Reconstruction of Small Multiple Inhomogeneities

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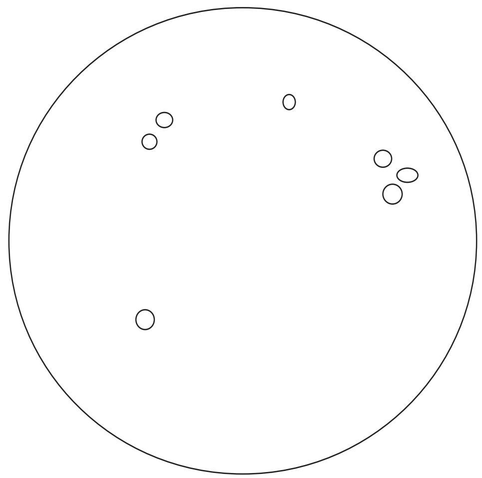
Configuration of Medium



$$D = \cup_{s=1}^{m} D_s = \cup_{s=1}^{m} \big(\cup_{l=1}^{m_s} D_s^l \big) = \cup_{s=1}^{m} \Big(\epsilon \cup_{l=1}^{m_s} B_s^l + z_s \Big).$$

Conductivity Profile of Ω:

$$\gamma(x) = \begin{cases} k_s^l, & x \in D_s^l, \quad l = 1, \ldots, m_s, \ s = 1, \ldots, m, \\ 1, & x \in \Omega \setminus \overline{D}. \end{cases}$$

Neumann Problem:

$$\begin{cases} \nabla \cdot (\gamma(x) \nabla u) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g. \end{cases}$$

Inverse Problem: Detect the inclusion D by means of finitely many g and $u|_{\partial\Omega}$ where u is the solution.

Asymptotic Expansion in Free Space

Suppose that $B = \cup_{s=1}^m B_l$ and the conductivity of B_l is k_l, and consider

$$\begin{cases} \nabla\cdot\Big(\chi(\Omega\setminus\overline{B}) + \displaystyle\sum_{s=1}^m k_s\chi(B_s)\Big)\nabla u = 0 \quad \text{in } \mathbb{R}^d, \\ u(x) - H(x) = O(|x|^{1-d}) \quad \text{as } |x|\to\infty, \end{cases}$$

where H is a harmonic function in $\mathbb{R}^d$.

The solution u is represented as

$$u(x) = H(x) + \sum_{l=1}^{m} \mathcal{S}_{B_l}\varphi^{(l)}(x), \quad x \in \mathbb{R}^d,$$

where $\varphi^{(l)} \in L^2_0(\partial B_l)$, $l = 1, \ldots, m$, is the solution of the integral equation

$$(\lambda_l I - \mathcal{K}^*_{B_l})\varphi^{(l)} - \sum_{k \neq l} \frac{\partial(\mathcal{S}_{B_k}\varphi^{(k)})}{\partial\nu^{(l)}}\Big|_{\partial B_l} = \frac{\partial H}{\partial\nu^{(l)}}\Big|_{\partial B_l} \quad \text{on } \partial B_l, \quad l = 1, \ldots, m.$$

Here $\nu^{(l)}$ denotes the outward unit normal to ∂B_l and

$$\lambda_l = \frac{k_l + 1}{2(k_l - 1)}, \quad l = 1, \ldots, m.$$

Layer Potential : Let $\Gamma(x)$ be the fundamental solution of the Laplacian Δ:

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d}, & d \geq 3, \end{cases}$$

where ω_d is the area of the $(d-1)$-dimensional unit sphere. The single layer potential of the density function ϕ on D is defined by

$$\mathcal{S}_D \phi(x) := \int_{\partial D} \Gamma(x-y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^d,$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$ where

$$\mathcal{K}_D \phi(x) := \frac{1}{\omega_d} \text{p.v.} \int_{\partial D} \frac{\langle x-y, \nu_y \rangle}{|x-y|^d} \phi(y) d\sigma(y), \quad x \in \partial D.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ be multi-indices. For $l = 1, \ldots, m$ let $\varphi_\alpha^{(l)}$ be the solution of

$$(\lambda_l I - \mathcal{K}_{B_l}^*)\varphi_\alpha^{(l)} - \sum_{k \neq l} \frac{\partial(\mathcal{S}_{B_k}\varphi_\alpha^{(k)})}{\partial \nu^{(l)}}\Big|_{\partial B_l} = \frac{\partial x^\alpha}{\partial \nu^{(l)}}\Big|_{\partial B_l} \quad \text{on } \partial B_l.$$

Since $H(x) = \sum_\alpha \frac{1}{\alpha!}\partial^\alpha H(0) x^\alpha$ where the series converges uniformly and absolutely on any compact set, we get

$$\varphi^{(l)} = \sum_{|\alpha| \geq 1} \frac{1}{\alpha!}\partial^\alpha H(0)\varphi_\alpha^{(l)}, \quad l = 1, \ldots, m.$$

By Taylor expansion, we have

$$\Gamma(x-y) = \sum_\beta \frac{1}{\beta!}\partial^\beta \Gamma(x) y^\beta, \quad y \text{ in a compact set}, \quad |x| \to \infty.$$

Thus

$$u(x) = H(x) + \sum_{|\alpha|\geq 1} \sum_{\beta} \frac{1}{\alpha!\beta!} \partial^\alpha H(0) \partial^\beta \Gamma(x) \sum_{l=1}^{m} \int_{\partial B_l} y^\beta \varphi_\alpha^{(l)}(y) d\sigma(y)$$

for $|x|$ large. Since

$$\sum_{|\alpha|\geq 1} \frac{1}{\alpha!} \partial^\alpha H(0) \sum_{l=1}^{m} \int_{\partial B_l} \varphi_\alpha^{(l)}(y) d\sigma(y) = \sum_{l=1}^{m} \int_{\partial B_l} \varphi^{(l)}(y) d\sigma(y) = 0,$$

Asymptotic Expansion:

$$u(x) = H(x) + \sum_{|\alpha|\geq 1} \sum_{|\beta|\geq 1} \frac{1}{\alpha!\beta!} \partial^\alpha H(0) \partial^\beta \Gamma(x) m_{\alpha\beta}, \quad |x| \to \infty,$$

where

$$m_{\alpha\beta} := \sum_{l=1}^{m} \int_{\partial B_l} y^\beta \varphi_\alpha^{(l)}(y) d\sigma(y). \tag{1}$$

The asymptotic expansion formula shows that the perturbation of the electric potential in $\mathbb{R}^d$ due to the presence of inclusions are completely described by $m_{\alpha\beta}$.

This $m_{\alpha\beta}$, $\alpha, \beta \in \mathbb{N}^d$, is called the *generalized polarization tensor* (GPT) associated with the inclusions $\cup_{l=1}^m B_l$. It should be observed that $m_{\alpha\beta}$ also depends on the conductivities k_l.

If $|\alpha| = |\beta| = 1$, we denote $m_{\alpha\beta}$ by m_{ij}, $i, j = 1, \ldots, d$. We call m_{ij} the *first-order polarization tensor.*

Remark.

- The first order PT was introduced by Pólya-Schiffer-Szegö.
- Low frequency asymptotic of waves: Kleinman and Senior.
- Inverse Problems: Cedio.Fenya-Moskow-Vogelius, Friedman-Vogelius.
- Our definition includes higher order tensors as well as multiple inclusions.

Theorem 1 (Single Inclusions). *[Ammari-K]*

(1) *(Symmetry) Suppose that* a_α *and* b_β *are constants such that* $\sum_\alpha a_\alpha y^\alpha$ *and* $\sum_\beta b_\beta y^\beta$ *are harmonic polynomials. Then*

$$\sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta} = \sum_{\alpha,\beta} a_\alpha b_\beta m_{\beta\alpha}.$$

(2) *(Positivity) There exists a constant* C *depending only on the Lipschitz character of* B *such that if* $\sum_{\alpha\in I} a_\alpha x^\alpha$ *is a harmonic polynomial, then*

$$\begin{aligned}\int_B |\nabla(\sum_{\alpha\in I} a_\alpha x^\alpha)|^2 dx &\le \frac{k+1}{|k-1|}\Big|\sum_{\alpha,\beta\in I} a_\alpha a_\beta m_{\alpha\beta}\Big| \\ &\le C\int_B |\nabla(\sum_{\alpha\in I} a_\alpha x^\alpha)|^2 dx.\end{aligned}$$

In particular, if $|\alpha| = |\beta| = 1$*, then*

$$|B| \le \frac{k+1}{|k-1|}\Big|\sum_{\alpha,\beta\in I} a_\alpha a_\beta m_{\alpha\beta}\Big| \le C|B|.$$

(3) *(Dirichlet-to-Neumann map) Let Ω be a domain compactly containing $\overline{B}$. Then the GTP uniquely determines the Dirichlet-to-Neumann map on $\partial\Omega$, and hence k and B.*

Remark.

- It would be interesting to find out what kinds of geometric information each $m_{\alpha\beta}$ carries.
- There is one-to-one correspondence between the class of ellipses and $m_{\alpha\beta}$, $|\alpha| = |\beta| = 1$ (Brühl-Henke-Vogelius, Movchan-Serkov).

Electromagnetic Polarization Tensors for Multiple Inclusions

Define

$$m_{\alpha\beta} := \sum_{l=1}^{m} \int_{\partial B_l} y^\beta \varphi_\alpha^{(l)}(y) d\sigma(y),$$

where $\varphi_\alpha^{(l)}$ be the solution of

$$(\lambda_l I - \mathcal{K}_{B_l}^*)\varphi_\alpha^{(l)} - \sum_{k \neq l} \frac{\partial(\mathcal{S}_{B_k}\varphi_\alpha^{(k)})}{\partial \nu^{(l)}}\Big|_{\partial B_l} = \frac{\partial x^\alpha}{\partial \nu^{(l)}}\Big|_{\partial B_l} \quad \text{on } \partial B_l.$$

Overall conductivity $\bar{k}$ of $B = \cup_{l=1}^{m} B_l$:

$$\frac{\bar{k}-1}{\bar{k}+1} \sum_{l=1}^{m} |B_l| := \sum_{l=1}^{m} \frac{k_l - 1}{k_l + 1} |B_l| \quad \text{Harmonic Average.}$$

Center $\bar{z}$ of $B = \cup_{l=1}^{m} B_l$:

$$\frac{\bar{k}-1}{\bar{k}+1} \bar{z} \sum_{l=1}^{m} |B_l| = \sum_{l=1}^{m} \frac{k_l - 1}{k_l + 1} \int_{B_l} x dx.$$

Note that if k_l is the same for all l then $\bar{k} = k_l$ and $\bar{z}$ is the center of mass of B.

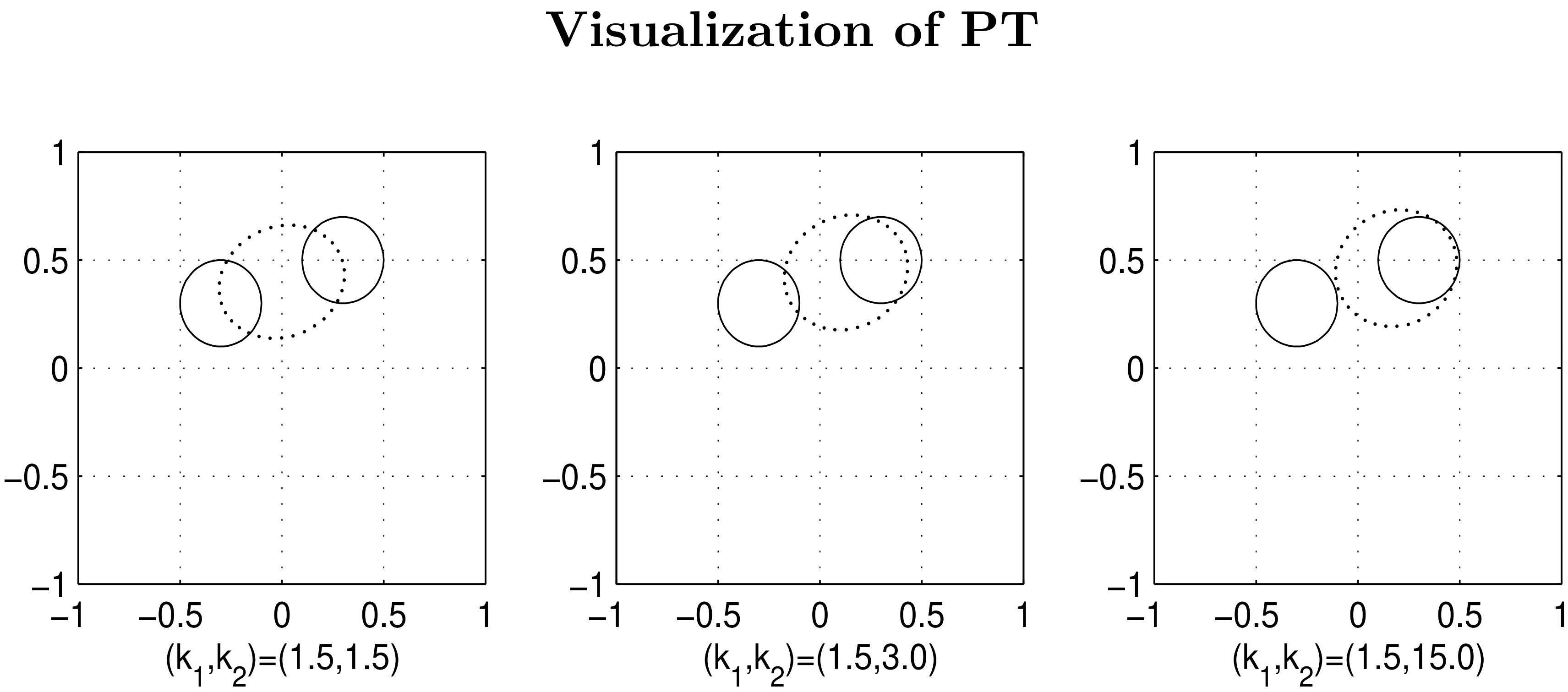


Figure 1: When the two disks have the same radius and the conductivity of the one on the right-hand side is increasing, the equivalent ellipse is moving toward the right inclusion.

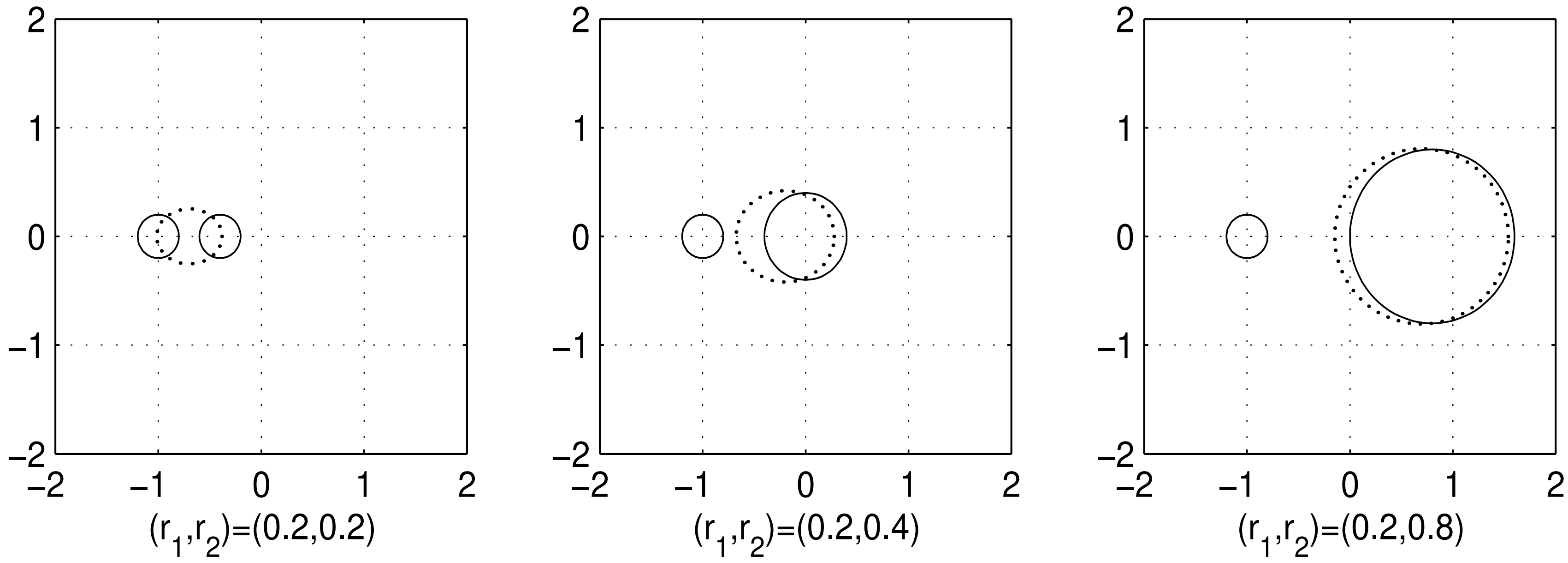


Figure 2: When the conductivities of the two disks is the same and the radius of the disk on the right-hand side is increasing, the equivalent ellipse is moving toward the right inclusion.

Theorem 2 (Ammari-K-Kim-Lim). (1) *The polarization tensor M is symmetric.*

(2) *Suppose that either $k_l - 1 > 0$ or $k_l - 1 < 0$ for all $l = 1, \dots, m$. Let*

$$\kappa := \max_{1 \le l \le m} \left| 1 - \frac{1}{k_l} \right| .$$

For any a_α such that $\sum_\alpha a_\alpha y^\alpha$ is harmonic,

$$\left| \sum_{\alpha,\beta} a_\alpha a_\beta m_{\alpha\beta} \right| \ge \frac{|\kappa - 1|}{m+1} \sum_{l=1}^{m} |k_l - 1| \int_{B_l} \left| \nabla (\sum_\alpha a_\alpha y^\alpha) \right|^2 dy.$$

(3) *Let Ω be a domain with smooth boundary such that $\cup_{l=1}^m \overline{B_l} \subset \Omega$. Then $m_{\alpha\beta}$ uniquely determines the Neumann-to-Dirichlet map on Ω, and hence $\cup_{l=1}^m B_l$ and $k_1, \dots, k_m$.*

Explicit Representation of Electric Potentials

$$\begin{cases} \nabla\cdot\Big(\chi(\Omega\setminus\bigcup_{l=1}^{m}\overline{B_l})+\sum_{l=1}^{m}k_l\chi(B_l)\Big)\nabla u=0 \quad \text{in } \mathbb{R}^2, \\ u(x)-H(x)=O(|x|^{1-d}) \quad \text{as } |x|\to\infty, \end{cases}$$

Suppose that B_l is a disk and let

$$(R_l f)(x) = f(R_l(x)), \quad R_l(x) := \frac{r_l^2(x - z_l)}{|x - z_l|^2} + z_l.$$

For $l = 1, ..., m$, let

$$S_l = \{\Theta = (k_1, \cdots, k_n) \mid n \in \mathbb{N},\ k_i \in \{1, \cdots, m\},\ k_1 \neq l,\ k_i \neq k_{i+1}\}.$$

For $\Theta = (k_1, \cdots, k_n) \in S_l$, let

$$R_\Theta = R_{k_1} R_{k_2} \cdots R_{k_n} \text{ and } \Lambda_\Theta = \prod_{i=1}^{n} \left(-\frac{1}{2\lambda_{k_i}} \right).$$

Theorem 3 (A-K-K-L).

$$u(x) = H(x) + \sum_{j=1}^{m} \mathcal{S}_{B_j} \varphi^{(j)}(x), \quad x \in \mathbb{R}^2,$$

where

$$\varphi^{(l)} = \frac{1}{\lambda_l} \sum_{\Theta \in S_l} \Lambda_\Theta \frac{\partial}{\partial \nu_l} (R_\Theta H)\Big|_{\partial B_l} + \frac{1}{\lambda_l} \frac{\partial H}{\partial \nu_l}\Big|_{\partial B_l}, \quad l = 1, \cdots, m,$$

provided that

$$\min_{1 \le i \ne j \le m} dist(B_i, B_j) > (\sqrt{m-1} - 1) \max_{1 \le i \le m} r_i.$$

The series converges absolutely.

It would be very useful in computation of electric fields in the presence of closely spaced inclusions to remove the assumption and to derive convergence estimates (Cheng-Greengard).

Asymptotic Expansion- Boundary Value Problem

$$D = \Big(\epsilon \cup_{s=1}^{m} B_l + z\Big).$$

$$\begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \bigcup_{j=1}^{m} \overline{D_j}) + \sum_{j=1}^{m} k_j \chi(D_j) \Big) \nabla u = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = g. \end{cases}$$

$$u(x) = U(x) - \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d} \frac{\epsilon^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} (\partial^\alpha U)(z) \partial_z^\beta N(x,z) m_{\alpha\beta} + O(\epsilon^{2d})$$

$$\Delta U = 0 \quad \text{in } \Omega, \quad \frac{\partial U}{\partial \nu}|_{\partial\Omega} = g,$$

$$\Delta_x N(x,z) = -\delta_z \quad \text{in } \Omega, \quad \frac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|}.$$

$m_{\alpha\beta}$: GPT for $\cup_{s=1}^{m} B_l$.

Detection of Inclusions

- Single Inclusion:
  - Conductivity: Ammari-K, Ammari-Seo, Cedio.Fenya-Moskow-Vogelius, Friedman-Vogelius, K-Kim-Kim, Kwon-Seo-Yoon,
  - Maxwell: Ammari-K, Ammari-Vogelius-Volkov
  - Elasticity: Ammari-K-Nakamura-Tanuma, K-Kim-Lee
- Multiple Well-Separated Inclusions: Ammari-Moskow-Vogelius, Bruhl-Hanke-Vogelius

For $g \in L_0^2(\partial\Omega)$, define

$$H[g](x) := -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(u|_{\partial\Omega})(x), \quad x \in \mathbb{R}^d \setminus \bar{\Omega}.$$

Then,

$$H[g](x) = -\sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d} \frac{\epsilon^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} (\partial^\alpha U)(z) \partial_z^\beta \Gamma(x-z) m_{\alpha\beta} + O(\epsilon^{2d}).$$

First order term,

$$H[g](x) = -\epsilon^d \nabla U(z) M \nabla_z \Gamma(x-z) + O(\epsilon^{d+1}), \tag{2}$$

where $M = (m_{ij})$ is the polarization tensor associated with $\cup_{l=1}^k B^l$.

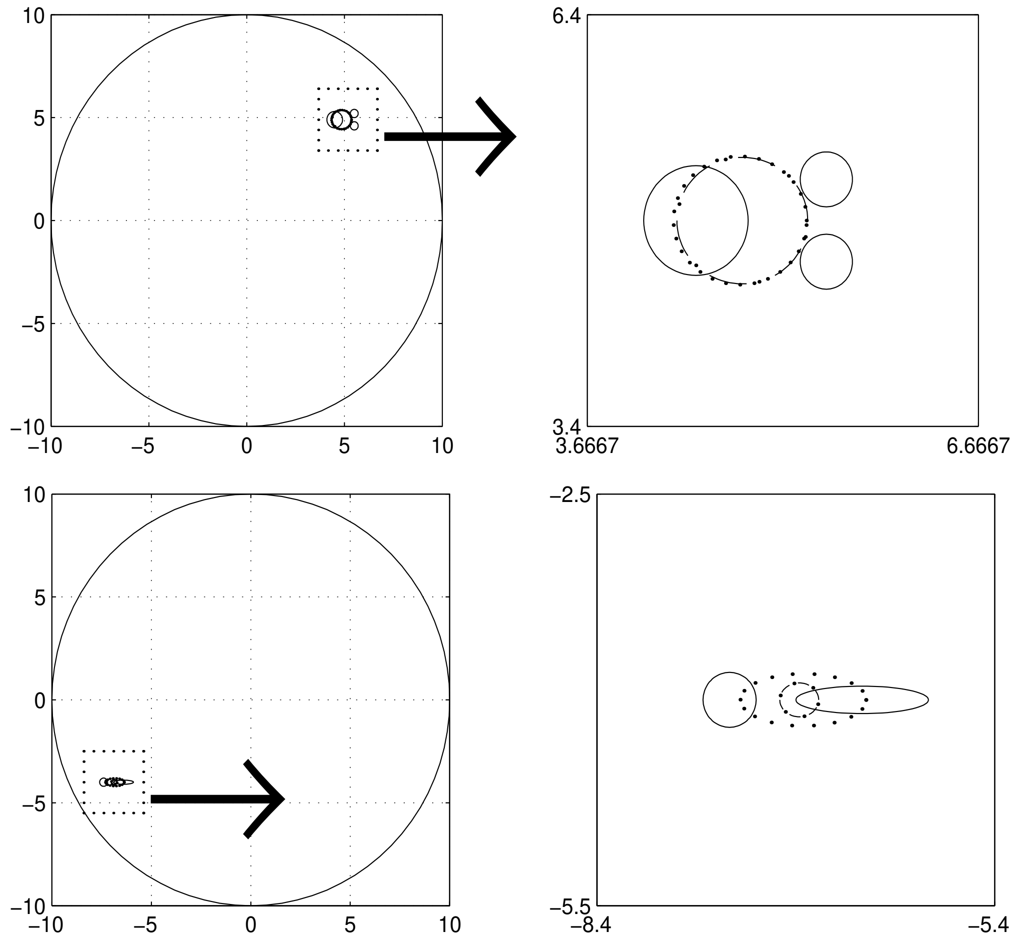
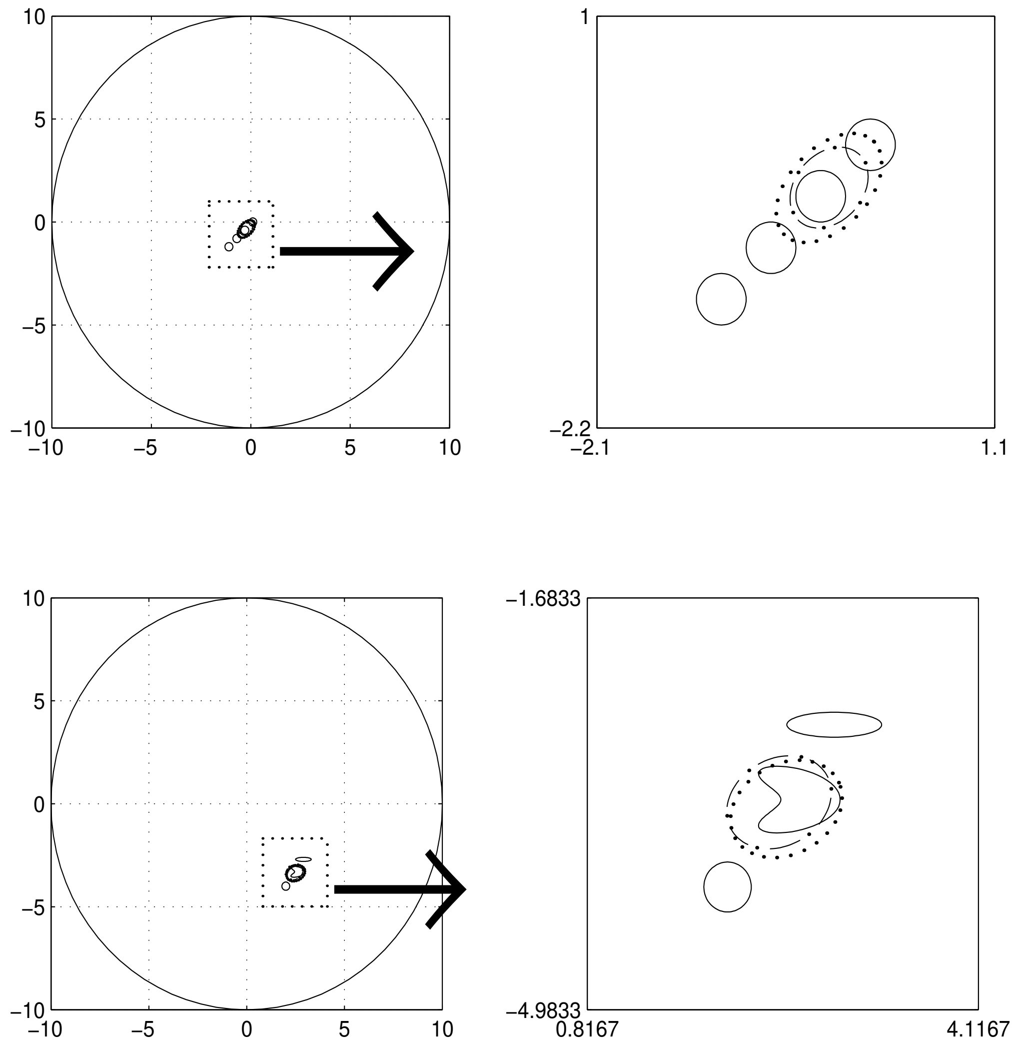


Figure 3: Reconstruction of closely spaced small inhomogeneities. The dash line is the equivalent ellipse and the dash-dot line is the detected ellipse.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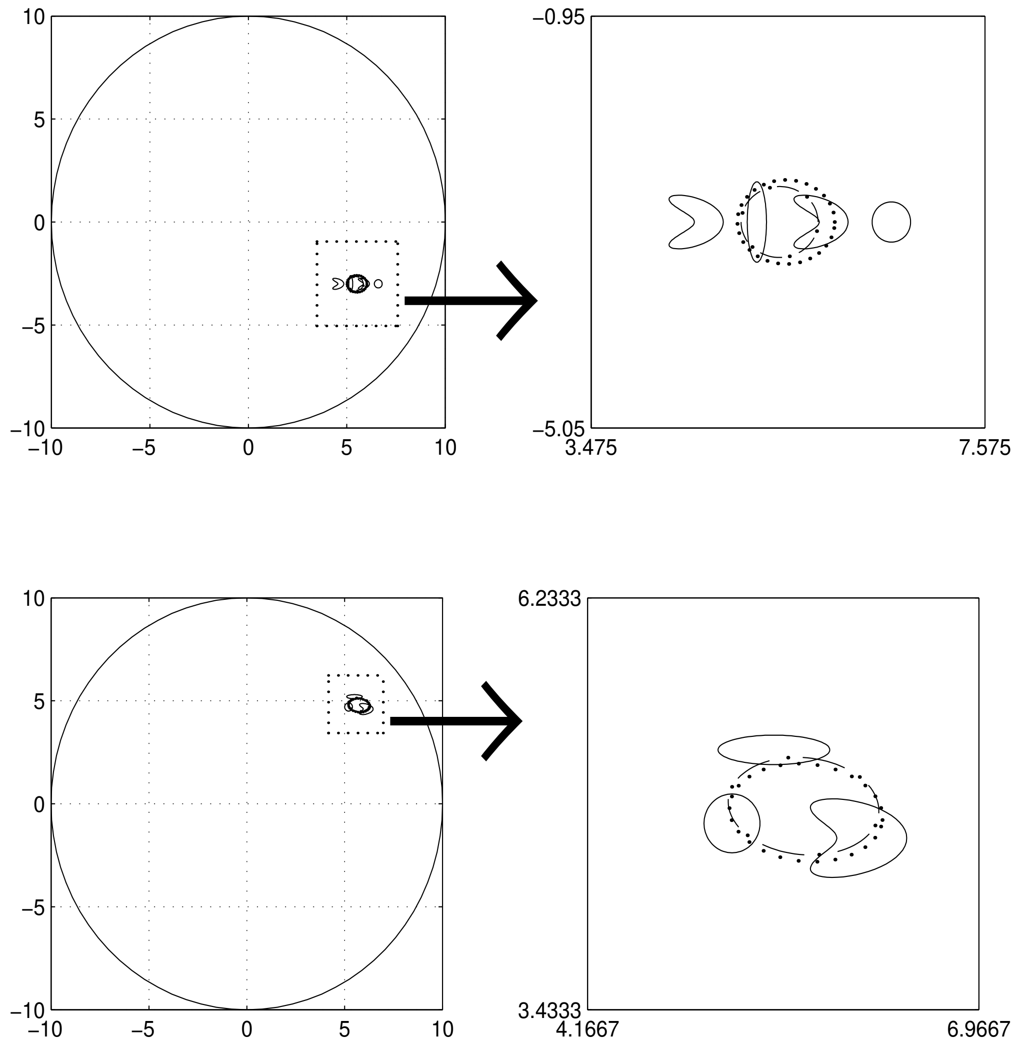
4.1167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−5
−10
−10
−5
0
5
10
6.2333
3.4333
4.1667
6.9667

Thank you.

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