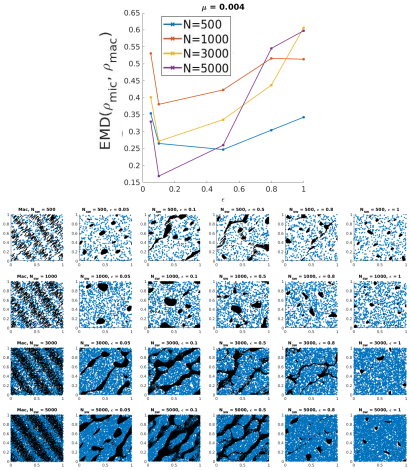
Real part of $\alpha(z)$:
$Re(\alpha(z)) = \frac{\rho_0}{\zeta} z^2 [\frac{z^2}{\kappa} \hat{\phi}_z^2 - \mu]$

Perturbations of the form $\rho_1(x,t) = e^{\alpha t + iz.x}$



Micro-macro comparison





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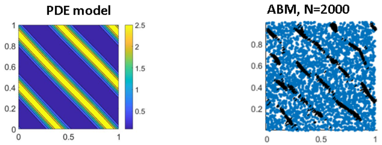

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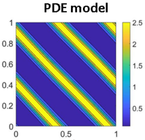

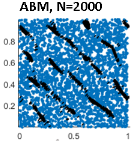

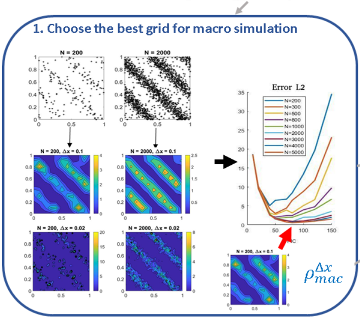

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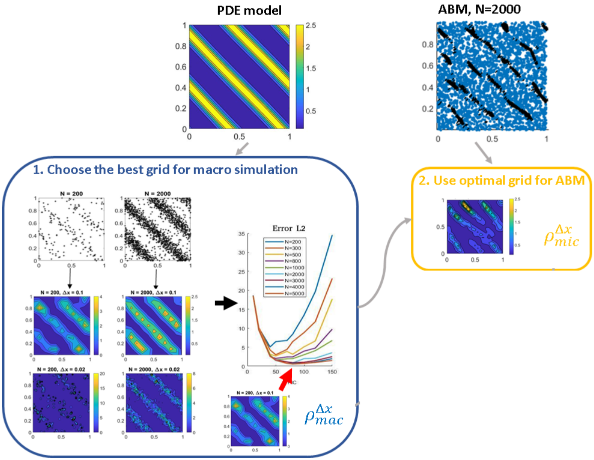

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