Dispersion Formula of Rayleigh Waves

joint work with

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1. Introduction

- NDT(nondestructive testing) using ultrasound (i.e. elastic waves)
- Rayleigh wave(R-wave)
  - propagate along the free boundary
  - speed < speed of body waves (i.e. subsonic)
  - elliptically polarized

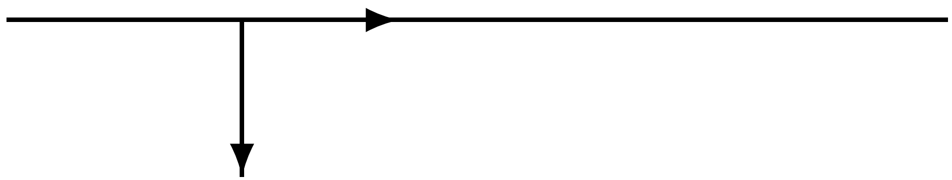
- Chi-Sing Man et al

  dispersion formula of the speed of R-wave

$$i.e. \quad v(k) = v_0 + \frac{1}{k} v_{-1} + \cdots$$

  under the conditions:

$$\begin{cases} \text{flat boundary} \\ \text{material is only depth dependent} \end{cases}$$



Assumptions:

- residual stress $\mathring{T}$ depends only on the depth z
- all components of the residual stress vanish except
$\mathring{T}_{11}(z) = \sigma_1(z), \quad \mathring{T}_{22}(z) = \sigma_2(z)$

Consider Rayleigh wave propagating along the surface in the 1-direction.

Constitutive Equation

$$
\begin{aligned}
\sigma &= \overset{\circ}{T} + C[E] + D[\overset{\circ}{T}, E] \\
&= \overset{\circ}{T} + \lambda(\text{tr } E)I + 2\mu E \\
&\quad +\beta_1(\text{tr } E)(\text{tr } \overset{\circ}{T})I + \beta_2(\text{tr } \overset{\circ}{T})E \\
&\quad +\beta_3((\text{tr } E)\overset{\circ}{T} + (\text{tr } E\overset{\circ}{T})I) + \beta_4(E\overset{\circ}{T} + TE)
\end{aligned}
$$

σ second Piola-Kirchhoff stress

$\overset{\circ}{T}$ residual stress

E infinitesimal strain tensor

I second-order identify tensor

C fourth-order elasticity tensor

D sixth-order acoustoelastic tensor bilinear in $\overset{\circ}{T}$ and E

High-Frequency Asymptotics

$$\begin{aligned}\frac{v - v_0}{v_0} &= K_{11}\sigma_1(0) + K_{12}\sigma_2(0) \\ &+ \frac{1}{k}(H_{11}\sigma_1'(0) + H_{12}\sigma_2'(0)) \\ &+ O\left(\frac{1}{k^2}\right)\end{aligned}$$

k	wave number
v	phase velocity of Rayleigh wave
v_0	phase velocity of Rayleigh wave when $\sigma_1 = \sigma_2 \equiv 0$
H_{ij}, K_{ij}	acoustoelastic constants
$\sigma_i'(0)$	derivative of σ_i at $z = 0$

- Cause of the dispersion of R-wave

$$\begin{cases} \text{curved boundary} \\ \text{inhomogeneity} \end{cases}$$

Question:

What about the cases

$$\left.\begin{array}{l}\cdot\ \text{curved boundary}\\ \cdot\ \text{inhomogeneous also in the tangential direction}\end{array}\right\}?$$

Aim of the study:

Want to derive the dispersion formula of R-wave in the general case.

- Known and related results

  (i) isotropic

  D.Gregory ('71)

  homogeneous, curved boundary

  (ii) anisotropic

  V.E. Nomofilov ('74)

  general case

  But its validity is unknown

  (iii) related result

  G. Nakamura ('91)

  necessary and sufficient condition for the existence of R-wave for the general case

  (i.e. generalized Barnett-Lothe condition)

2. Outline of deriving the formula

methods:

- semiclassical analysis

  pseudodifferential operator (ΨDO)

  Fourier integral operator (FIO)

- Stroh formalism
- stationary phase method

Main steps

- Poisson operator in the elliptic region
- factorization of the operator and Neumann operator
- block diagonalization of the Neumann operator

  (generalized Barnett-Lothe condition and extracting the hyperbolic part)
- stationary phase method

Set up

$\Omega \subset \mathbf{R}^3$; domain, $\partial\Omega$; C^∞

$\rho(x) \in C^\infty(\overline{\Omega})$; density

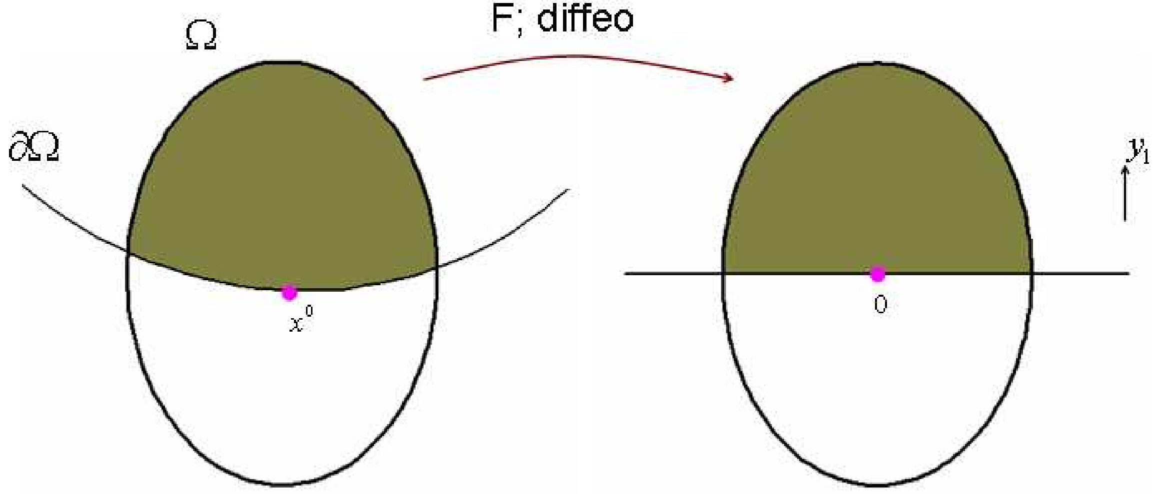
C=$(\mathrm{c}_{ijkl}(x)) \in C^\infty(\overline{\Omega})$; elasticity tensor

$$\begin{cases} \text{hyperelasticity (symmetry)} \\ \text{strong convexity} \end{cases}$$

Step 1

flattening the boundary

$x^0 \in \partial\Omega$; fix



Claim

$$\begin{cases} \rho {\partial_t}^2 u = div(C\nabla u) = \nabla \cdot (C\nabla u) \\ (C\nabla u)\nu = 0 \end{cases}$$

ν; outer unit normal vector

$$\Longrightarrow \begin{cases} \tilde{\rho} {\partial_t}^2 u = \nabla \cdot (B\nabla u) \\ (B\nabla u) \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = 0 \end{cases}$$

$\tilde{\rho} := \left|\text{det}\nabla F\right|^{-1}\rho$ etc.

B; hyperelastic, strongly convex

Notations

$\rho = \tilde{\rho}$

$D_t = -\sqrt{-1}\partial_t$, $D_{y_r} = -\sqrt{-1}\partial_{y_r}$

$y\prime = (y_2, y_3)$

$\eta_1 \longleftrightarrow y_1$, $\eta\prime = (\eta_2, \eta_3) \longleftrightarrow (y_2, y_3)$

$M = \rho {D_t}^2 + L$

$Lu = -D \cdot (BDu)$

(analogous to $\nabla \cdot (C\nabla u)$)

The rest of the arguments are microlocalized to $t = 0$, $y_1 = 0$, $y\prime = 0$, $\eta\prime = \eta^{0\prime}$

Goal

Speed of R-wave passing $y\prime = 0$ in the direction $\eta^{0\prime}$ at $t = 0$

$$= \lambda(0, \eta^{0\prime}) - \frac{1}{k}\mathrm{Re}\{\delta(y\prime, \lambda(0, \eta^{0\prime}), \eta^{0\prime})\},$$

where

$\delta = \delta(y\prime, \eta\prime, \tau)$; positive homog. deg. 0 part of the symbol of $D_{11}(k)$
$D_{11}(k)$; $(1, 1)$ block of the diagonalized symbol of Neumann operator

$\sigma(D_{11}(k)) = k(\tau - \lambda(y\prime, \eta\prime))$

Step 2

Construction of Poisson operator P(k):

$$(P(k)g)(t,y) = (\frac{k}{2\pi})^3 \iint e^{\sqrt{-1}k\{(t-s)\tau+(y\prime-z\prime)\cdot\eta\prime\}} P(t,y,\tau,\eta\prime,k)$$

$$g(s,z\prime,\tau,\eta\prime) ds dz\prime d\tau d\eta\prime$$

(ΨDO)

$$\begin{cases} M(P(k)g)(t,y) = O(k^{-\infty}) \quad \text{and } C^\infty \\ (P(k)g)|_{y_1=0} = g \end{cases}$$

$(t, y\prime, \tau, \eta\prime) \in \varepsilon$; elliptic region

i.e. det$(\rho\tau^2\delta_{ij} - \sum_{j,l=1}^{3} B_{ijkl}\eta_j\eta_l)|_{y_1=0} \neq 0 \quad (\eta_1 \in$ **R**$)$

Step 3

Computing the Neumann operator

By the factorization

$-M = (D_{y_1} - B^*(y, D_t, D_{y\prime}, k) + G_0(y, D_t, D_{y\prime}, k))T(y)(D_{y_1} - B(y, D_t, D_{y\prime}, k))$

and

the Stroh formalism,

the Neumann operator $N(k)$:

$$N(k)g := (B\nabla(P(k)g)) \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix}$$

has the principal part (i.e. principal symbol):

$$\sigma(N(k)) = k(|\tau| + |\eta\prime|)Z(y\prime, \tau, \eta\prime)$$

$Z(y\prime, \tau, \eta\prime)$; surface impedance tensor

$$Z(y\prime,\tau,\eta\prime) := -V^{-1}(y\prime,\tau,\eta\prime) - \sqrt{-1}V^{-1}(y\prime,\tau,\eta\prime)Y(y\prime,\tau,\eta\prime)$$

$$V(y\prime,\tau,\eta\prime) := -(2\pi)^{-1}\int_0^{2\pi} <\omega,\omega>^{-1}\, d\phi$$

$$Y(y\prime,\tau,\eta\prime) := -(2\pi)^{-1}\int_0^{2\pi} <\omega,\omega>^{-1}<\omega,\zeta>\, d\phi$$

$$<\zeta,\omega>=(<\zeta,\omega>_{ik})$$

$$<\zeta,\omega>_{ik}:=\sum_{j,l=1}^{3}(B_{ijkl}-|\eta\prime|^{-4}\rho\tau^2\eta_j\eta_l\delta_{ik})\zeta_j\omega_l$$

$$\zeta=(\zeta_1,\zeta_2,\zeta_3)=(\sin\phi,|\eta\prime|^{-1}(\cos\phi)\eta\prime)$$

$$\omega=(\omega_1,\omega_2,\omega_3)=(\cos\phi,-|\eta\prime|^{-1}(\sin\phi)\eta\prime)$$

Step 4

Extracting the hyperbolic part from the Neumann operator

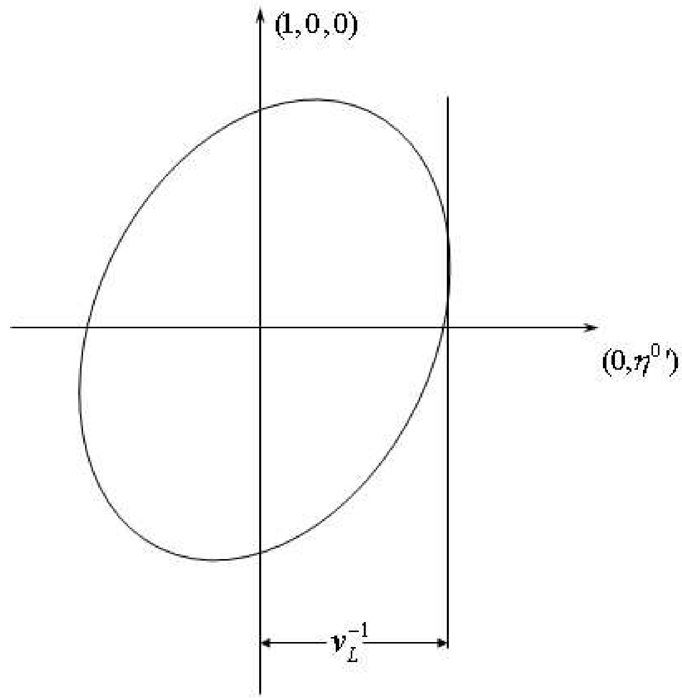
Assume the generalized Barnett-Lotte condition at $(y\prime, \eta\prime) = (0, \eta^{0\prime})$:

$$\lim_{v \uparrow v_L(0,\eta^{0\prime})} Z(0, v|\eta^{0\prime}|, \eta^{0\prime}) < 0$$

or

$$\lim_{v \uparrow v_L(0,\eta^{0\prime})} \{(\text{trace} Z(0, v|\eta^{0\prime}|, \eta^{0\prime}))^2 - \text{trace} Z^2(0, v|\eta^{0\prime}|, \eta^{0\prime})\} < 0$$

$v_L(0, \eta^{0\prime})$; limiting velocity



outermost slowness curve

By Nakamura's result, there exists an intertwining ΨDO $\Psi(k)$

$$N(k)\Psi(k) \sim \Psi(k)D(k)$$

$D(k) = \begin{bmatrix} D_{11}(k) & 0 \\ 0 & D_{22}(k) \end{bmatrix}$; block diagonal ΨDO

$\sigma(D_{11}(k)) = k(\tau - \lambda(y\prime, \eta\prime))$,

$\lambda(y\prime, \eta\prime)$; positive homog. deg. 1, C^∞, real valued

$\left.\begin{array}{l} \sigma(D_{22}(k)) \\ \sigma(\Psi(k)) \end{array}\right\}$; positive homog. deg. $\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}$, C^∞, invertible

Moreover, there exists an intertwining ΨDO $q(k)$:

$D_{11}(k)q(k) \sim q(k)O_p(\sigma(D_{11}(k)))$

$O_p(\sigma(D_{11}(k)))$; the principal part of $D_{11}(k)$

$(-iH_{\tau-\lambda} + \delta)\sigma(q(k)) = 0$

$H_{\tau-\lambda} = \partial_t - \nabla_{\eta\prime}\lambda \cdot \nabla_{y\prime} + \nabla_{y\prime}\lambda \cdot \nabla_{\eta\prime}$; Hamilton field of $\tau - \lambda$

δ; positive homog. deg. 0 part of the symbol of $D_{11}(k)$

Note that, symbolically

$\sigma(q(k)) = e^{-iH_{\tau-\lambda}^{-1}\delta}$

Step 5

Solving $O_p(\sigma(D_{11}(k)))h = 0$ for h

h can be given by FIO $a_\phi f$:

$$(a_\phi f)(t, y\prime, k) = (\frac{k}{2\pi})^2 \iint e^{\sqrt{-1}k(\phi(t,y\prime,\eta\prime)-z\prime\cdot\eta\prime)} a(t, y\prime, z\prime, \eta\prime) f(z\prime, k) dz\prime d\eta\prime$$

$$\begin{cases} \partial_t \phi - \lambda(y, \nabla_{y\prime}\phi) = 0 & \text{Hamilton-Jacobi eq.} \\ \phi|_{t=0} = y\prime \cdot \eta\prime \end{cases}$$

$$D_t\sigma(a) - \nabla_{\eta\prime}\lambda(y\prime, \nabla_{y\prime}\phi) D_{y\prime}\sigma(a) + im\sigma(a) = 0$$

$$m = \frac{1}{2} \sum_{|\alpha|=2} \partial^\alpha_{\eta\prime} \lambda(y\prime, \nabla_{y\prime}\phi) \partial^\alpha_{y\prime}\phi$$

Step 6

Summing up Step 2$\sim$ Step 5, the solution u describing R-wave is given by:

$$u = P(k)\Phi(k)\begin{bmatrix} q(k)a_\phi f \\ 0 \\ 0 \end{bmatrix}$$

N.B. $\begin{cases} N(k)\Phi(k) \sim \Phi(k)D(k) \\ D_{11}(k)q(k) \sim q(k)O_p(\sigma(D_{11}(k))) \\ O_p(\sigma(D_{11}(k)))\, a_\phi f = 0 \end{cases}$

Step 7

Analyze the asymptotic of $u|_{y_1=0}$ using the stationary phase method

Let the initial source f of R-wave be

$$f = e^{iky\prime\cdot\eta^{0\prime}}\tilde{f}(y\prime)$$

$\tilde{f}(y\prime)$; supported near $y\prime = 0$

Then, the leading term of $u|_{y_1=0}$ is

$$\exp\left[ik(\phi(t,y\prime,\eta^{0\prime}) - k^{-1}\int_0^t \delta(H_{s,\tau}(z\prime,\zeta\prime),\tau)ds)\Big|_{\substack{(z\prime,\zeta\prime)=H^{-1}_{t,\tau}(y\prime,\eta^{0\prime}) \\ \tau=\phi_t(t,y\prime,\eta\prime^0)}}\right]$$

$$\exp\left(\int_0^t m(s,V_{s,\eta^{0\prime}}(z\prime),\eta^{0\prime})ds\right)\Big|_{z\prime=V^{-1}_{s,\eta^{0\prime}}(y\prime)}$$

$$\tilde{f}(\nabla_{\eta\prime}\phi(t,y\prime,\eta\prime^0))$$

$$\sigma(\Phi(k))_{[1]}(t,y\prime,\phi_t(t,y\prime,\eta^{0\prime}),\nabla_{y\prime}\phi(t,y\prime,\eta^{0\prime}))$$

where $\sigma(\Phi(k))_{[1]}$; 1 st column vector of $\sigma(\Phi(k))$ (polarization vector)

$$m = \frac{1}{2}\sum_{|\alpha|=2}\partial^\alpha_{\eta\prime}\lambda(y\prime,\nabla_{y\prime}\phi)\partial^\alpha_{y\prime}\phi$$

$$H_{t,\tau} : (z\prime, \zeta\prime) \longmapsto (y\prime, \eta\prime)$$

$$\begin{cases} \dfrac{dy\prime}{dt} = -\nabla_{\eta\prime}\lambda, & \dfrac{d\eta\prime}{dt} = \nabla_{y\prime}\lambda \\ (y\prime, \eta\prime)|_{t=0} = (z\prime, \zeta\prime) \end{cases}$$

$$V_{t,\eta^{0\prime}} : z\prime \longmapsto y\prime$$

$$\begin{cases} \dfrac{dy\prime}{dt} = -\nabla_{\eta\prime}\lambda(y\prime, \nabla_{y\prime}\phi(t, y\prime, \eta^{0\prime})) \\ y\prime|_{t=0} = z\prime \end{cases}$$

Speed of R-wave passing $y\prime = 0$ in the direction $\eta^{0\prime}$ at $t = 0$

$$= \lambda(0, \eta^{0\prime}) - \frac{1}{k}\text{Re}\{\delta(y\prime, \lambda(0, \eta^{0\prime}), \eta^{0\prime})\} \, ,$$

where $\delta = \delta(y\prime, \eta\prime, \tau)$; positive homog. deg. 0 part of the symbol of

$$D_{11}(k)$$

Conclusion and future problems

- We derived the dispersion formula in the coord.

  flattening the boundary
- Have to separate the geometric and material quantities.
- Apply it to identify residual stress or texture.