Perfectly Matched Layers Techniques: stability and instability results.

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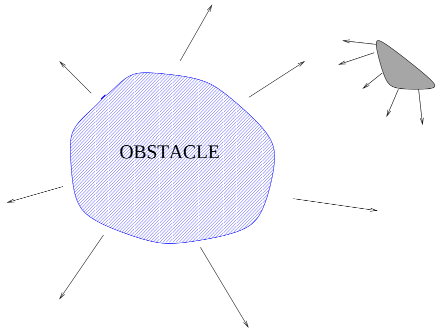
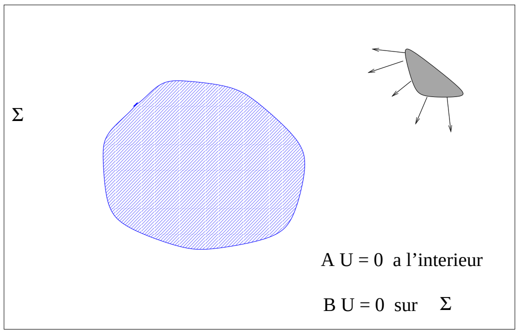
Absorbing Boundary Conditions

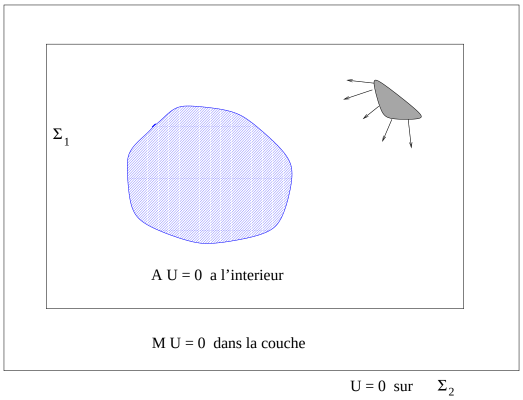
Absorbing Layers



- Absorbing boundary conditions (ABC): add a boundary condition on Σ, artificial boundary



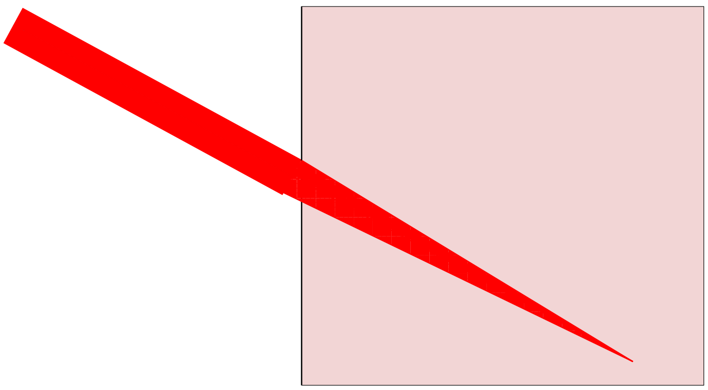
- Absorbing Layers : surround the computational domain by a layer in which the wave is damped.



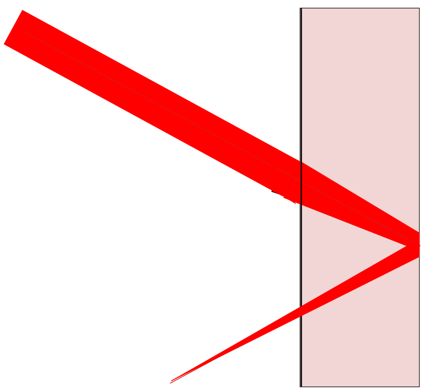
Perfectly Matched Layers

The general principle (Bérenger)

1- Juxtapose the propagation medium and an absorbing medium which generates no reflection at the interface : the restriction of the solution to the "propagative domain" coïncides with the exact solution exacte and the transmitted wave decays exponentially during its propagation.



2- Bound the absorbing layer with some quasi-arbitrary boundary condition.



The formal construction

This is the direct extension of the original work by Bérenger for Maxwell's equations.

The propagation model

$$\frac{\partial U}{\partial t} + A_x \frac{\partial U}{\partial x} + A_y \frac{\partial U}{\partial y} = 0,$$

The field splitting:

$$\begin{cases} U = U^x + U^y, \\ \dfrac{\partial U^x}{\partial t} + A_x \dfrac{\partial U}{\partial x} = 0 \\ \dfrac{\partial U^y}{\partial t} + A_y \dfrac{\partial U}{\partial y} = 0 \end{cases}$$

Le PML model in the x-direction:

$$\begin{cases} \dfrac{\partial U^x}{\partial t} + \sigma U^x + A_x \dfrac{\partial}{\partial x}(U^x + U^y) = 0, \\ \dfrac{\partial U^y}{\partial t} + A_y \dfrac{\partial}{\partial y}(U^x + U^y) = 0, \\ U = U^x + U^y, \quad \sigma = \sigma(x) \geq 0. \end{cases}$$

It is an anisotropic absorption .

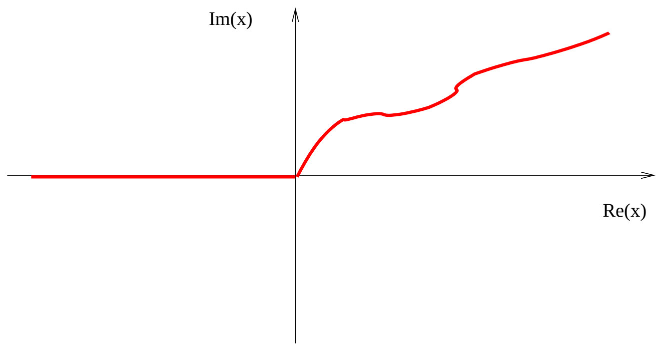
The formal construction (I)

Go to the frequency domain ($U(x,y,t) \to U(x,y,\omega)$)

$$i\omega U + A_x \frac{\partial U}{\partial x} + A_y \frac{\partial U}{\partial y} = 0,$$

One can remark that the solution can be extended to the complex plane $x \in \mathbb{C}$ (analytically) and one looks at the solution along another path of the complex plane:

$$x \to x + iF(x).$$



To find the new equation, one applies then the (complex) change of variable :

$$x \to x + \frac{1}{i\omega}\int_0^x \sigma(\xi)\, d\xi \quad \Rightarrow \quad \frac{\partial}{\partial x} \to (1 + \frac{\sigma}{i\omega})\frac{\partial}{\partial x}$$

which lets invariant the half-space $x < 0$ and one studies:

$$\widetilde{U}(x) = U\left(x + \frac{1}{i\omega}\int_0^x \sigma(\xi)\, d\xi\right)$$

The formal construction (II)

Remarking that:

$$\frac{\partial \widetilde{U}}{\partial x}(x) = \frac{i\omega}{i\omega + \sigma} \frac{\partial U}{\partial x} \left(x + \frac{1}{i\omega} \int_0^x \sigma(\xi)\, d\xi \right),$$

one obtains the equation:

$$i\omega\, \widetilde{U} + A_y \frac{\partial \widetilde{U}}{\partial y} + \left(\frac{i\omega}{i\omega + \sigma}\right) A_x \frac{\partial \widetilde{U}}{\partial x} = 0,$$

that can be rewritten as:

$$\left| \begin{array}{ccccc} \widetilde{U} & = & \left(-\dfrac{1}{i\omega + \sigma}\right) A_x \dfrac{\partial \widetilde{U}}{\partial x} & + & \left(-\dfrac{1}{i\omega}\right) A_y \dfrac{\partial \widetilde{U}}{\partial y} \\ & = & \widetilde{U}^x & + & \widetilde{U}^y \end{array} \right.$$

where by construction

$$(i\omega + \sigma)\, \widetilde{U}^x + A_x \frac{\partial \widetilde{U}}{\partial x} = 0, \quad i\omega\, \widetilde{U}^y + A_y \frac{\partial \widetilde{U}}{\partial y} = 0.$$

One gets the “Bérenger” system by going back to the time variable.

The case of acoustic waves

One writes the acoustic wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \text{div}\,(\mu \nabla u) = 0$$

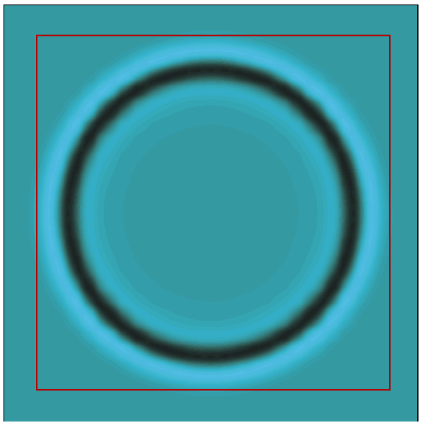
as a first order system

$$\begin{cases} \rho \, \dfrac{\partial u}{\partial t} - \dfrac{\partial v_x}{\partial x} - \dfrac{\partial v_y}{\partial y} = 0 \\[2ex] \mu^{-1} \, \dfrac{\partial v_x}{\partial t} - \dfrac{\partial u}{\partial x} = 0 \\[2ex] \mu^{-1} \, \dfrac{\partial v_y}{\partial t} - \dfrac{\partial u}{\partial y} = 0 \end{cases}$$

One realizes that one can avoid the "splitting" of the variables v_x et v_y and one gets:

$$\begin{cases} \rho \, (\dfrac{\partial u^x}{\partial t} + \sigma u^x) - \dfrac{\partial v_x}{\partial x} = 0 \\[2ex] \mu^{-1} \, (\dfrac{\partial v_x}{\partial t} + \sigma v_x) - \dfrac{\partial}{\partial x}(u^x + u^y) = 0 \\[2ex] \rho \, \dfrac{\partial u^y}{\partial t} - \dfrac{\partial v_y}{\partial y} = 0 \\[2ex] \mu^{-1} \, \dfrac{\partial v_y}{\partial t} - \dfrac{\partial}{\partial y}(u^x + u^y) = 0 \end{cases}$$

A numerical experiment



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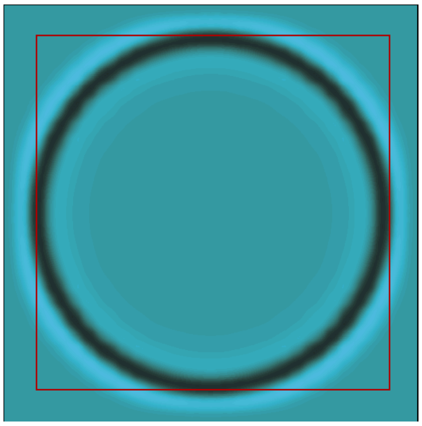

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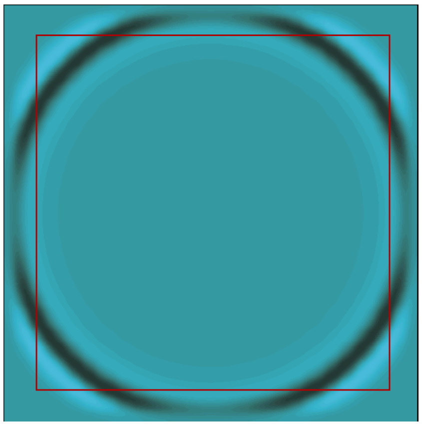

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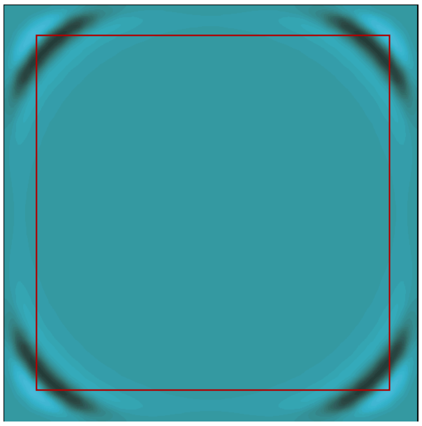

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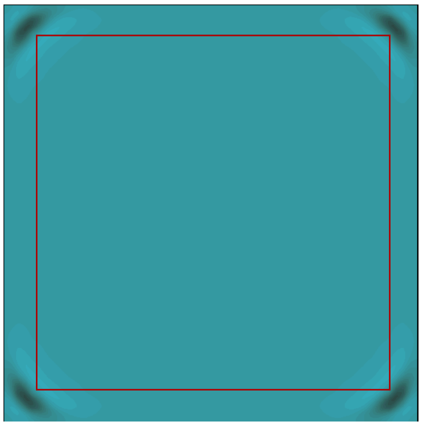

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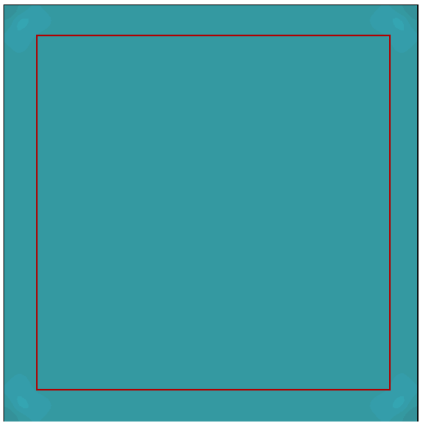

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Comparison PML / ABC

The PML technique seems to have scored a lot of points in the competition.

Among the advantages of PML's sont:

- Their systematic derivation.
- Their easy implementation.
- The treatment of corners simple.
- Very good performances.
- Work in practice in complex situations (heterogeneous media, surface waves,...)

However

- Their mathematical analysis is not completely understood.
- There are examples of instabilities.

Mathematical Background (1)

We wish to study the Cauchy problem associated to the PML system. This system appears as a zero order perturbation of a first order system (for our application, $V = (U^x, U^y)$):

$$(P)\begin{cases} \dfrac{\partial V}{\partial t} + A_x \dfrac{\partial V}{\partial x} + A_y \dfrac{\partial V}{\partial y} + BV = 0, (x,y) \in \mathbb{R}^2, t > 0, \\[2ex] V(x,y,0) = V_0(x,y), \quad (x,y) \in \mathbb{R}^2, \\[2ex] V \in \mathbb{R}^m, (A_x, A_y, B) \in \mathcal{L}(\mathbb{R}^m)^3. \end{cases}$$

For all $k = (k_x, k_y) \in \mathbb{R}^2$, one sets:

$$A(k) = k_x A_x + k_y A_y \quad \in \mathcal{L}(\mathbb{R}^m)$$

Definition 1: The unperturbed system (P_0) ($B = 0$) is hyperbolic if and only if:

- $\forall k \in \mathbb{R}^2$, the eigenvalues of $A(k)$ are real

It is strongly hyperbolic if moreover:

- $\forall k \in \mathbb{R}^2$, $A(k)$ est diagonalizable

Mathematical Background (2)

Theorem 2: (The perturbed case)

- If (P_0) is strongly hyperbolic, the problem (P) is strongly well posed and

$$(1) \qquad \forall\, t > 0, \quad \|V(t)\|_{L^2} \leq \; C\, e^{Kt}\, \|V_0\|_{L^2}$$

- If (P_0) is weakly hyperbolic, for some matrices B, the problem (P) is strongly ill posed.

($\mathcal{I}m\, \omega(k) \to -\infty \quad (|k| \to +\infty)$ as $|k|^{1/p}, \quad p = s + 1$).

The well-posedness concept is not satisfactory. In particular, it does not prevent from exponentially growing solutions.

Definition 3: The problem (P) is strongly (resp. weakly) stable if and only if,

$$\forall\, t > 0, \quad \|U(t)\|_{L^2} \leq C\, (1+t)^s\, \|U_0\|_{H^s}$$

with $s = 0$ (resp. $s > 0$).

Remark: In the unperturbed case, the stability estimate holds where $s + 1$ is the maximal size of the Jordan blocks of $A(k)$.

Mathematical Background (3)

The basic tool is the Fourier analysis, i.e. the study of particular solutions of the form:

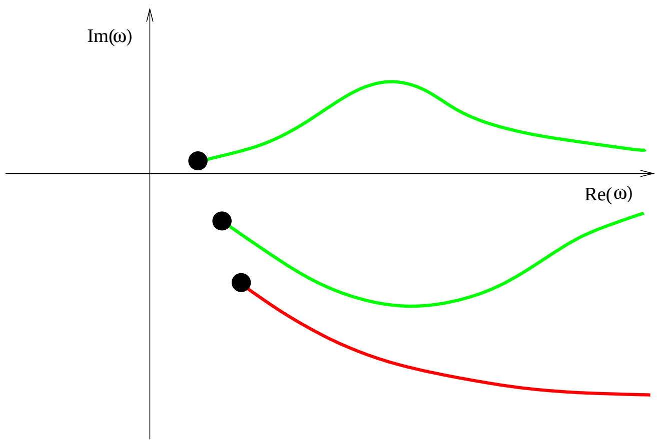
$$\begin{cases} V(x,y,t) = V(k)\ \exp i(k_x x + k_y x)\ e^{i\omega(k)t}, \\ k = (k_x, k_y) \in \mathbb{R}^2, \quad \omega(k) \in \mathbb{C} \end{cases}$$

The analysis is reduced to study of the branches of solutions $\omega = \omega(k)$ of the dispersion equation:

$$\det\ (A(k) - iB - \omega I) = 0,$$

$$\begin{cases} \text{well-posedness} & \Longleftrightarrow & \mathcal{I}m\ \omega(k) \text{ bounded from below.} \\ \text{stability} & \Longleftrightarrow & \mathcal{I}m\ \omega(k) \geq 0. \\ \text{ill-posedness} & \Longleftrightarrow & \mathcal{I}m\ \omega(k) \to -\infty \text{ as } |k|^{1/p}, \quad p = s+1 \end{cases}$$

Geometry of the curves : $|k| \mapsto \omega(|k| \cdot K) \in \mathbb{C},\ |K| = 1.$



Well-posedness theory

- For the PML model, it is easy to show that one is in the unconfortable situation corresponding to the fact that the unperturbed system is only weakly hyperbolic. This was first pointed out for Maxwell's equations by Abarbanel-Gottlieb.

- For the Maxwell system, it has been shown (Zhao-Cangellaris, Petropoulos, Rahmouni,...) that the PML system could be rewritten in an equivalent form (with another choice of unknowns) as a first order perturbation of a weakly hyperbolic system, which ensures well-posedness.

- For general systems, under general assumptions (including acoustics, electromagnetism, elasticity,...) that the PML perturbation does not generate strong ill-posedness: the PML model remains weakly well-posed (Bécache-Fauqueux-Joly).

- However, this type of result is not sufficient (or not pertinent) for practical applications as it can be shown on the example of anisotropic elastic waves (Collino-Tsogka).

Anisotropic elastic waves

$$\rho\frac{\partial^2 u}{\partial t^2} - A(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y})\, u = 0$$

$$A(K_x, K_y) = \begin{pmatrix} c_{11}K_x^2 + c_{33}K_y^2 & (c_{12}+c_{33})K_xK_y \\ (c_{12}+c_{33})K_xK_y & c_{33}K_x^2 + c_{22}K_y^2 \end{pmatrix}$$

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{33} > 0, \quad c_{11}c_{22} - c_{12}^2 > 0$$

Isotropic medium : $c_{11} = c_{22} = \lambda + 2\mu$, $c_{12} = \lambda$, $c_{33} = \mu$.

The dispersion relation is:

$$(1) \quad F(\omega, k) \quad \left(\equiv \det\left(A(k_x, k_y) - \rho\, \omega^2 I\right)\right) = 0$$

Note that F is homogeneous (of degree 4). The slowness curves are the curves in the plane $(k_x/\omega, k_y/\omega)$ of equation:

$$F(1, \frac{k_x}{\omega}, \frac{k_y}{\omega}) = 0$$

From the solutions $\omega(k)$ of (1) one defines the group velocity

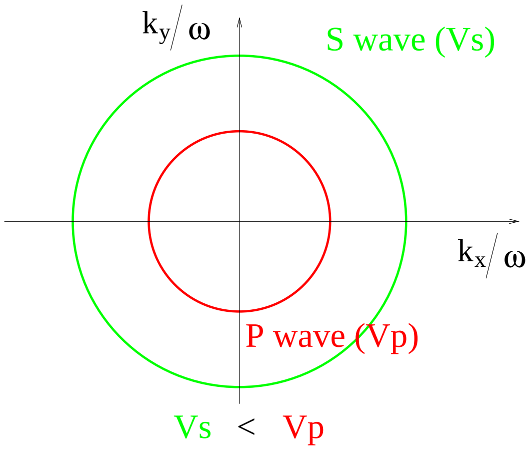
$$V(k) = \nabla_k \omega\,(k) \quad (= V(K), \quad K = k/|k|\,)$$

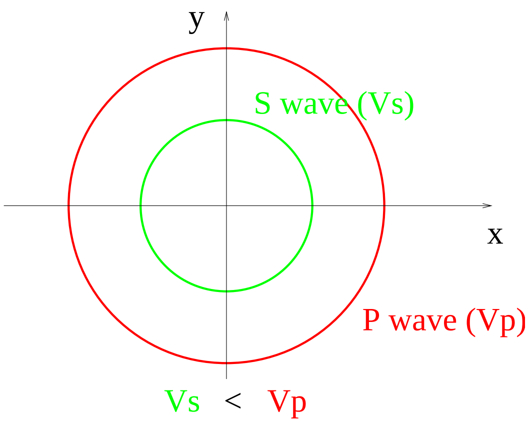
which is orthogonal to the slowness curves:

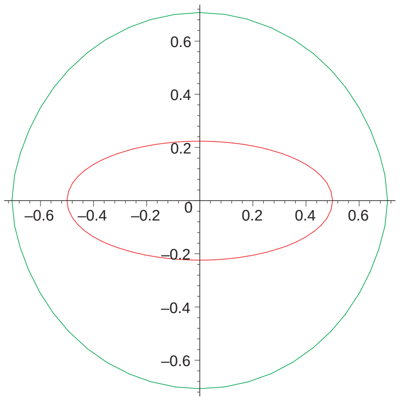
$$V(k) = -\left(\frac{\partial F}{\partial \omega}(\omega, k)\right)^{-1} \nabla_k F(\omega, k).$$

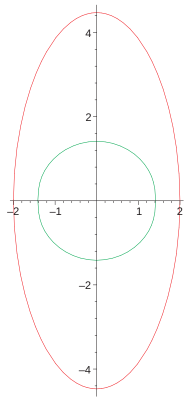
Anisotropic elastic waves

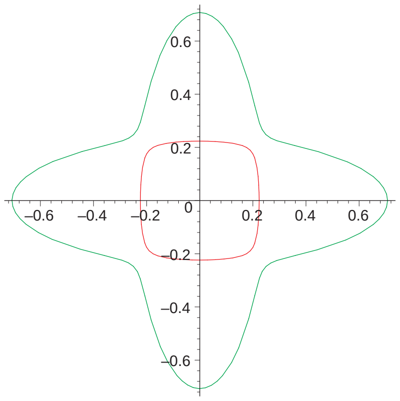
Slowness curves and wave fronts

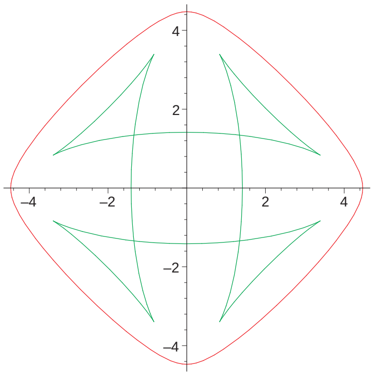








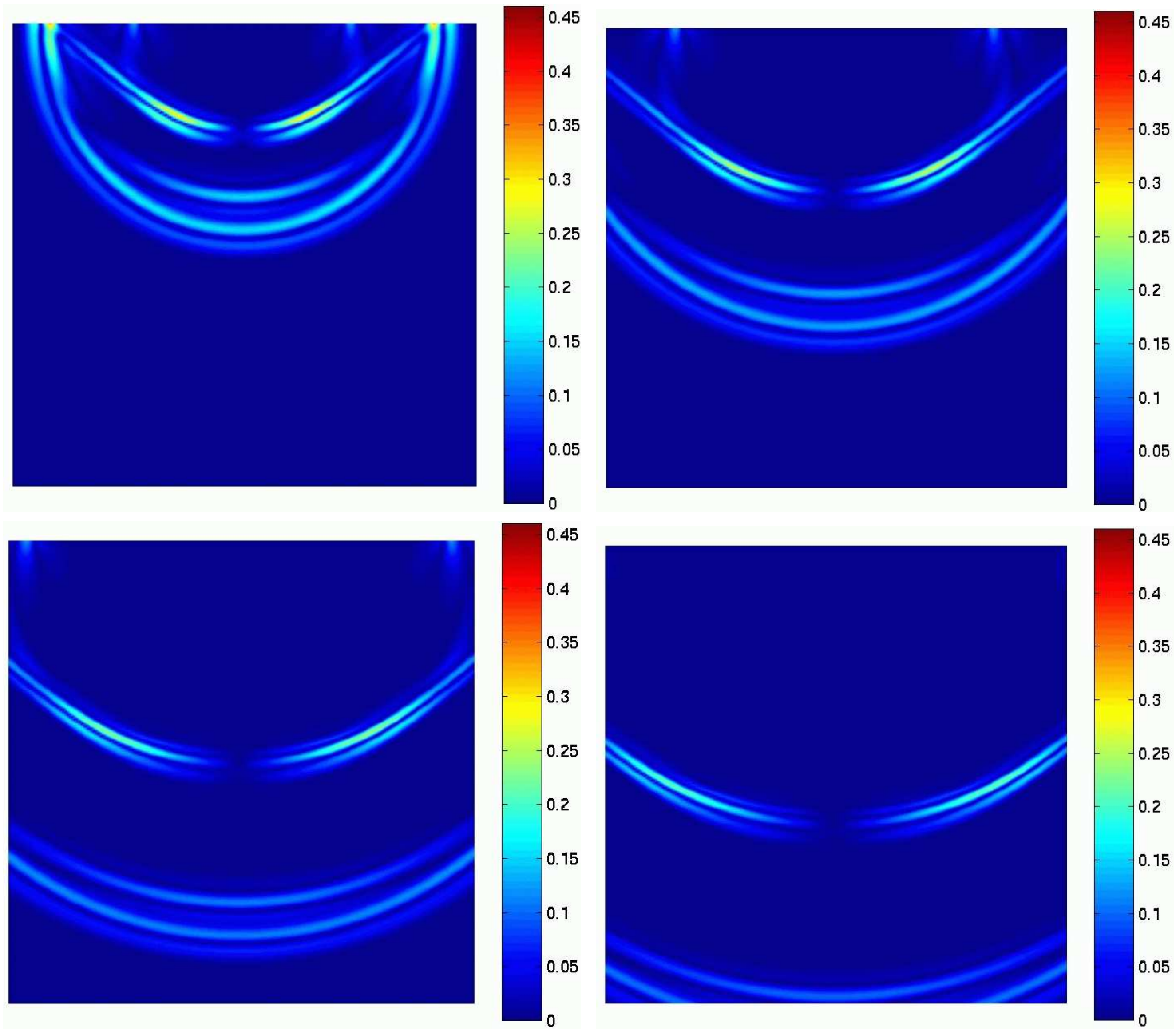




Anisotropic elastic waves

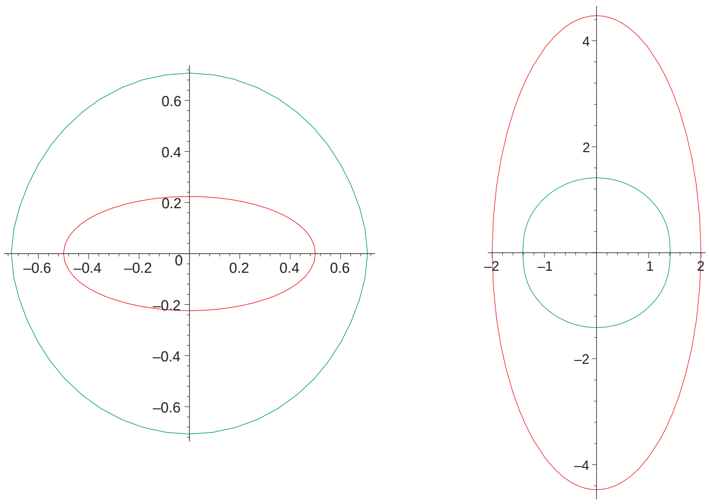
Numerical experiment (1)

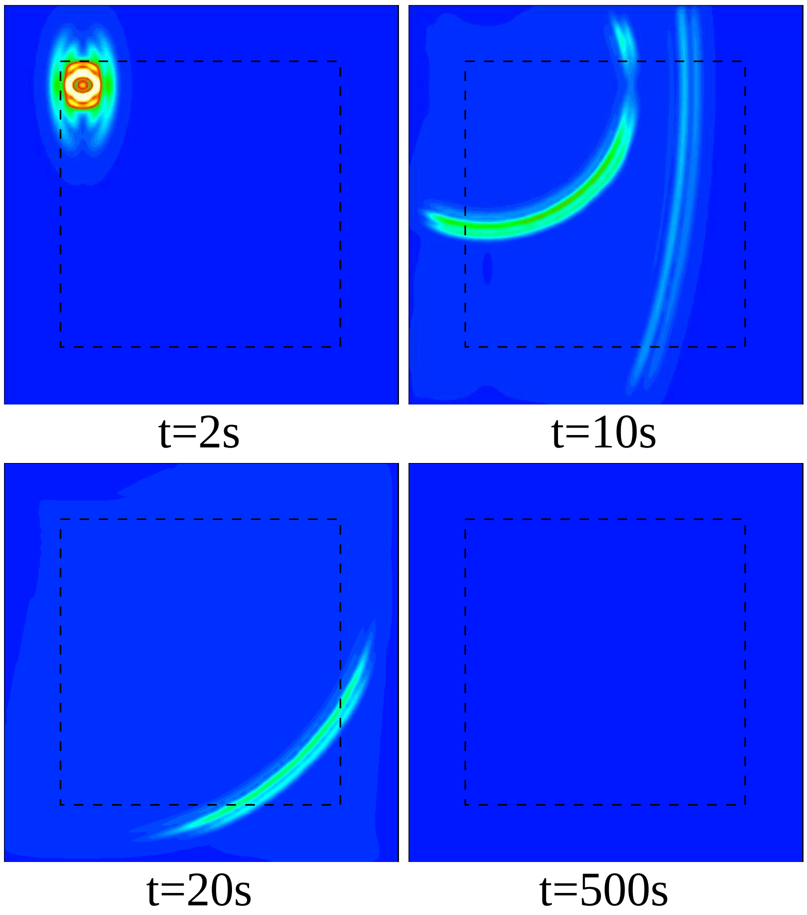
Isotropic media and Rayleigh wave



Anisotropic elastic waves

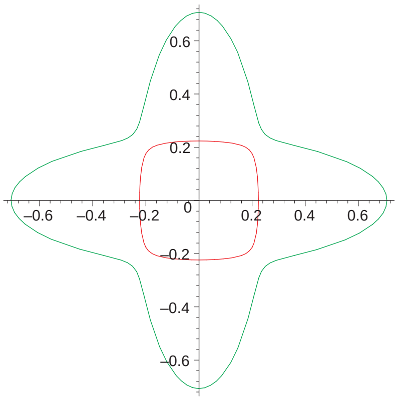
Numerical experiment (2)

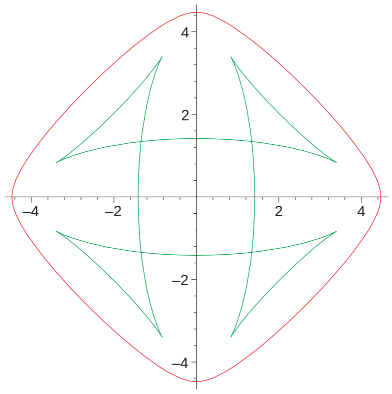


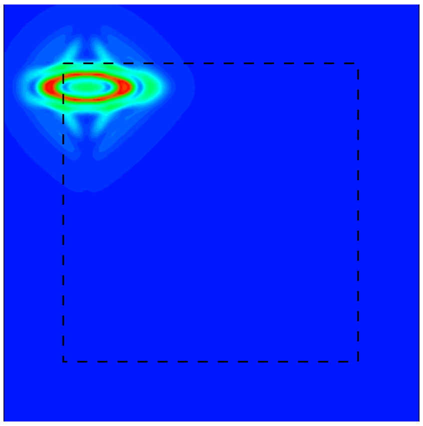


Anisotropic elastic waves

Numerical experiment (3)







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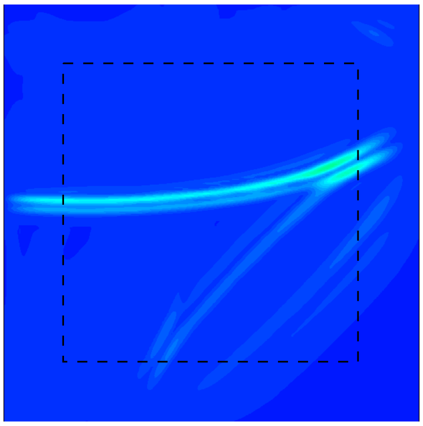

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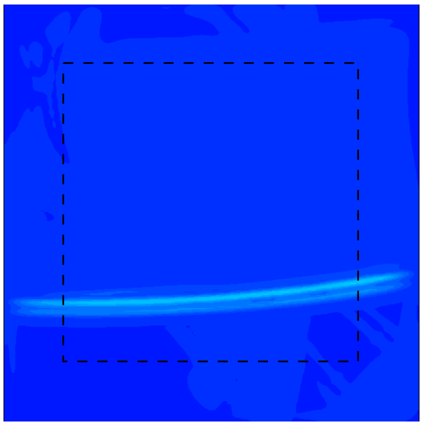

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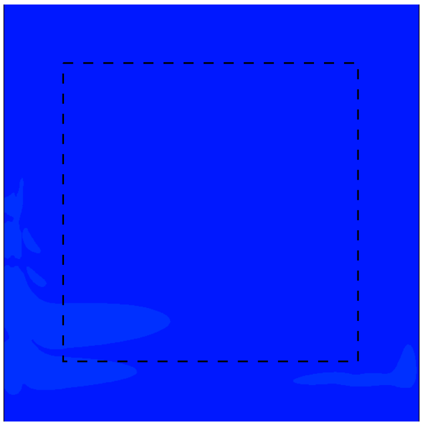
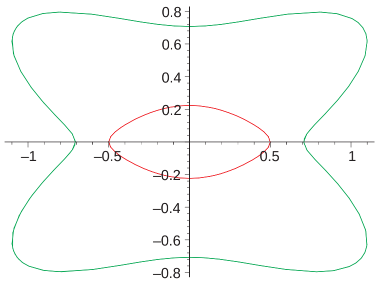
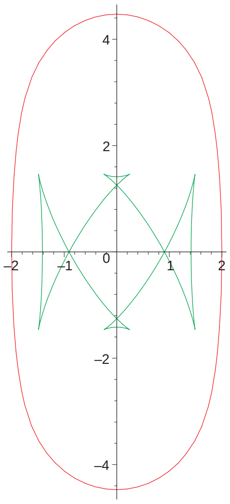


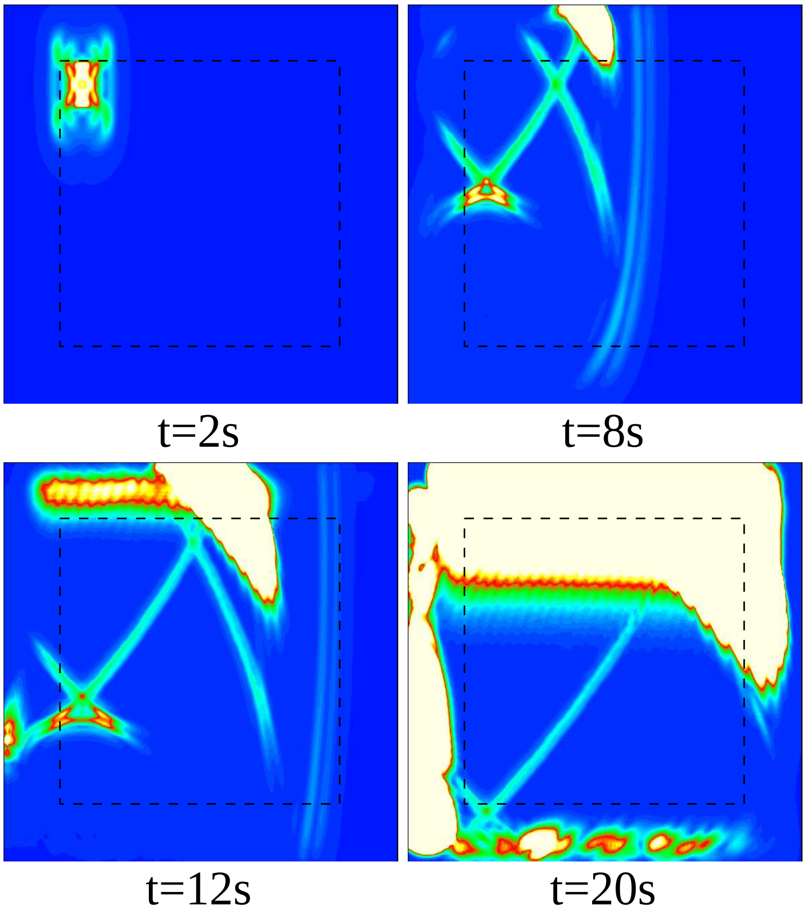
t=500s

Anisotropic elastic waves

Numerical experiment (4)







Stability of PML's for acoustic waves (1)

To simplify we assume $c = 1$. The acoustic PML dispersion relation is:

$$(\omega^2 - k_y^2)(i\omega + \sigma)^2 + \omega^2 k_x^2 = 0$$

which becomes for the unperturbed system ($\sigma = 0$):

$$\omega^2(\omega^2 - k_y^2 - k_x^2) = 0$$

One easily shows that, if $\sigma > 0$, for any solution $\omega(k)$:

$$0 \leq \mathcal{I}m\ \omega(k) \leq \sigma, \quad \forall k \in \mathbb{R}^2$$

and to establish (with additional work) the following stability result (which is also valid for isotropic Maxwell's equations):

Theorem (Bécache-Joly)

The (u^x, u^y, v_x, v_y) of the Cauchy problem for the acoustic PML model satisfies (we set $u = u^x + u^y$)

$$\begin{cases} \|u(t)\|_{L^2} + \|v_x(t)\|_{L^2} + \|v_y(t)\|_{L^2} \leq C\ \|U_0\|_{L^2} \\ \\ \|u_x(t) - u_y(t)\|_{H^{-1}} \leq C\ t\ \|U_0\|_{L^2} \end{cases}$$

with $U_0 = (\ (u^x)_0, (u^y)_0, (v_x)_0, (v_y)_0\ )$.

Stability of PML's for acoustic waves (1)

The previous result can be revisited with the help of an energy analysis.

In the case $\sigma = 0$, it is well known that:

$$\frac{1}{2}\frac{d}{dt}\int \{\rho\, |u|^2 + \mu^{-1}\left(|v_x|^2 + |v_y|^2\right)\}\, dx = 0.$$

In the case $\sigma > 0$, one shows that (Bécache-Joly):

$$\left|\begin{array}{l} \dfrac{1}{2}\dfrac{d}{dt}\left\{\displaystyle\int \rho\, |\frac{\partial u}{\partial t} + \sigma u|^2\, dx + \int \sigma^2 \mu^{-1} |v_x|^2\, dx\right\} \\ +\dfrac{1}{2}\dfrac{d}{dt}\left\{\displaystyle\int \mu^{-1}\left(|\frac{\partial v_x}{\partial t}|^2 + |\frac{\partial v_y}{\partial t} + \sigma\, v_y|^2\right)\, dx\right\} \\ +\, 2\displaystyle\int \sigma\, \mu^{-1} |\frac{\partial v_x}{\partial t}|^2\, dx = 0. \end{array}\right.$$

which represents a dissipation result for $\sigma > 0$..

- The proof relies on the Zhao-Cangellaris formulation.
- What about σ variable ? $^{(*)}$

$^{(*)}$ One easily gets estimates in $\exp\{t\|\sigma\|_\infty\}$)

A general necessary stability condition (1)

We consider a propagation model whose dispersion relation is:

$$F(\omega, k_x, k_y) = 0, \quad (F \text{ homogeneous}).$$

and defines physical modes:

$$\omega = \omega(k) = |k| \, \omega(K) > 0, \quad K = k \,/\, |k|.$$

The group velocity only depends on K:

$$V(K) = \nabla\omega \; (k) = -\left(\frac{\partial F}{\partial \omega}(\omega, k)\right)^{-1} \; \nabla_k F(\omega, k)$$

The dispersion relation of the corresponding PML model is simply:

$$F(\omega, k_x(\frac{i\omega}{i\omega + \sigma}), k_y) = 0.$$

One obtains a high frequency necessary stability conditions by looking at the asymptotic behaviour of the modes which, for large $|k|$, which is equivalent to small σ, approach the physical modes:

$$\omega(\sigma, k) = \omega(k) + i \; \alpha(K) \; \sigma + O(\frac{1}{|k|}).$$

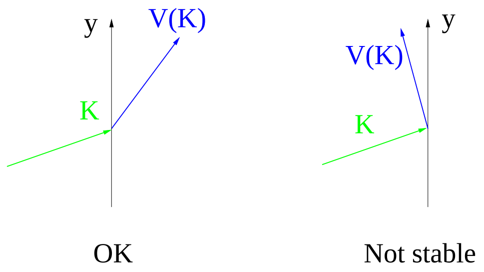
$$\alpha = -\frac{K_x}{c(K)} \; (\frac{\partial F}{\partial \omega})^{-1} (\frac{\partial F}{\partial k_x}) = \frac{K_x \cdot V_x(K)}{c(K)}$$

A general necessary stability condition (2)

A necessary stability condition can thus easily be expressed with the help of the group velocity $^{(*)}$:

$$\forall K = (K_x, K_y) \;/\; |K| = 1, \quad K_x \cdot V_x(K) \geq 0.$$

which can be expressed geometrically on the slowness curves:



In other words, high frequency instabilities are due to the existence of "back propagating" waves in the original model.

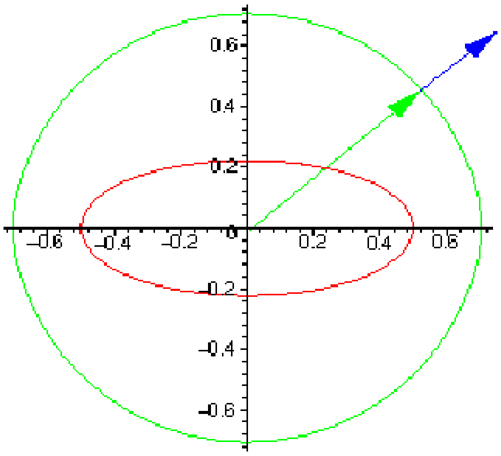
For the elasticity system, this condition corresponds to:

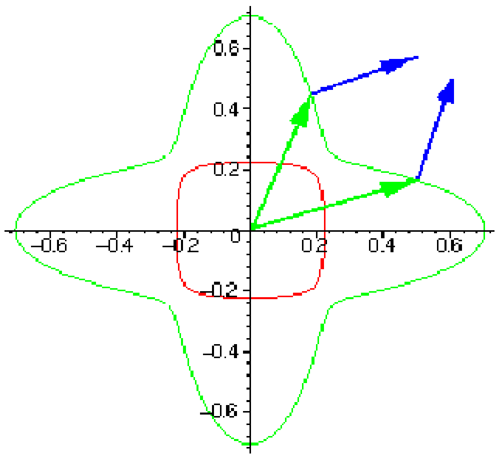
$$(HF) \qquad (c_{12} + c_{33})^2 \leq \sup\,\{c_{11}(c_{22} - c_{33}), -c_{33}(c_{22} - c_{33})\}$$

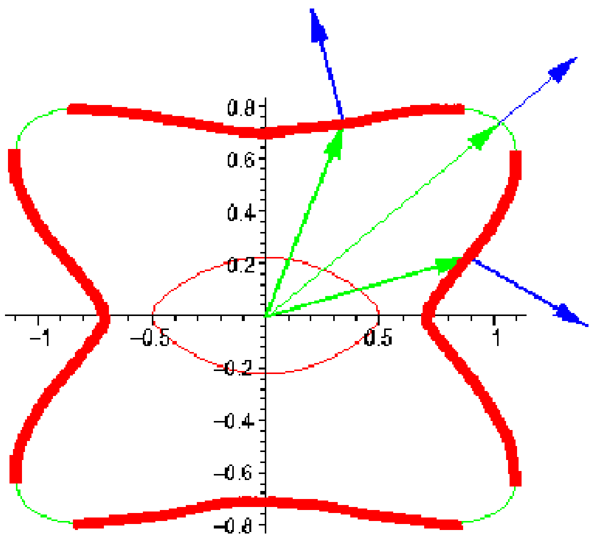
$^{(*)}$ The rôle of group velocities has already been emphasized:

- by Trefethen for the behaviour of numerical schemes
- by Higdon for the well-posedness of boundary conditions.

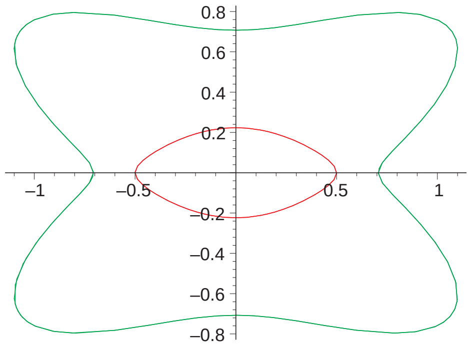
Back to the numerical experiments



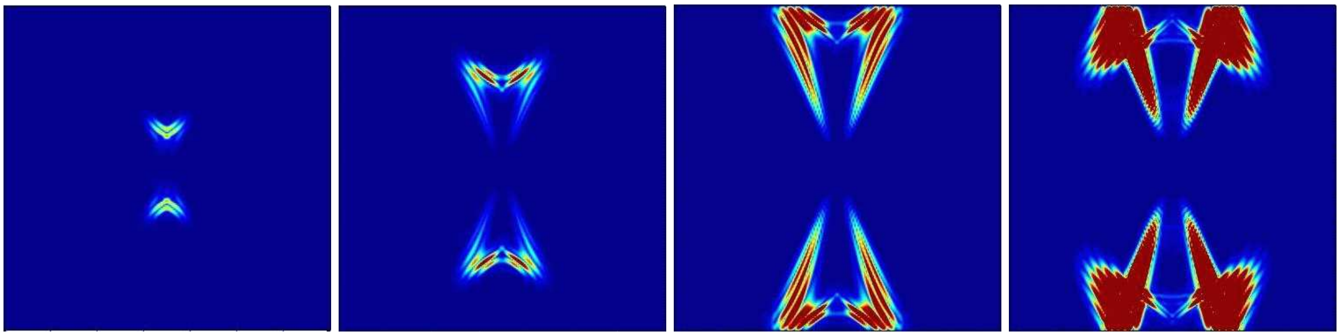




Numerical illustration of instability

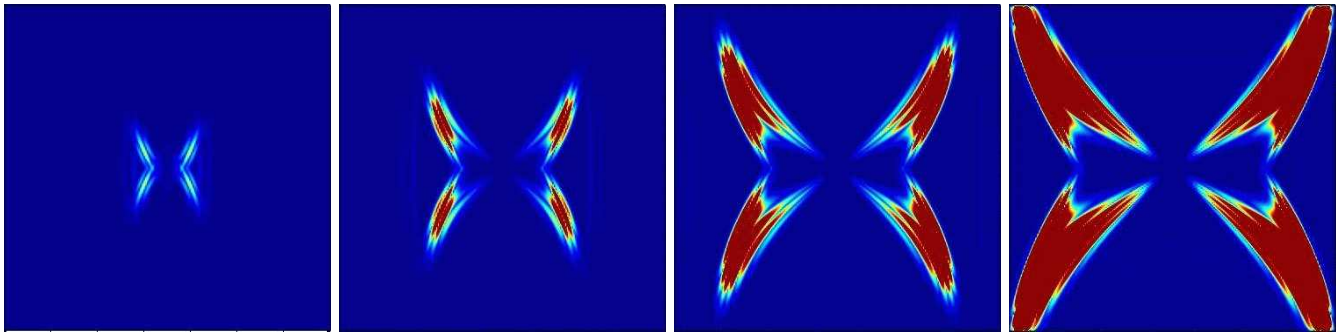


PML suivant x



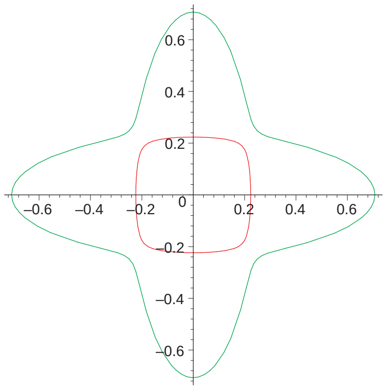
t=4s t=8s t=12s t=16s

PML suivant y

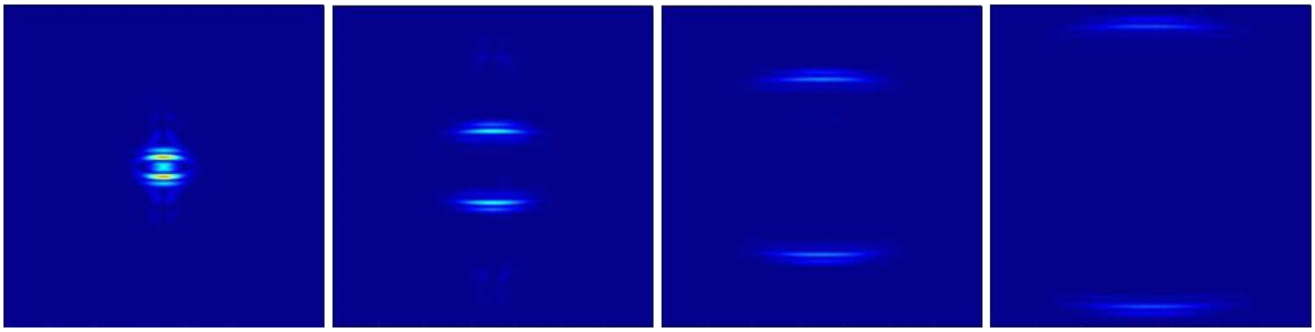


t=3s t=6s t=9s t=12s

Numerical illustration of stability

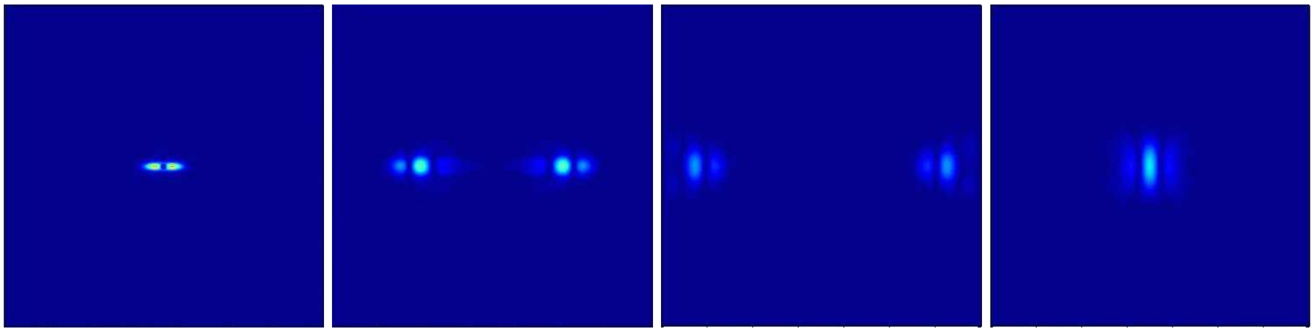


PML suivant x



t=2s t=4s t=8s t=12s

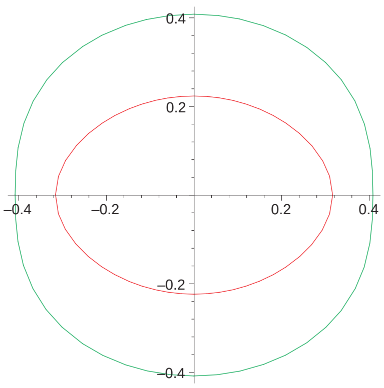
PML suivant y

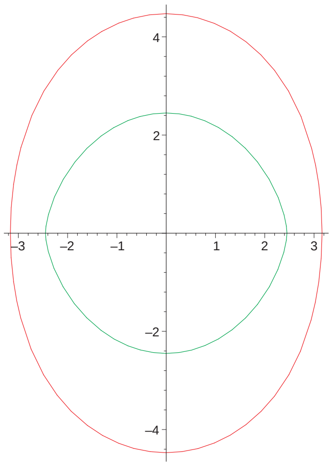


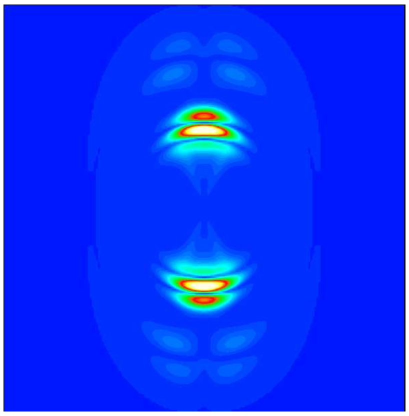
t=1s t=3s t=6s t=9s

Anisotropic elastic waves

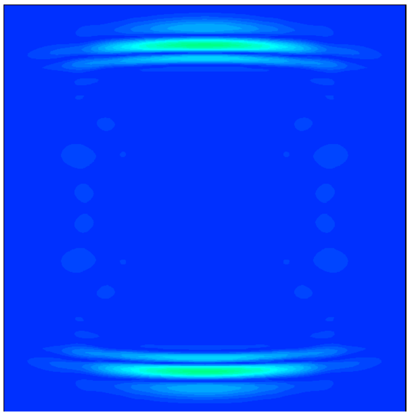
Numerical experiment (5)



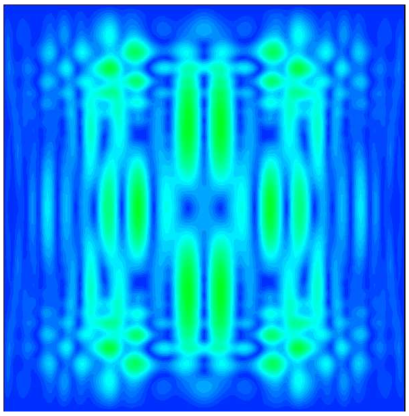




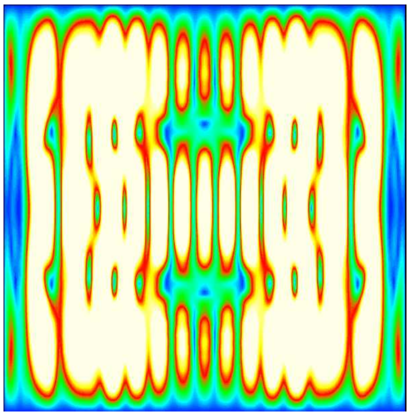
t=4s



t=50s



t=150s



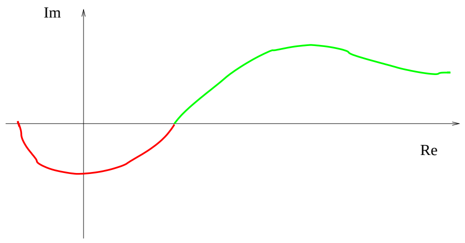
t=200s

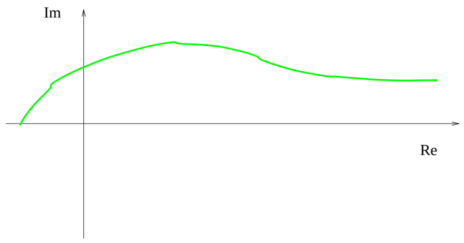
Anisotropic elastic waves

A sufficient stability condition

One obtains a sufficient stability condition by ensuring, that for all modes:

1. $\omega(\sigma, k)$ is in the right complex half-space for large $|k|$.
2. $\omega(\sigma, k)$ never becomes real





Theorem (Bécache-Fauqueux-Joly)

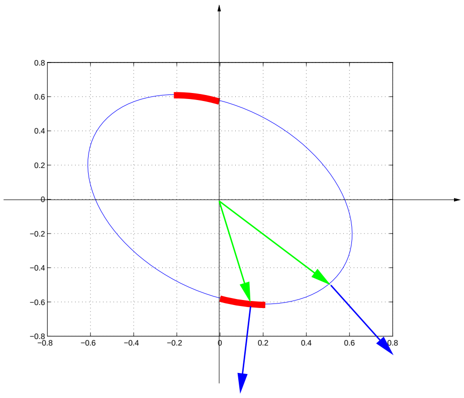
The x-PML model for anisotropic elastic waves is stable as soon as (HF) holds as well as one of the following conditions :

$$
\left\{
\begin{array}{l}
(c_{12}+c_{33})^2 < (c_{11}-c_{33})(c_{22}-c_{33}) \qquad \text{or} \\[2ex]
\left\{
\begin{array}{l}
(c_{11}-c_{33})(c_{22}-c_{33}) \leq (c_{12}+c_{33})^2 \\[2ex]
(c_{11}-c_{33})\left(c_{11}c_{22}-c_{33}^2\right) < (c_{11}+c_{33})(c_{12}+c_{33})^2 \\[2ex]
(c_{12}+2c_{33})^2 < c_{11}c_{22}
\end{array}
\right.
\end{array}
\right.
$$

Other examples

- Anisotropic waves (see Fauqueux-Joly):

$$\frac{\partial^2 u}{\partial t^2} - \text{div}\,(A\nabla u) = 0, \quad A \text{ not diagonal}$$



- Linearized Euler equations (see Diaz-Joly, Hu, Hestaven, Métral-Vacus, Hagström-Nazarov):

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial x} - \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} = 0 \\ &\frac{\partial v_x}{\partial t} + M\frac{\partial v_x}{\partial x} - \frac{\partial u}{\partial x} = 0 \\ &\frac{\partial v_y}{\partial t} + M\frac{\partial v_y}{\partial x} - \frac{\partial u}{\partial y} = 0 \end{aligned}\right.$$