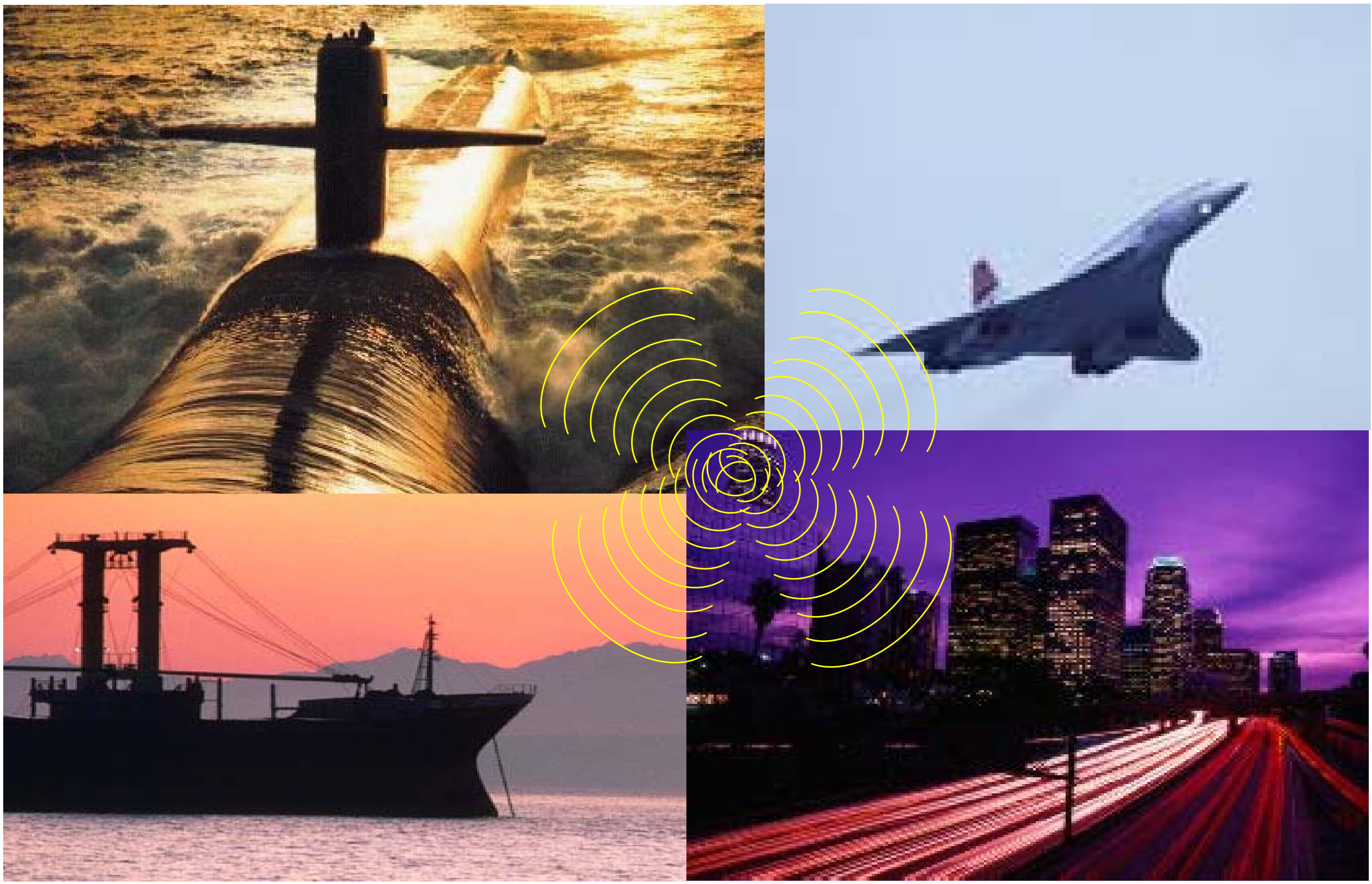
New high-order, high-frequency methods in computational electromagnetism

Oscar P. Bruno
Caltech

JPL
NSF
TRW
AFOSR
DARPA
Lockheed Martin

Surfaces and volumes in 3D



Topics

- *Direct integral solvers ……………(Bruno & Kunyansky, [2001])*
  - *Regular-surface, singular-kernel integration*
  - *Acceleration*
  - *Singular surfaces and kernels*

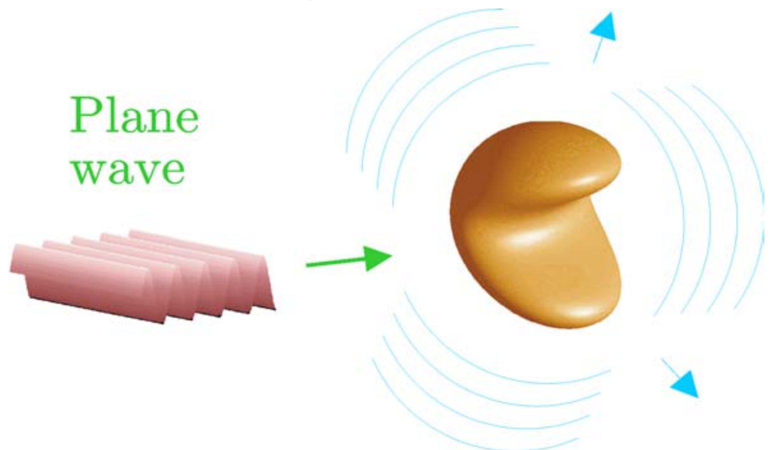
- *High order surface representation…(Bruno & Pohlman, [2003])*

- *High-frequency, high-order, O(1) integral solvers*

  Convex obstacles…………(Bruno, Geuzaine and Monro, [2002])

  Non-convex obstacles………… (Bruno and Reitich, [2002-03])

Governing Equations



$$\Delta\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = 0$$

$$\nabla \times E = i\omega\mu H$$

$$\nabla \times H = -i\omega\varepsilon H$$

$$\frac{1}{2}\varphi(\mathbf{r}) + (K\varphi)(\mathbf{r}) - i\gamma(S\varphi)(\mathbf{r}) = \psi^i(\mathbf{r}), \qquad \mathbf{r} \in \partial D$$

$$\Phi(\mathbf{r}, \mathbf{r}') = e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} / 4\pi\left|\mathbf{r}-\mathbf{r}'\right|$$

$$(K\varphi)(\mathbf{r}) = \int_{\partial D} \varphi(\mathbf{r}') \frac{\partial}{\partial \boldsymbol{\nu}(\mathbf{r}')} \Phi(\mathbf{r}, \mathbf{r}') \, dS(\mathbf{r}')$$

$$(S\varphi)(\mathbf{r}) = \int_{\partial D} \Phi(\mathbf{r}, \mathbf{r}')\varphi(\mathbf{r}') dS(\mathbf{r}')$$

These fast high-order solvers resulted from a number of innovations, including:

1) *Use of smooth Partitions-of-Unity and Local Smooth Parameterizations -which make the trapezoidal rule a high-order integrator*

2) *Analytic Resolution of Singularities - to avoid costly refinement strategies*

3) *Use of Dual Grids and Equivalent Sources located on a sparsely distributed Planar Grids - to reduce convolutions to sparse three-dimensional FFT's*

4) *Convergent evaluation of oscillatory integrals, through stationary phase and critical points*

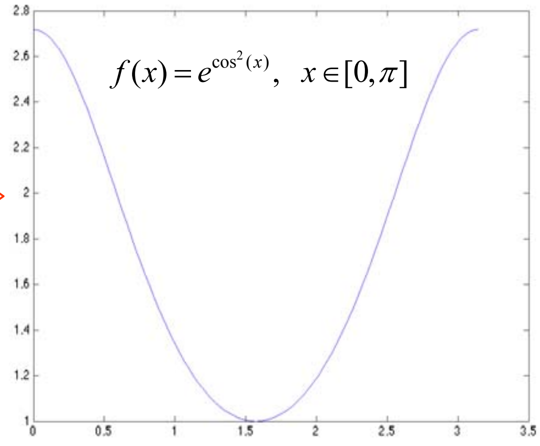
High-order Integration and the Trapezoidal Rule

$$\int_0^{\pi/4} f(x)\,dx \approx 1.8009$$

N	Rel. Error	Ratio
1	4.77(-2)	
2	1.19(-2)	4.03
4	2.95(-3)	4.02
8	7.36(-4)	4.01
8192	7.01(-10)	

$$\int_0^{\pi} f(x)\,dx \approx 5.5084$$

N	Rel. Error	Ratio
1	5.50(-1)	
2	6.03(-2)	9.12
4	3.10(-4)	1.95(2)
8	7.17(-10)	4.32(5)
16	2.10(-23)	3.42(13)



Fast, High-Order Direct Solver

Relation to other methods

Described towards the end of this presentation

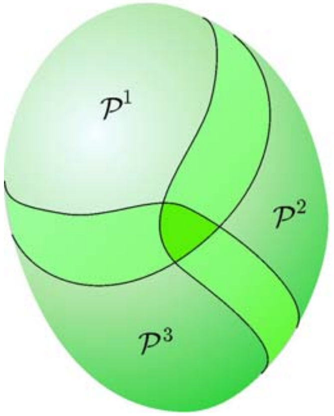
Present Approach

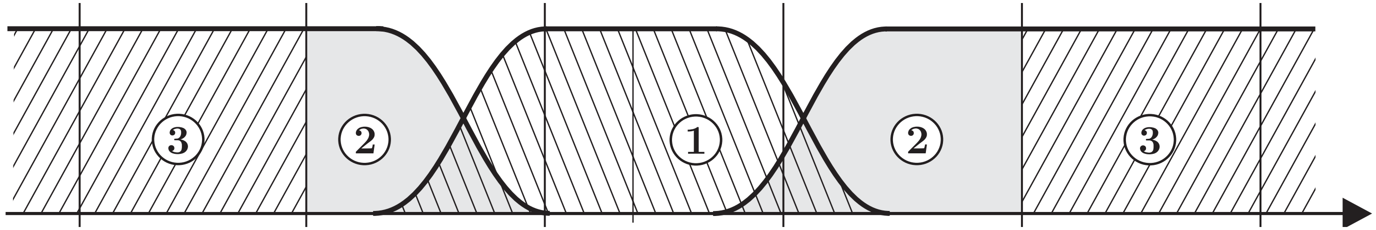
High-Order, Fast, Stable, Accurate $O(N^{6/5}\log(N))$ ---
$O(N^{4/3}\log(N))$ *operations*

(Acceleration strategy does not lead to accuracy breakdowns)

Bruno and Kunyansky, [2001]

Partitions of Unity...





→ *localize integration problem:*

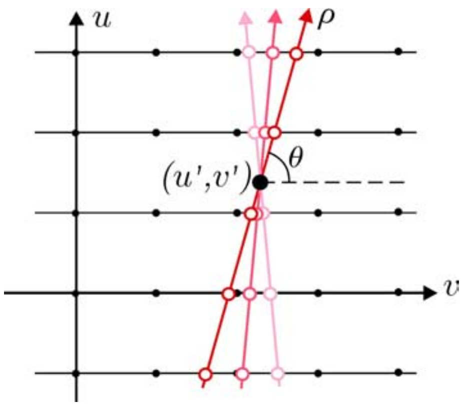
$$\int_{\partial \mathcal{D}} \ldots ds = \sum_j \int_{\mathcal{P}j} \ldots w_j(u_j, v_j)\, du_j dv_j$$

Resolution of singularities

(Basic, high-order solver; adjacent interactions)

$$\cos k\,|\mathbf{R}|\,\frac{\mathbf{R}\cdot\boldsymbol{\nu}(r)}{\mathbf{R}^3}$$

A polar-coordinate jacobian regularizes the integration problem



$$L(u',v',\theta)=\int_{-r_1}^{r_1} f_k^*(\rho,\theta)\frac{|\rho|}{|\mathbf{R}|}\cos k\,|\mathbf{R}|\,\frac{\mathbf{R}\cdot\boldsymbol{\nu}(r)}{\mathbf{R}^2}d\rho$$

Accuracy of the basic non-accelerated solver

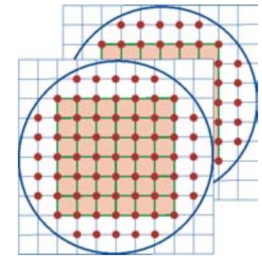
Scattering by a sphere of radius 2.7 λ

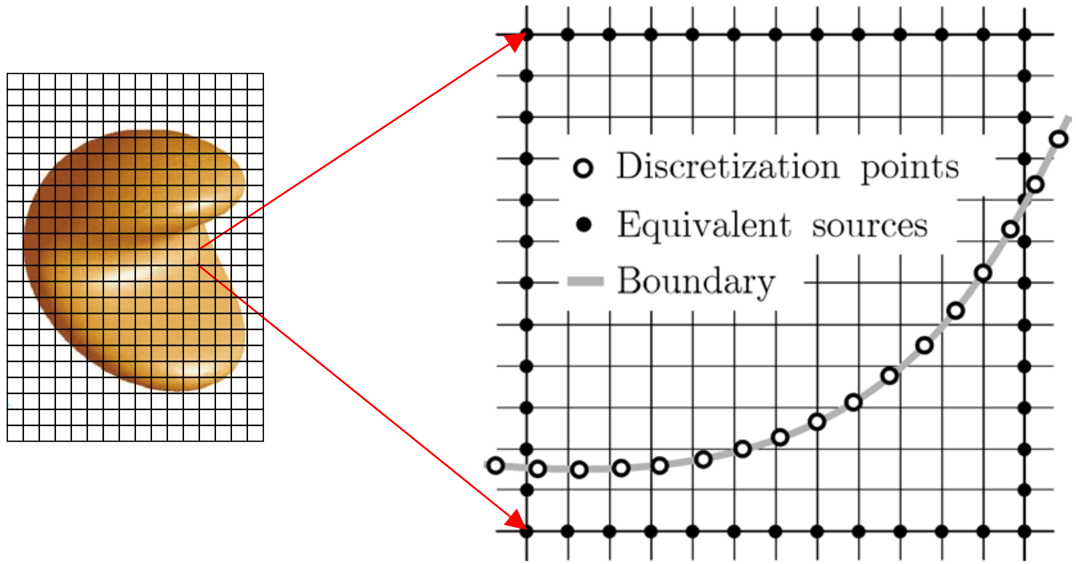
Patches	Unknowns	Discretization density	Max Error	RMS
$6 \times 17 \times 17$	1350	3 per 1λ	0.1	2.9×10^{-2}
$6 \times 33 \times 33$	5766	6 per 1λ	9.0×10^{-4}	1.8×10^{-4}
$6 \times 65 \times 65$	23790	12 per 1λ	3.6×10^{-6}	1.4×10^{-6}
$6 \times 129 \times 129$	93726	24 per 1λ	1.6×10^{-8}	5.6×10^{-9}

Doubling the discretization density improves the accuracy by 200 to 300 times!

Equivalent Sources

(Acceleration; Non-adjacent interactions)





$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n} h_n^{(1)}(k\,|\mathbf{r}|)Y_n^m(\mathbf{r}/\,|\mathbf{r}|)j_n(k\,|\mathbf{r}'|)\overline{Y_n^m(\mathbf{r}'/\,|\mathbf{r}'|)}$$

Bruno and Kunyansky, [2001]

Previous Work

- *Integral-equations; Fast Methods (low order)*

- *Finite-difference/finite-element methods*

- *None of the existing algorithms have been designed to perform in a fast <u>and</u> high-order fashion*

Present Approach

- *High-Order, Fast, Stable, Accurate* $O(N^{6/5}\log(N))$ --- $O(N^{4/3}\log(N))$ *operations*

- *Acceleration strategy does not lead to accuracy breakdowns*

→ *Examples...*

Remark 5. The last theorem proves the convergence of the discretized approximated kernel which is used numerically. Unfortunately, because of roundoff errors, this convergence is not numerically attained...

$$G(x;x') \approx G_N^D(x;x0) := \frac{1}{2\pi N_T} \sum_{n_T=1}^{N_T} e^{(ik(x-z_i)\cdot U(\theta_{n_T}))}$$

$$\cdot e^{(-ik(x'-z_j)U(\theta_{n_T}))} \cdot$$

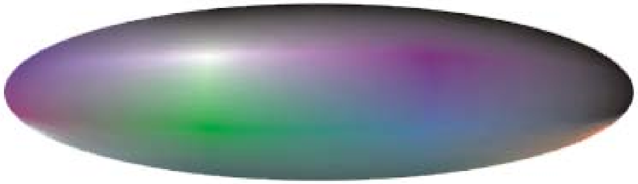
$$\left[\sum_{m=-N}^{N} e^{(im(\theta_{n_T}-arg(z_i-z_j)))} K_{|m|}(-ik|z_i - z_j|)\right] \qquad (6)$$

$$\left(\theta_{n_T} = \frac{2\pi}{N_T} n_T\right)$$

The main difficulty we face in studying Rokhlin's method lies in the fact that, even if from a theoretical point of view (see Theorems 2, 4, 6 and 7) the greater N the more accurate the approximation, N must (in numerical simulations) belong to a fixed range of integers. If N is too small, the overall accuracy is not good, which is quite logical. But if N is too large, then (6) is not numerically accurate... Hopefully, there is a range of integer values N such that the accuracy of Rokhlin's formula (6) is quite good...

C. Labreuche, "A convergence theorem for the fast multipole method for 2-dimensional scattering problems", Math. Comp. 67, 553-

Large ellipsoids



Size	# It	T/it	RAM	Unknowns	Max Error	RMS Error
$80\lambda \times 20\lambda \times 20\lambda$	15	5h 22m	$600M$	691206	$1.4 \cdot 10^{-4}$	$2.9 \cdot 10^{-5}$
$100\lambda \times 25\lambda \times 25\lambda$	15	5h 29m	$600M$	691206	$1.1 \cdot 10^{-3}$	$2.4 \cdot 10^{-4}$

One of the largest scattering problems ever solved!

Scattering from bodies of similar sizes has been evaluated using:

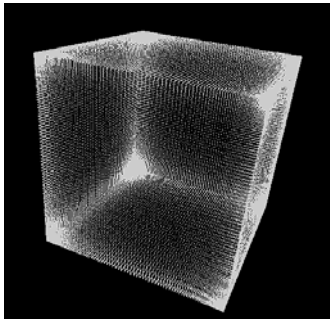
- 40 IBM SP2 nodes (AIM, E. Bleszynski et al, 1996);
- 256 IBM SP nodes (FVTD, J. S. Shang et al, 2000);
- SGI Origin 2000 (8 proc.) (FISC, J. M.Song et al, 1998).

The present results are obtained on 400 MHz 1G Pentium II PC.

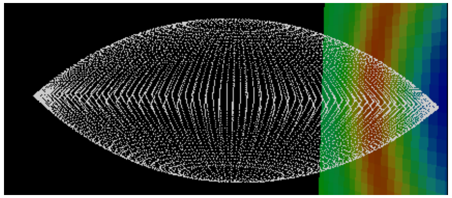
Large spheres

(comparison w/ O(N log(N)) FISC)

Algorithm	Diameter	Time	RAM	Unknowns	RMS Error	Computer
FISC	120λ	$32 \times 14.5h$	$26.7Gb$	9,633,792	4.6%	SGI Origin 2000 (32 proc.)
Present	80λ	$55h$	$2.5Gb$	1,500,000	0.005%	AMD 1.4GHz (1 proc.)
Present	100λ	$68h$	$2.5Gb$	1,500,000	0.03%	AMD 1.4GHz (1 proc.)

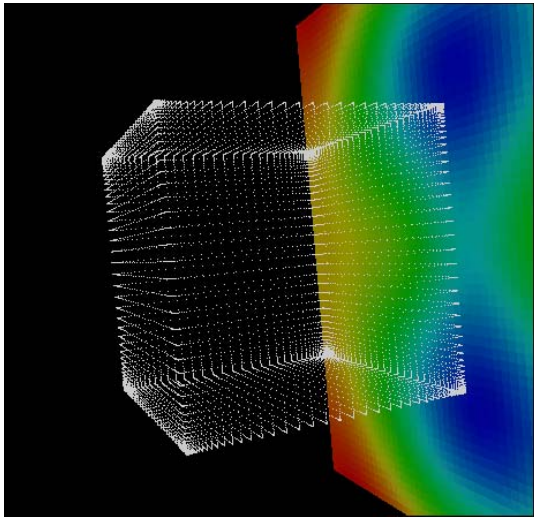


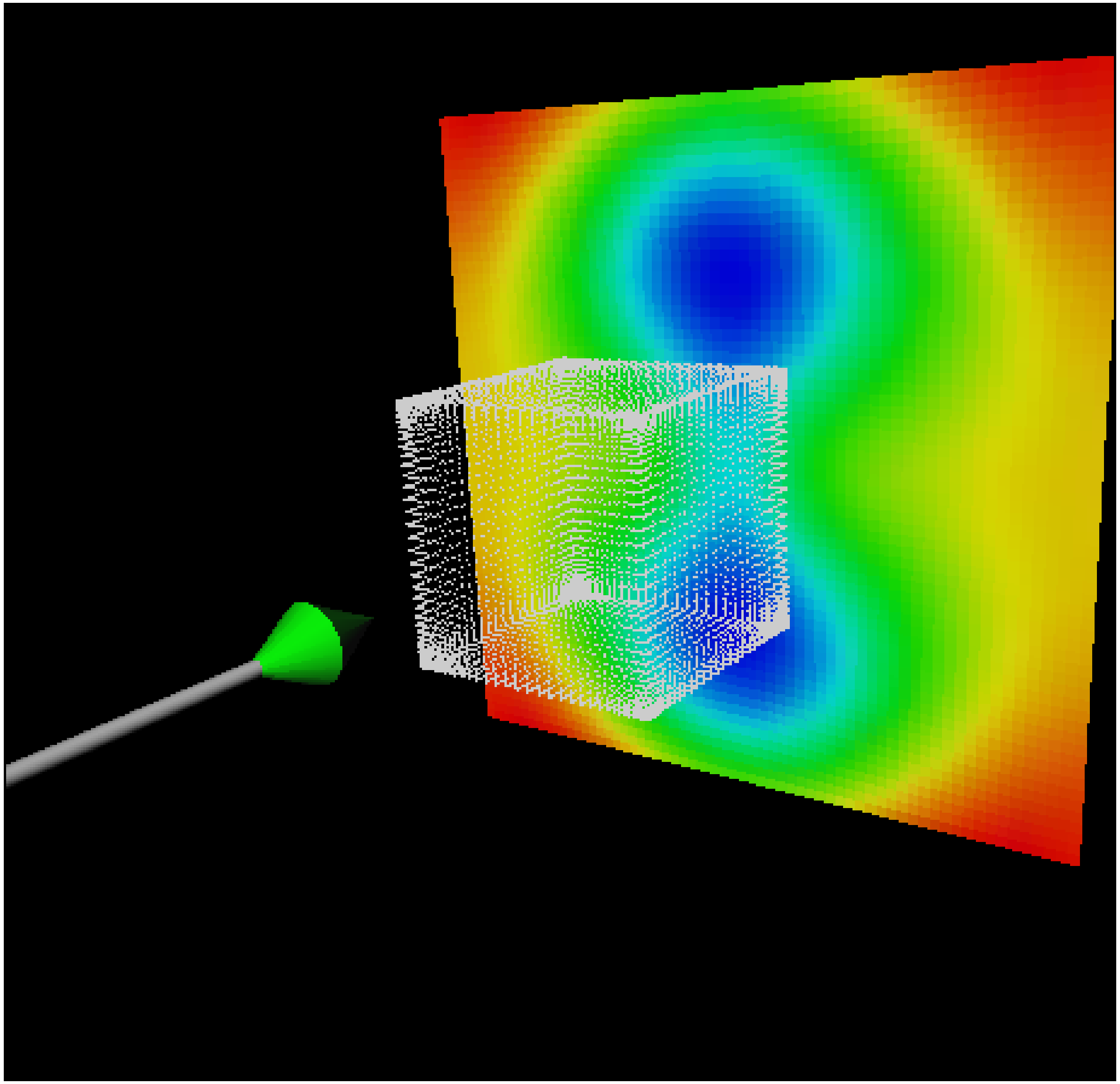
Singular Scatterers



Geometry	Diameters	Time	Unknowns	RMS Error	Computer
Cube (Present work)	$10\lambda \times 10\lambda \times 10\lambda$	$21h$	96,774	0.049%	AMD 1.4GHz (1 proc.)
Flying Saucer (Present work)	$42\lambda \times 42\lambda \times 17\lambda$	$53h$	290,874	0.0045%	AMD 1.4GHz (1 proc.)

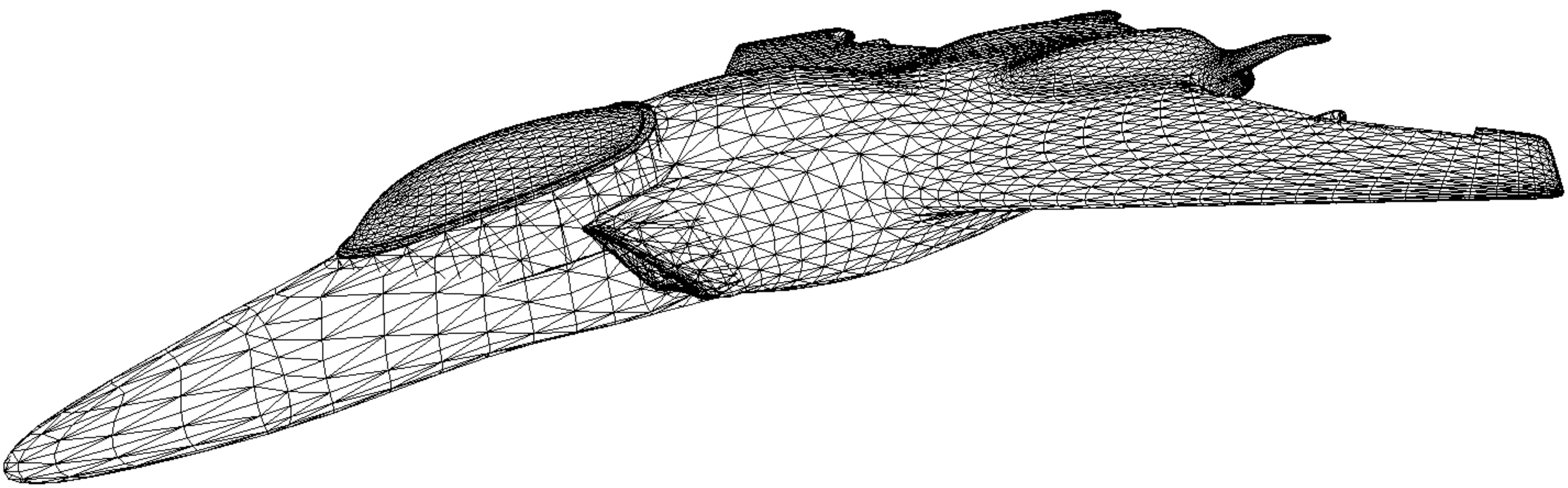
Electromagnetic Cube; 1.0 e-4





$$ka = 3.4$$

High-Order Surface Representation



Bruno and Pohlman, [2003]

Generation of Smooth Surfaces

A problem of present interest in the computer science literature

For general irregular triangulations, previous methods produce (at best) <u>C^1 surfaces only</u>

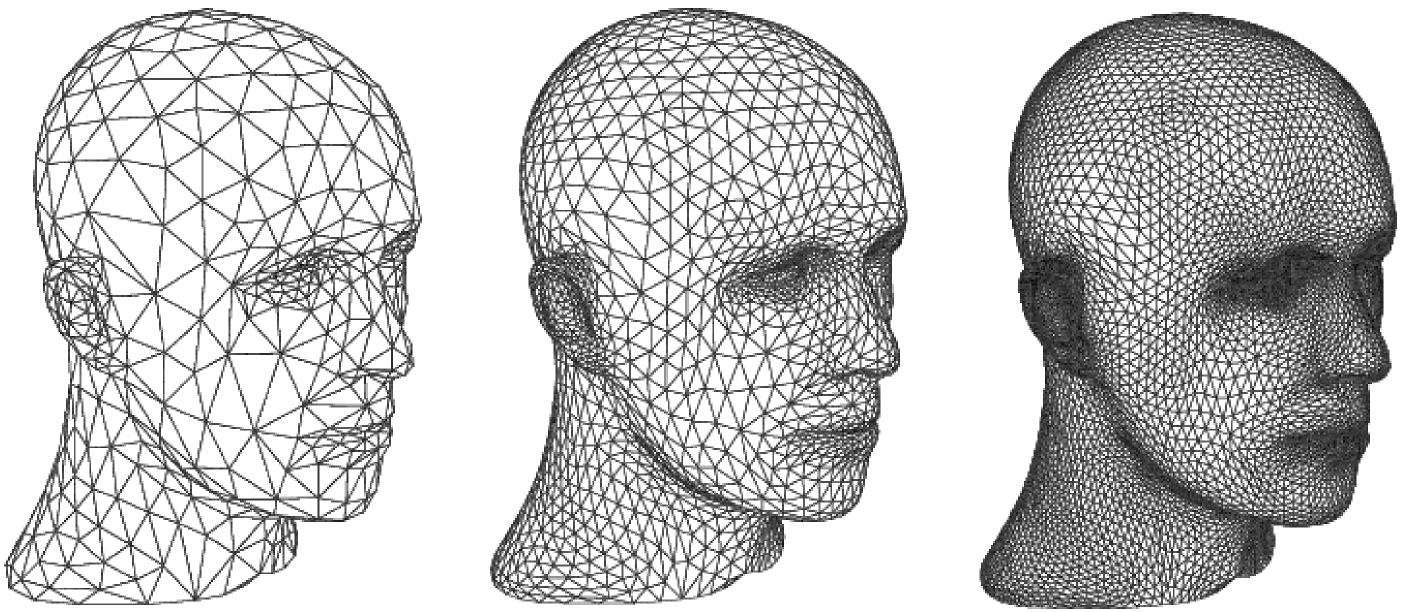


Figure 4. Two-dimensional loop subdivision is used to generate smooth surfaces from a coarse description.

Daubechies, Guskov, Schröder and Sweldens, [1999]

Present Approach

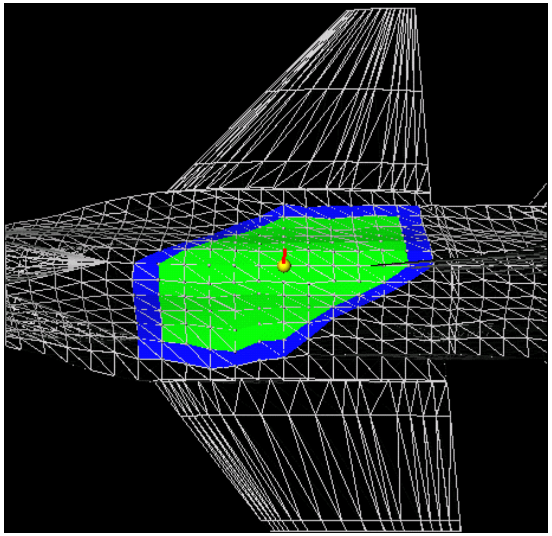
Interpolation via Fourier series, using

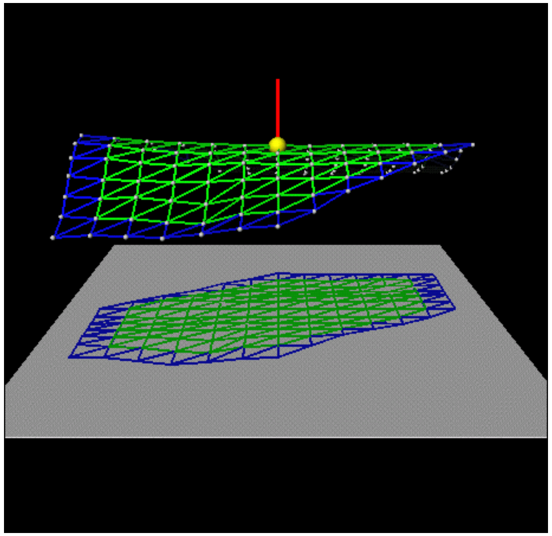
- *Unequally spaced FFTs (USFFT), and*

- *A "Continuation Method" for trigonometric representation of non-periodic functions with spectral accuracy (thus, overcoming the Gibbs phenomenon)*

Fourier Representation 1:

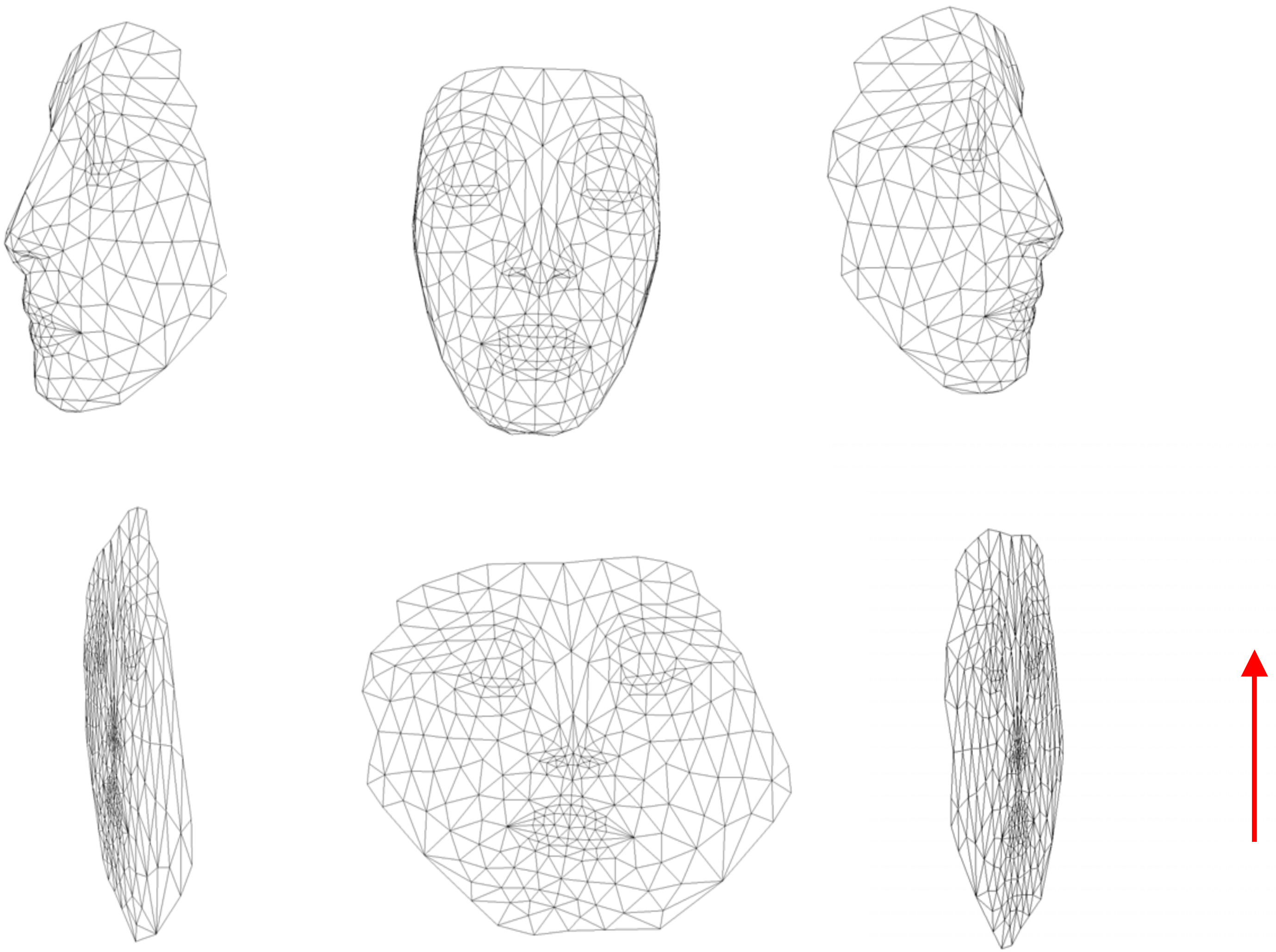
Patches





Difficult Patches → Intrinsic Parameterizations

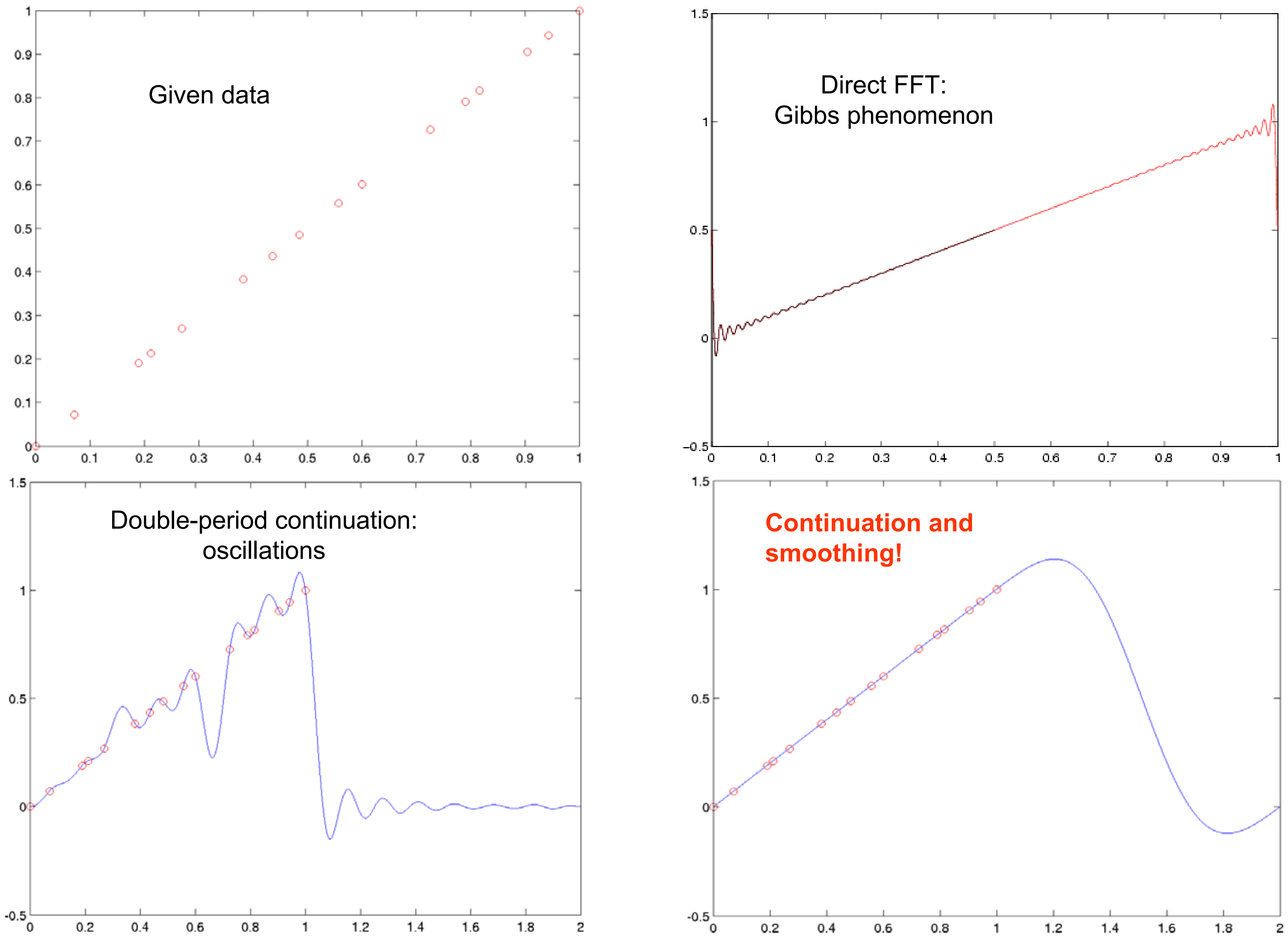
Intrinsic Parameterizations



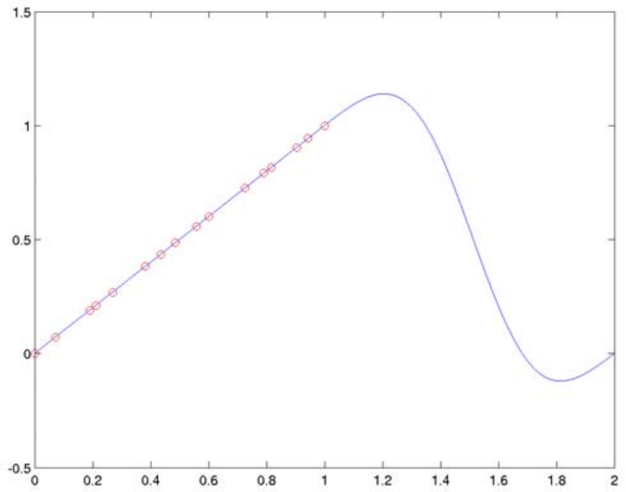
Desbrun, Meyer and Alliez, [2002]

Fourier Representation 2

POUs for boundary regions (Gibbs resolution)



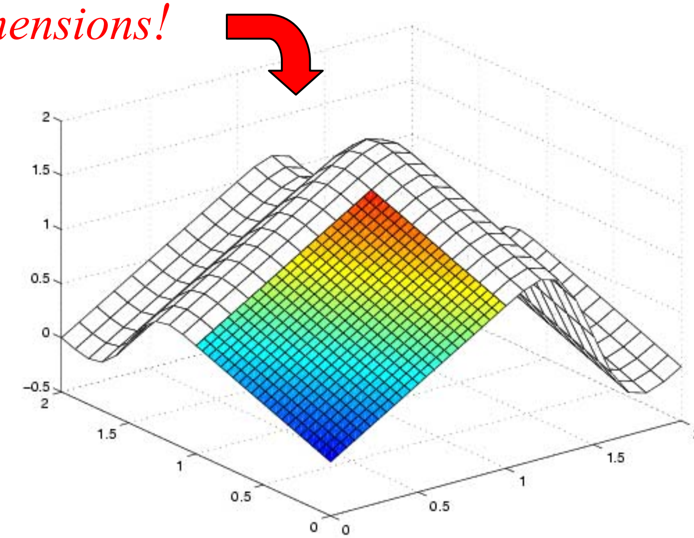
Fourier Representation 2 (Continued):



N	$f(x)$	ratio	df/dx	ratio	d^2f/dx^2	ratio
8	3.3e-03		1.3e-01		3.3e-00	
16	1.1e-05	3.0e+2	1.3e-03	9.9e+1	1.0e-01	3.2e+1
32	5.1e-10	2.2e+4	1.5e-07	8.6e+3	3.1e-05	3.3e+3
64	2.8e-13	1.8e+3	1.5e-10	9.7e+2	6.0e-08	5.3e+2
128	8.8e-15	3.2e+1	8.4e-12	1.9e+1	4.6e-09	1.3e+1

Generalizes to any number of dimensions!

N	$f(x,y)$	ratio	$\partial f/\partial x$	ratio	$\partial^2 f/\partial x^2$	ratio
8^2	2.9e-02		8.2e-01		1.4e+1	
16^2	3.5e-03	8.4e+1	2.7e-01	3.0e+0	1.4e+1	1.0e+0
32^2	1.2e-07	2.8e+4	3.0e-05	9.0e+3	4.8e-03	2.9e+3
64^2	2.8e-12	4.4e+4	1.4e-09	2.1e+4	4.2e-07	1.1e+4



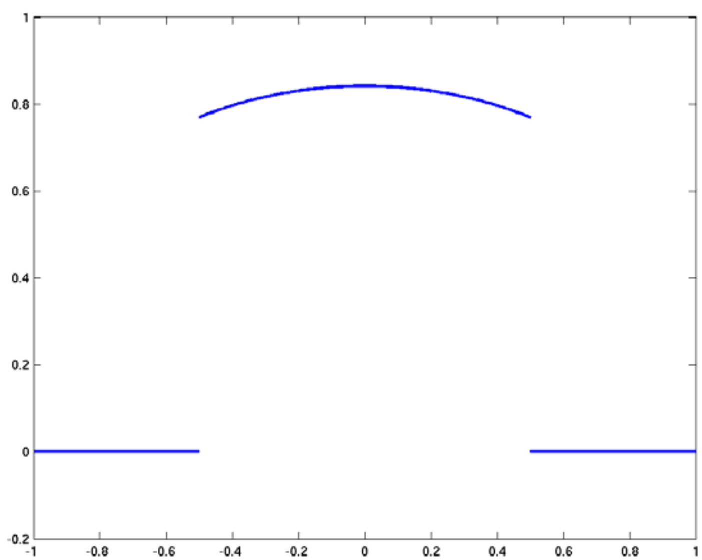
Also useful for coarse inner discretizations.

Does not require domain to be a square!!!!

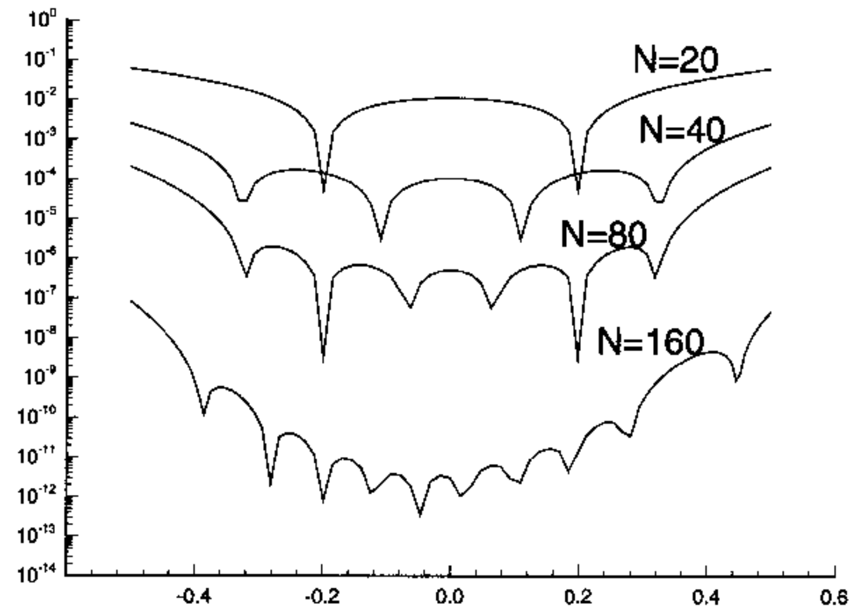
Comparison w/other methods:

Gegenbauer-polynomial approach

- *Do not use information about discontinuity location* (+)
- *Require much finer discretizations for given error tolerance* (-)
- *Only applies to square domains* (-)
- *Require use of data at (generally unavailable) data points* (-)



Function

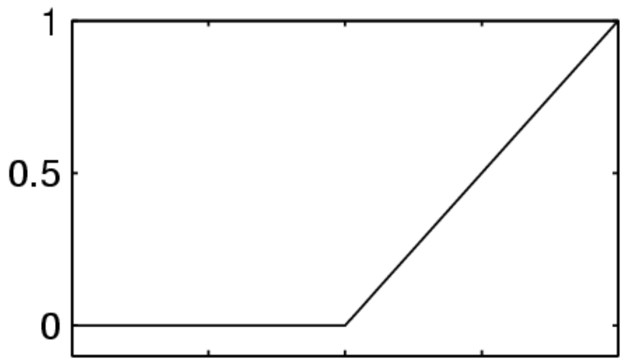


Error

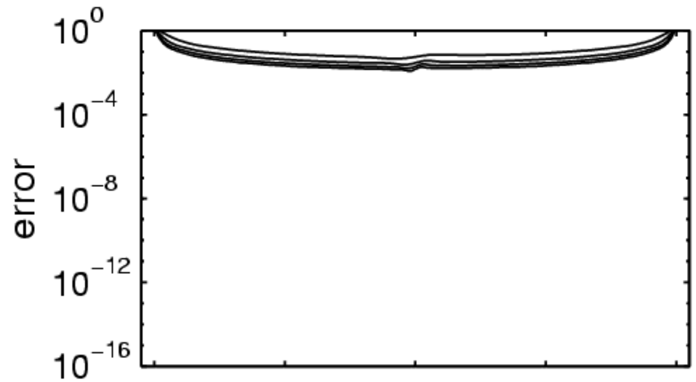
Gottlieb and Shu [1992]-…

Comparison w/other methods (contd.):

Singular Padè-Fourier approach

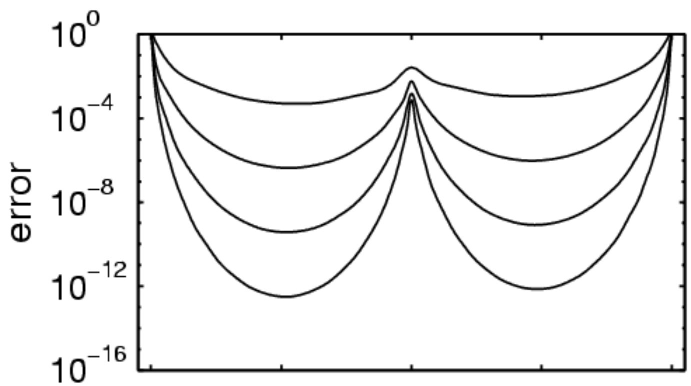


Function

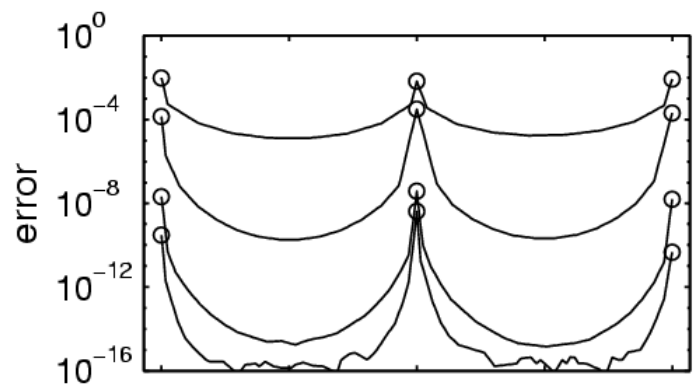


Fourier sum

$$N = 16, 32, 48, 64$$



Padè-Fourier sum

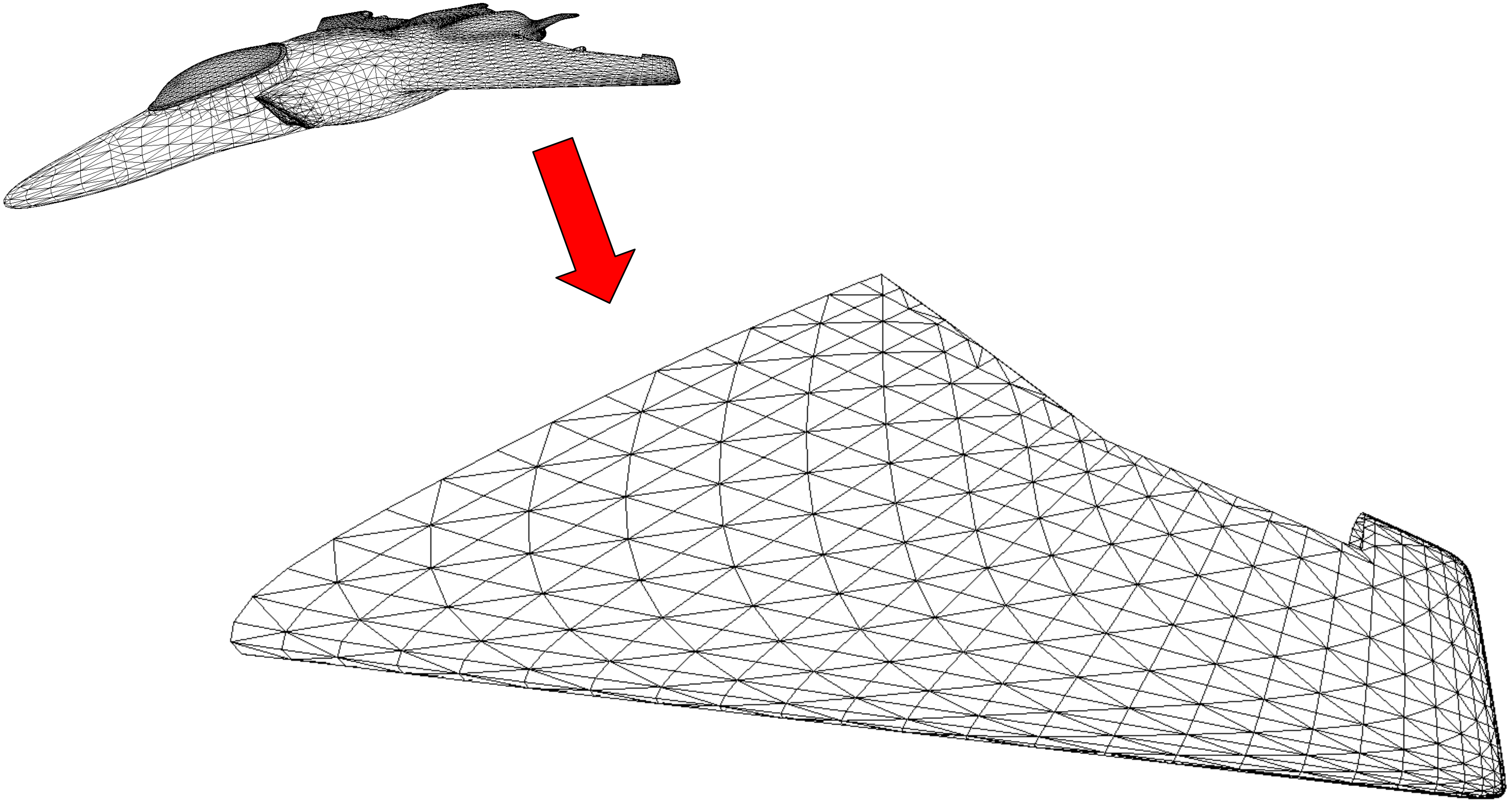


Singular Padè-Fourier sum

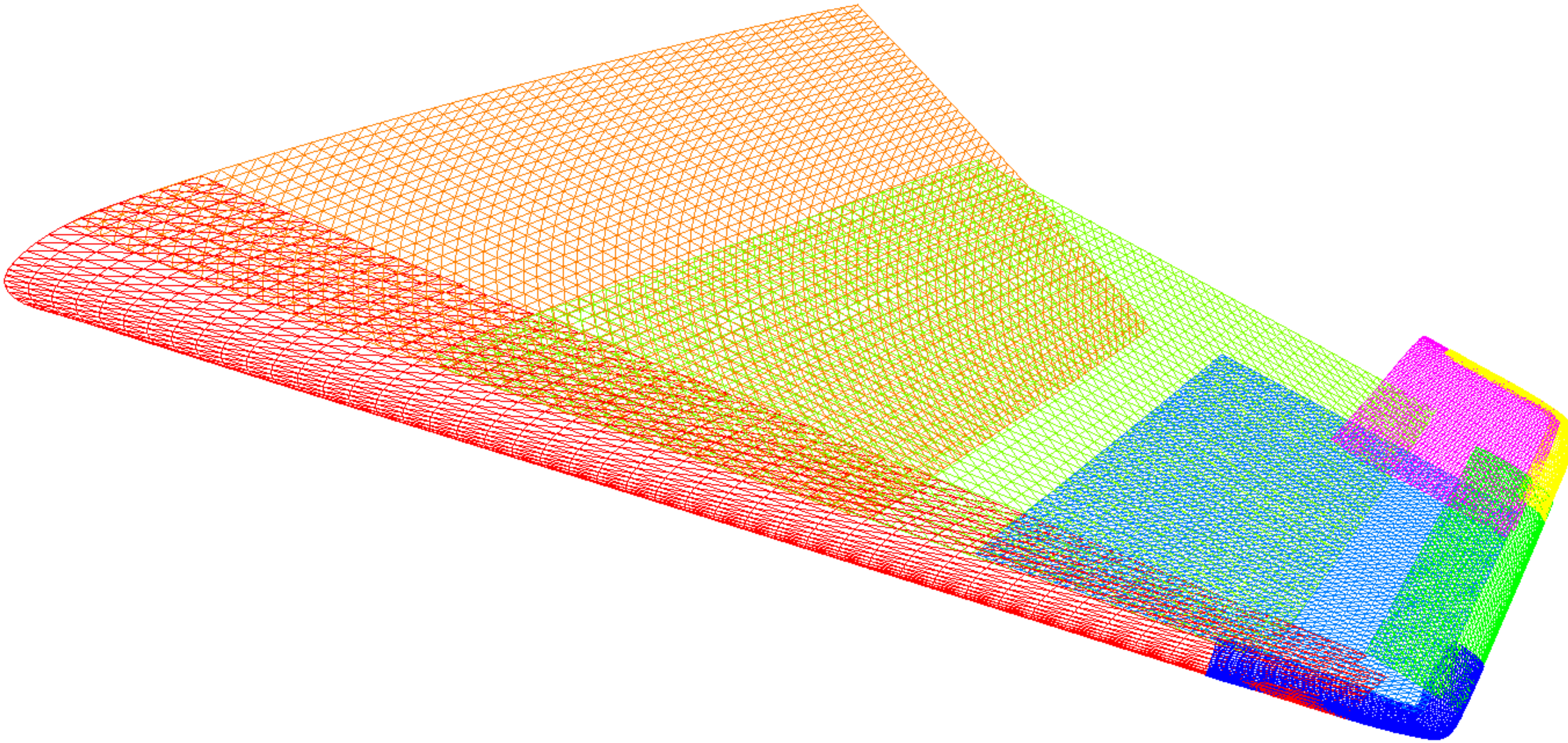
Driscoll and Fornberg [2001]

"The proposed approach exhibits the significant advantage of being able to deal with arbitrary data sets (non-square domains, non-uniformly-spaced data, arbitrary dimensionality), and yet, it yields more accurate results than other available methods"

Wing Patch



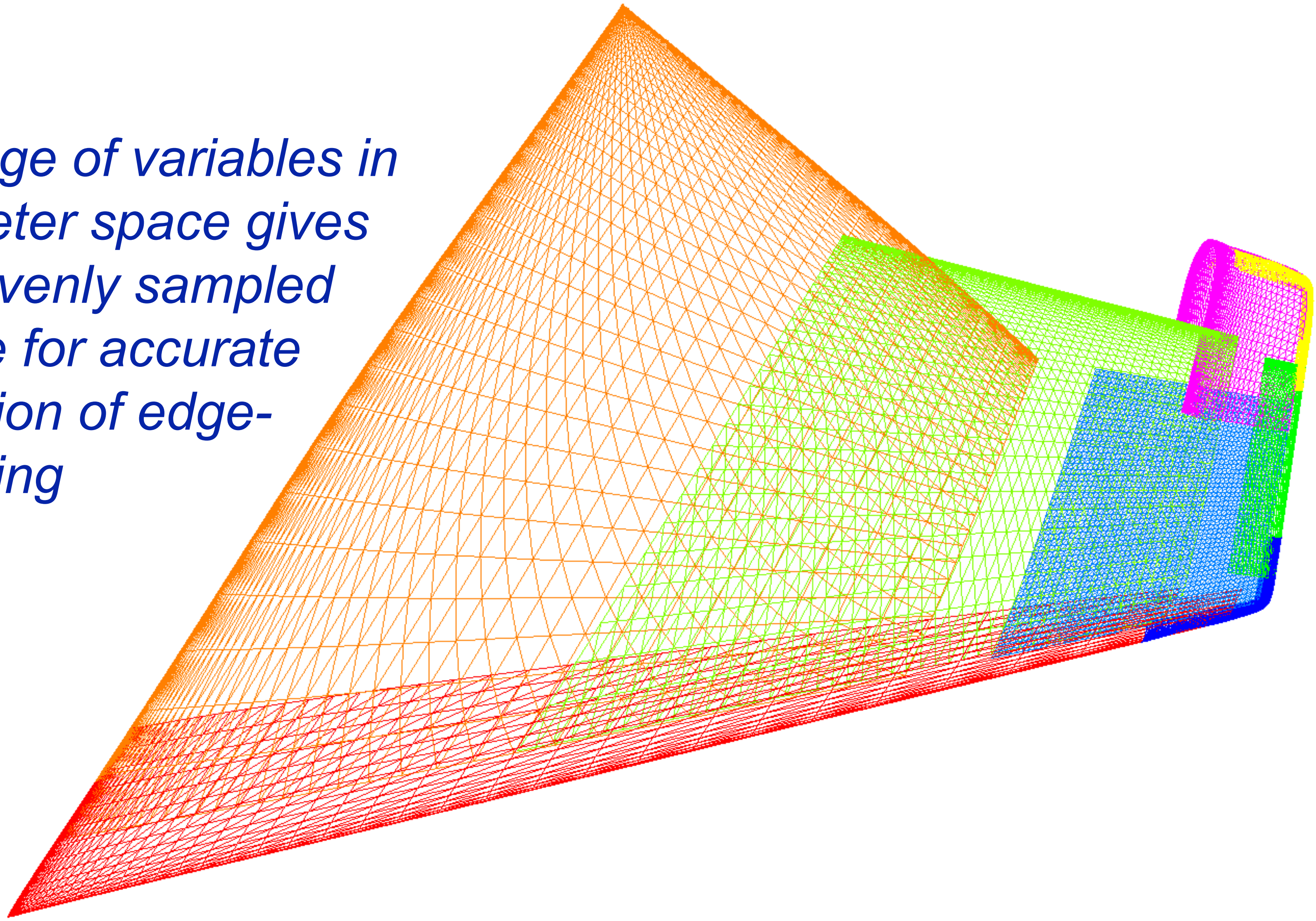
Surface Interpolation of Wing



Wing represented by eleven overlapping patches, each patch given explicitly by three coordinate functions (Fourier Series!)

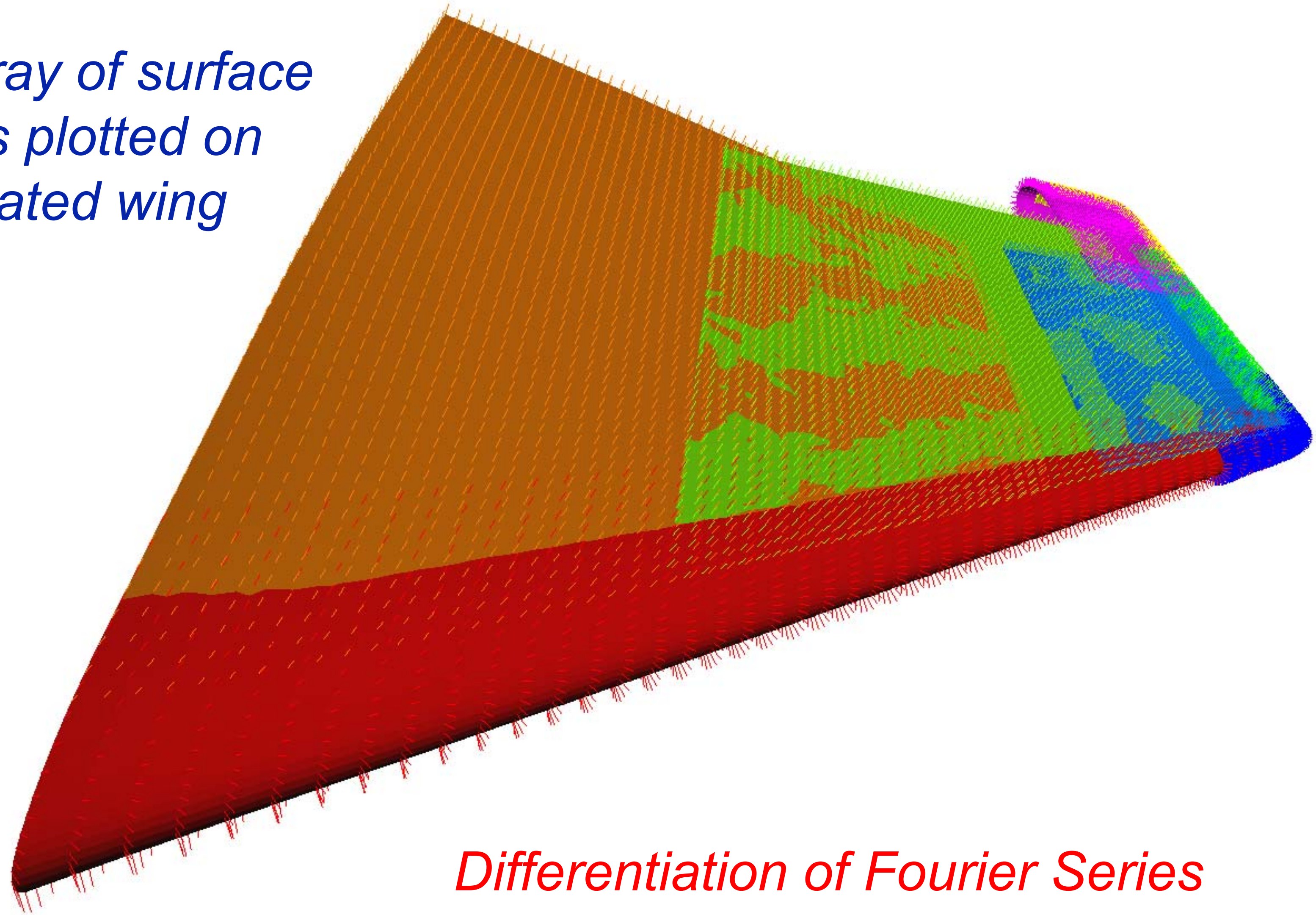
Wing Edges

A change of variables in parameter space gives an unevenly sampled surface for accurate resolution of edge-scattering



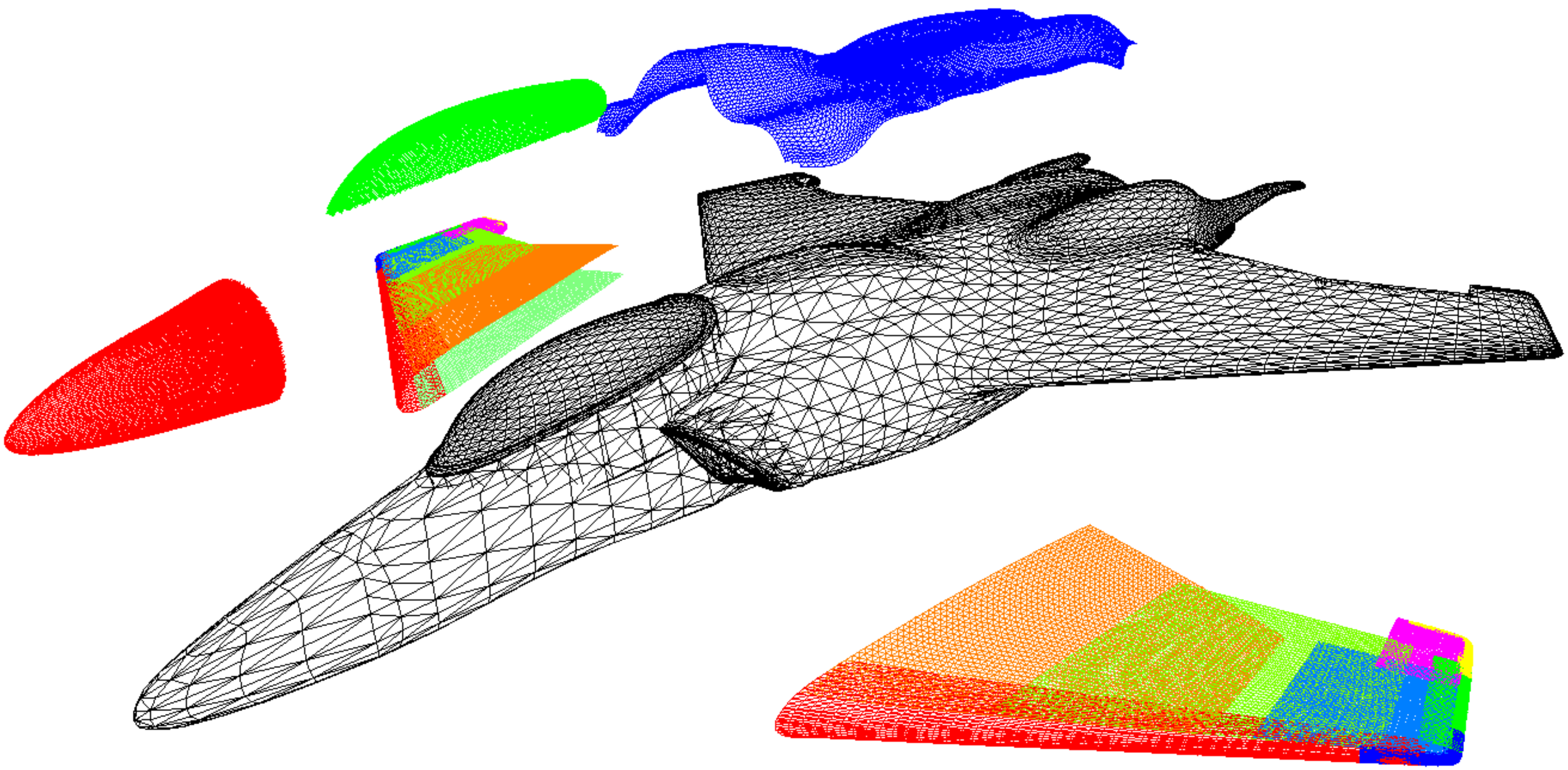
Wing Normals

Fine array of surface normals plotted on interpolated wing surface



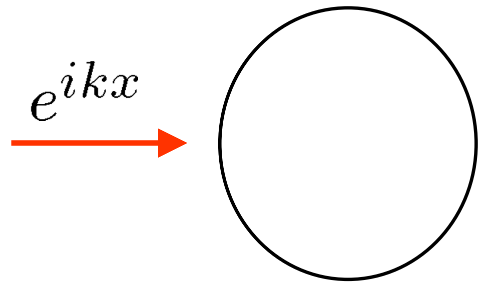
Differentiation of Fourier Series representation!

F-15 Aircraft



Oscar P. Bruno and Matthew M. Pohlman, [2003]

High Frequencies:
Phase extraction



$$\int_S H_0^1(k|x - x'|)\mu(x')dx' = f_{slow}(x)e^{ikx}$$

Ansatz: $\mu(x) = \mu_{slow}(x)e^{ikx}$

Highly oscillatory

$$\int_S \left[H_0^1(k|x - x'|)e^{ik(x'-x)} \right] \mu_{slow}(x')dx' = f_{slow}(x)$$

Previous Work
(Convex scatterers)

- *Melrose & Taylor, [1985]*
- *Abboud, Nédélec & Zhou, [1994], $O(k^{2/3})$ operations*
- *Lagreuche and Bettess, [2000], $O(k^{2/3})$ operations*

Present Approach

- *O(1) operations*
- *Convex and non-convex scatterers*

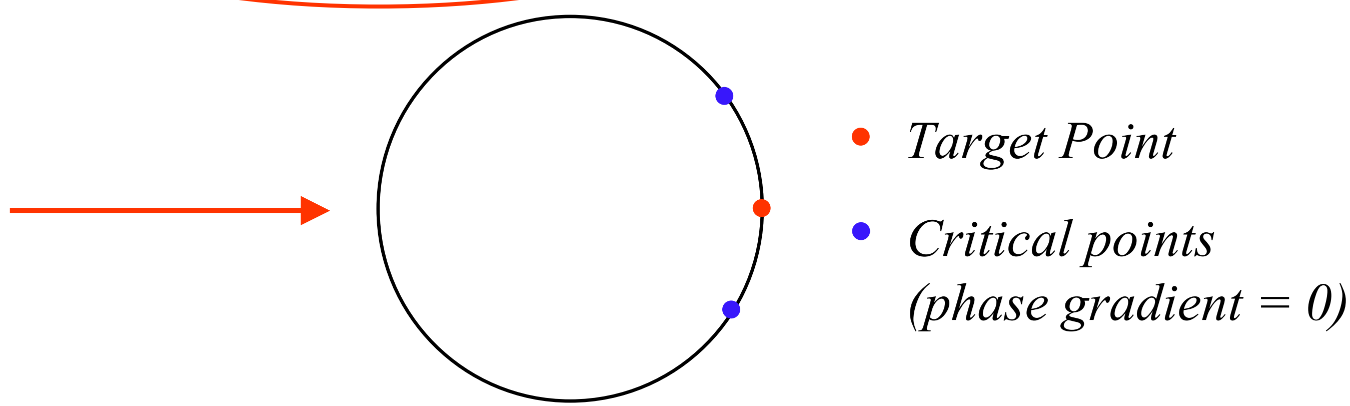
Convex obstacles………………………… (Bruno, Geuzaine and Monro, [2002])
Non-convex obstacles (work in progress)………(Bruno and Reitich, [2002-03])

O(1)-methods for high-frequency scattering

Integration exercise

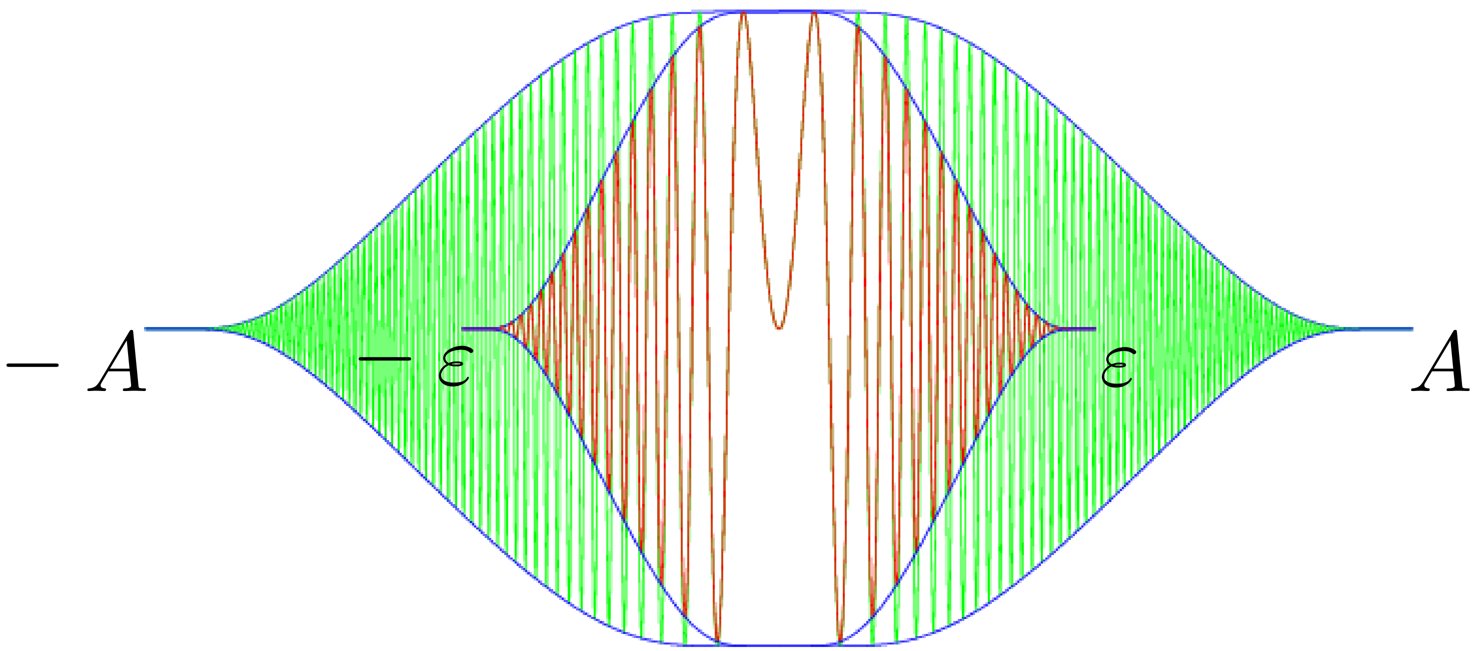
Highly oscillatory

$$\int_S \left[H_0^1(\kappa|x - x'|)e^{i\kappa x'} \right] \cos(x')dx'$$



- Critical points?
- Asymptotically? Want convergence!!
- Idea: Why compute integral at other points?

Thus our proposed approach:
Localized Integration

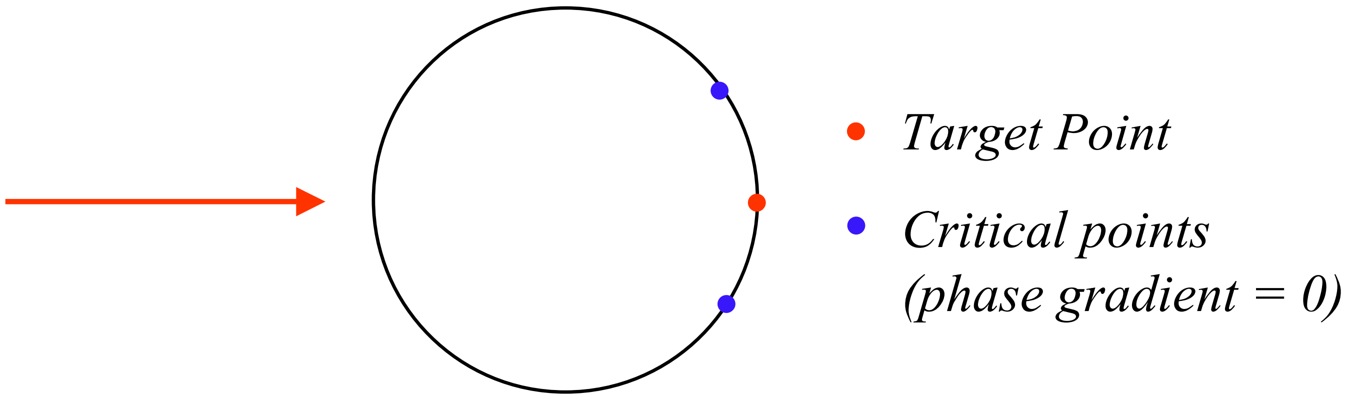


$$\int_{-A}^{A} f_A(x) e^{ikx^2} = \int_{-\varepsilon}^{\varepsilon} f_\varepsilon(x) e^{ikx^2} + \mathcal{O}((k\varepsilon^2)^{-n})$$

for all n!

Integration exercise

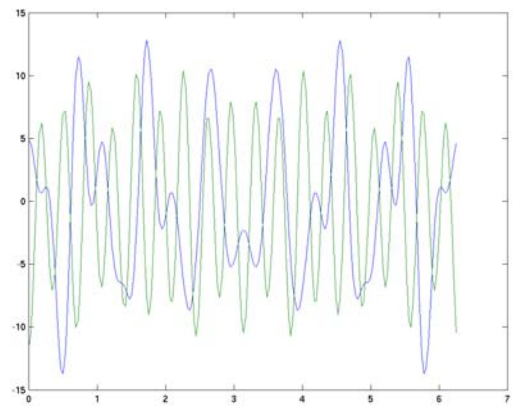
$$\int_S \left[H_0^1(\kappa|x - x'|) e^{i\kappa x'} \right] \cos(x') dx'$$

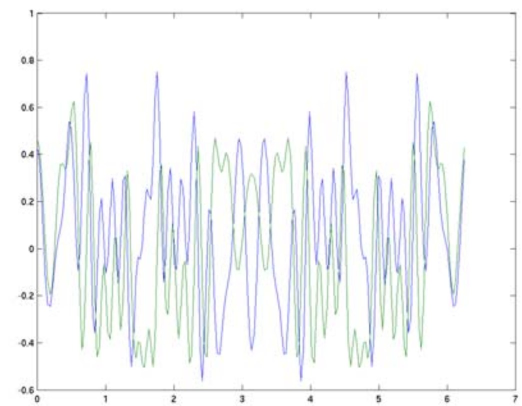


κ	N	ϵ	c	$Error$
1000	2100	1.0	0.5	1.5e-6
2000	2100	0.5	0.5	4.8e-8
4000	2100	0.25	0.5	1.2e-7
8000	2100	0.125	0.5	9.8e-7
16000	2100	0.0625	0.5	1.5e-6

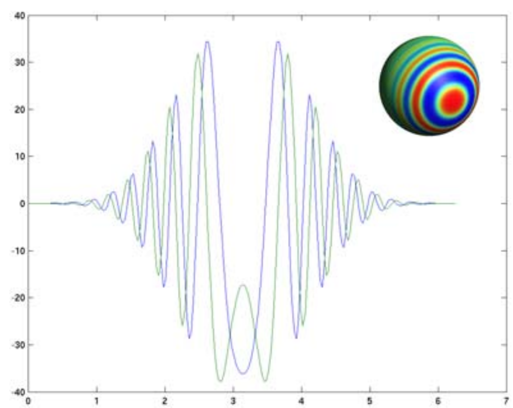
Is μ_{slow} actually slow?

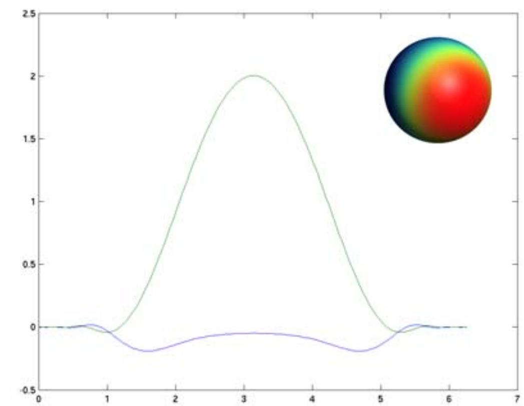
$$\frac{\varphi(\mathbf{r})}{2} = u^i(\mathbf{r}) - \int_{\partial D} \frac{\partial \Phi(\mathbf{r},\mathbf{r}')}{\partial \vec{\nu}_{\mathbf{r}'}} \varphi(\mathbf{r}')\, ds(\mathbf{r}') + i\gamma \int_{\partial D} \Phi(\mathbf{r},\mathbf{r}')\varphi(\mathbf{r}')\, ds(\mathbf{r}')$$





$$\frac{1}{2}\frac{\partial u}{\partial \vec{\nu}_{\mathbf{r}}}(\mathbf{r}) = \left(\frac{\partial u^i}{\partial \vec{\nu}_{\mathbf{r}}}(\mathbf{r}) + i\gamma u^i(\mathbf{r})\right) + \int_{\partial D} \frac{\partial \Phi(\mathbf{r},\mathbf{r}')}{\partial \vec{\nu}_{\mathbf{r}}} \frac{\partial u}{\partial \vec{\nu}_{\mathbf{r}'}}(\mathbf{r}')\, ds(\mathbf{r}') + i\gamma \int_{\partial D} \Phi(\mathbf{r},\mathbf{r}') \frac{\partial u}{\partial \vec{\nu}_{\mathbf{r}'}}(\mathbf{r}')\, ds(\mathbf{r}')$$





Key: Physical Density!

Multiple reflections: three-d, full Maxwell

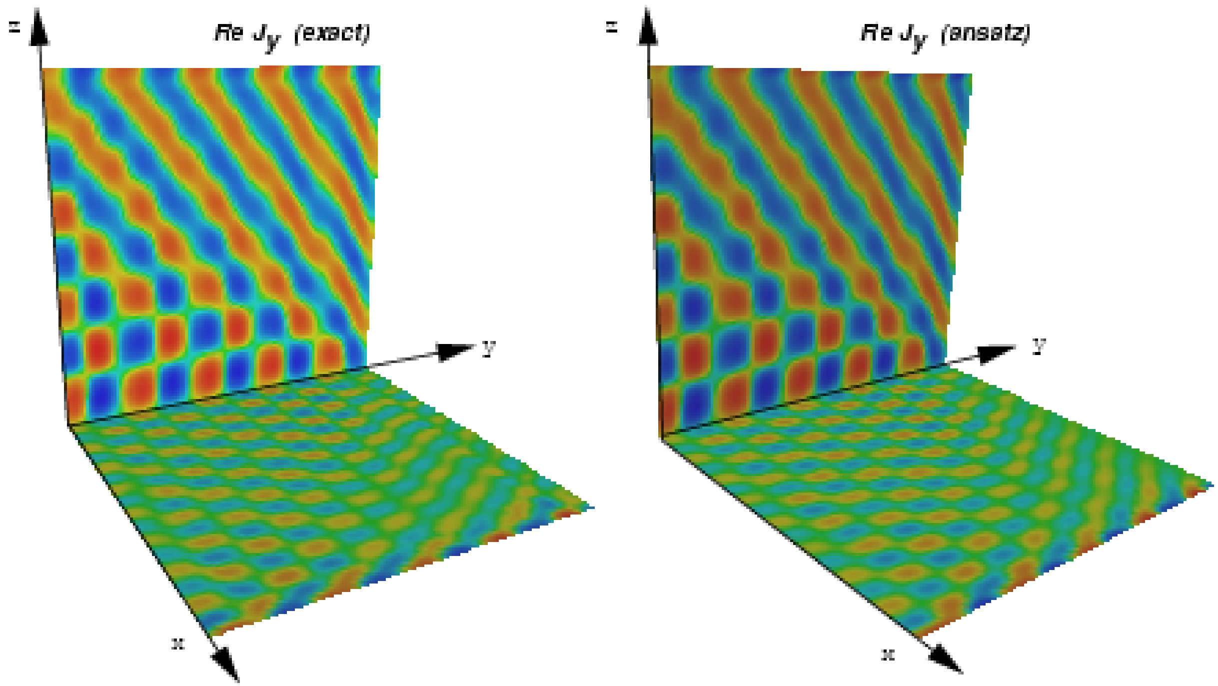
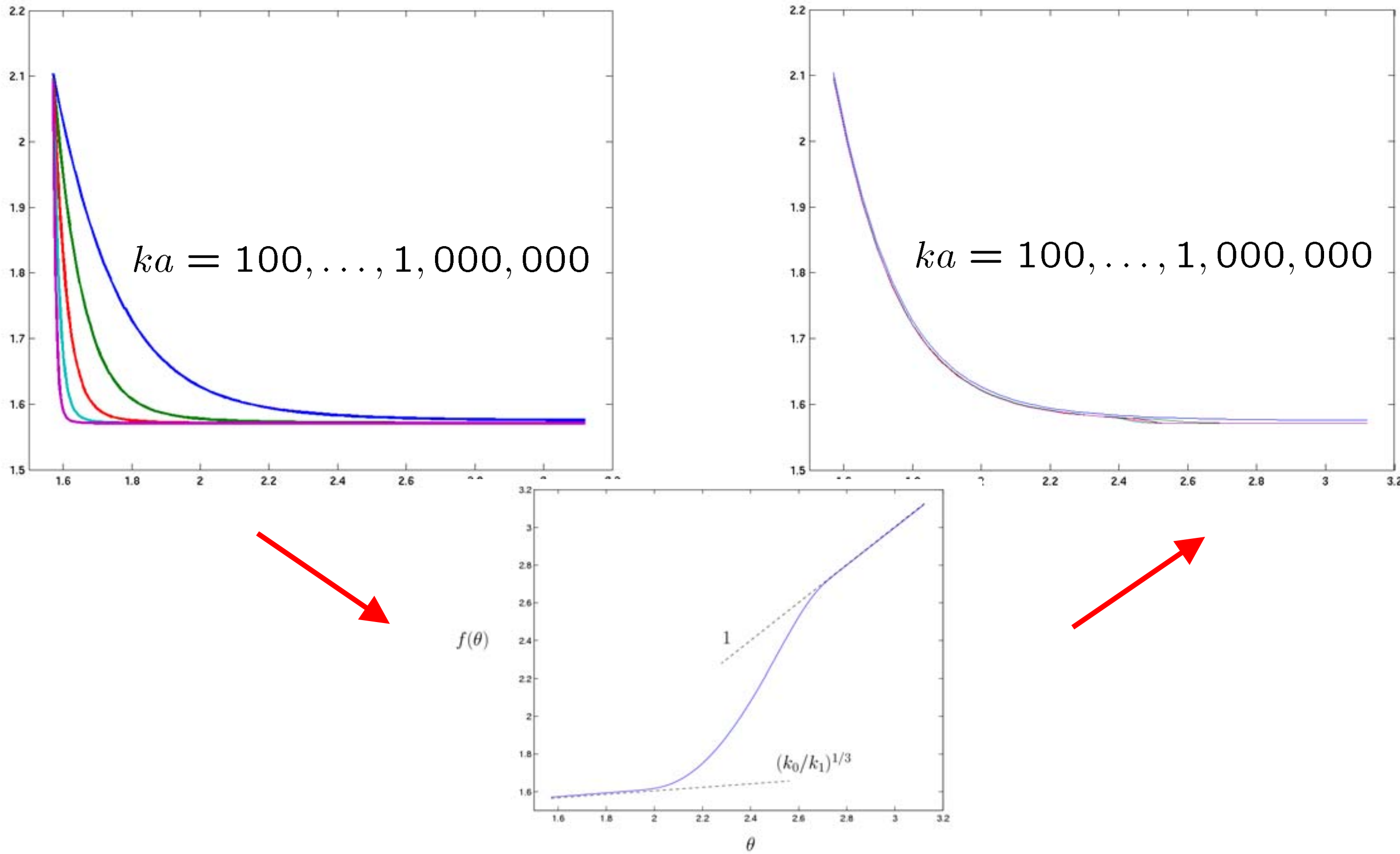


Fig. 20: Distribution of the y-component of the current on a dihedral consisting of two $10\,\lambda \times 10\,\lambda$ perfectly conducting plates for a horizontally polarized plane wave incident at the angles $\theta = 70°$ and $\phi = 30°$.

Bleszinsky, Bleszinsky and Jaroszewicz [2002]

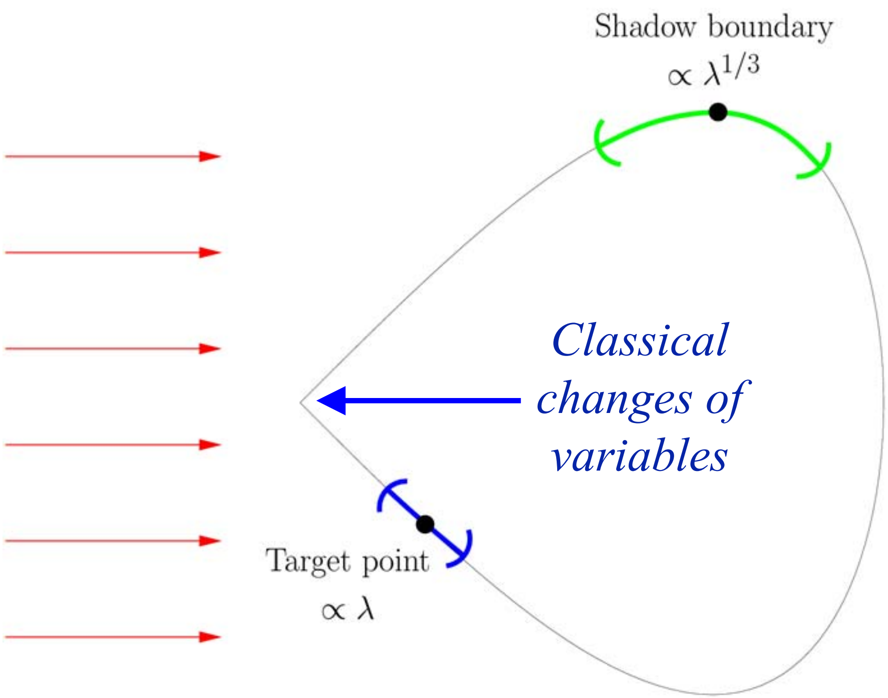
Cubic root ratios in the slow-density slopes
around shadow boundaries



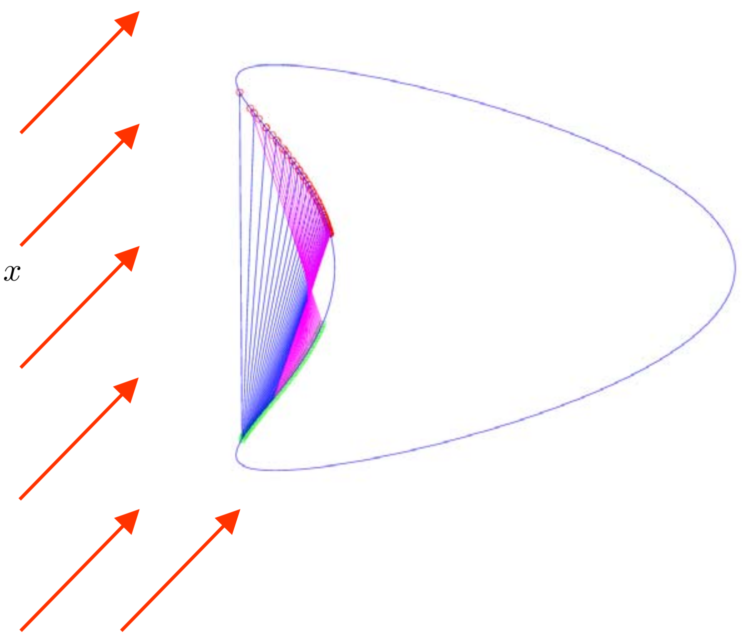
# of Fourier modes needed to represent μ_{slow} with a fixed accuracy

k	w/out chg. of vars.	w/ chg. of vars.
100	110	110
1000	230	220
10000	310	280
100000	350	280
1000000	> 500	280

Overall high-frequency algorithm



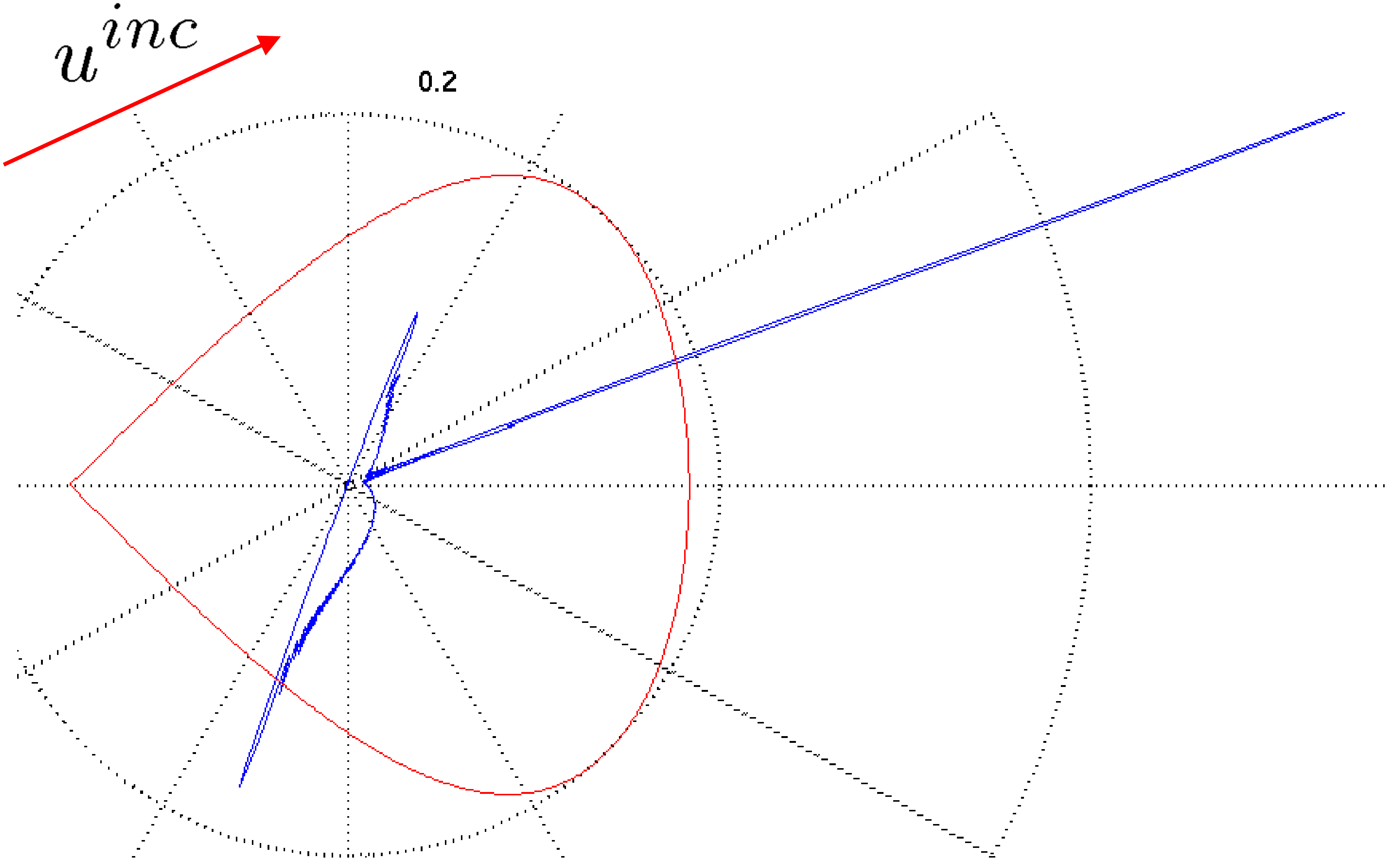
$$\mu(x) = \mu_{slow}(x)e^{ik\cdot x}$$



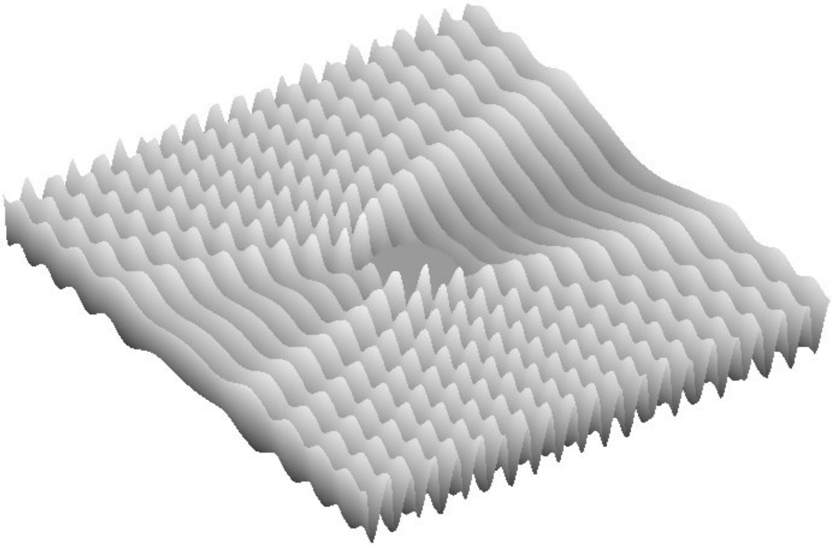
$$\mu(x) = \mu^0_{slow}(x)e^{ikx} + \mu^1_{slow}(x)e^{ik^1\cdot x} + \mu^2_{slow}(x)e^{ik^2\cdot x}$$

Unknowns	Max. Err. in μ^j_{slow}
512	5.0 e(-3)
768	2.0 e(-4)
925	3.0 e(-6)

DROP: Far Field; ka = 1000



Example: Combined Field IE



Prescribed error in bounded time from $\lambda = 6.28m$ to $\lambda = 0.68mm$

$\Downarrow$

$\mathcal{O}(1)$!

ka	Unknowns	Iter.	Max. Error	Mean Square Err.	CPU (s)
1	100	9	1.8e−12	8.8e−12	< 1
10	100	17	2.0e−12	9.2e−12	< 1
100	100	31	5.0e−5	2.5e−5	8
1000	100	30	7.8e−4	2.1e−4	84
10000	100	33	2.6e−3	6.6e−4	83

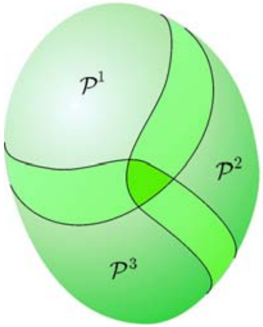
Convergence (Combined Field IE)

$$ka = 150$$

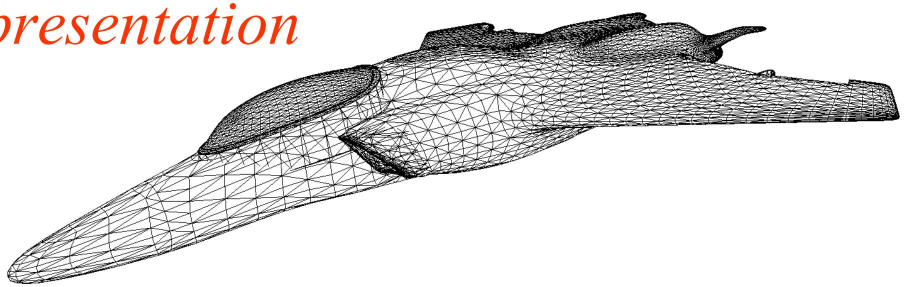
Unknowns	GMRES Iterations	Max. Error
25	13	4.4e−3
50	23	1.2e−3
100	31	1.2e−4
200	34	4.4e−6
400	39	1.0e−9
800	45/56	1.0e−12/1.3e−13

Recap

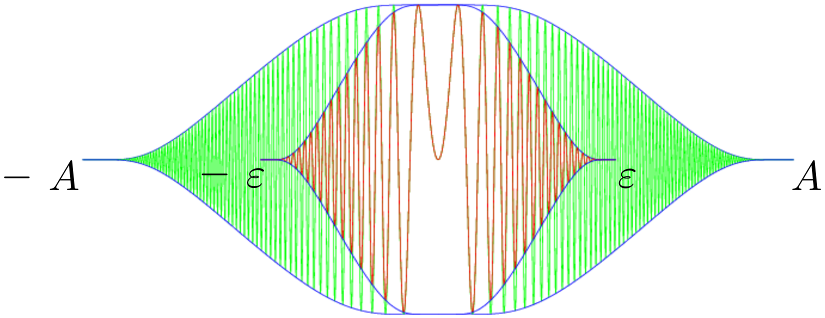
1) *Direct integral solvers*



2) *High-Order Surface Representation*



3) *Convergent **O(1)** High-Frequency Integral Method*



Conclusions

- *General solvers*
- *Fast: $O(N^{6/5} \log(N))$-$O(N^{4/3} \log(N))$ operations +*
  O(1) HF solver
- *Very High Order (Spectrally accurate), no accuracy breakdowns of any kind*
- *Orders of magnitude higher accuracy than leading solvers (in fast runs on 400 MHz PCs!)*
- *Innovative solution for high-order geometry representation – based on use of partitions of unity and non-uniform FFT*