Wavelet Galerkin Algorithms for Option Pricing in Lévy models

"hypersingular integral equations in computational finance"

Christoph Schwab

RiskLab and Seminar for Applied Mathematics ETH Zürich

joint with Ana-Maria Matache

J.C. Nédélec's 60th birthday, Paris, June 2003

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Outline

- Parabolic PIDE for Pricing European Vanillas
- Examples of Lévy processes: Variance Gama (VG) and CGMY
- Variational formulation
- Localization/Truncation
- Galerkin Discretization: Wavelet basis/Matrix compression/Preconditioning
- American options: Pricing problem as optimal stopping problem
- Formulation as parabolic integro-differential inequality/Variational formulation
- Truncated problem for the excess to pay-off
- Galerkin discretization/Wavelet preconditioning of the sequence of LCPs

Lévy processes

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbf{P})$: filtered probability space under the usual hypothesis.
- $X = (X_t)_{0 \leq t < \infty}$ adapted with $X_0 = 0$ a.s. is a **Lévy process** iff
  1. X has **increments independent** of the past, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$
  2. X has **stationary increments**, i.e. $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$
  3. X_t is **continuous in probability**.

Lévy-Khintchine formula

X_t Lévy process ➔ $E_{\mathbb{Q}}[e^{-iuX_t}] = e^{-t\psi(u)}$

$$\psi(u) = \frac{\sigma^2}{2}u^2 + i\alpha u + \int_{|x|<1}(1 - e^{-iux} - iux)\nu_{\mathbb{Q}}(dx) + \int_{|x|\geq 1}(1 - e^{-iux})\nu_{\mathbb{Q}}(dx)$$

for $\sigma, \alpha \in \mathbb{R}$ and for a measure $\nu_{\mathbb{Q}}$ on $\mathbb{R}\backslash\{0\}$ satisfying $\int \min(1, x^2)\nu_{\mathbb{Q}}(dx) < \infty.$

- $\nu_{\mathbb{Q}}$ is called the Lévy measure of X.
- Let $\mu_X(dx, dt)$ denote the integer valued random measure (the *jump measure*) that counts the number of jumps in space-time ➔ $\nu_{\mathbb{Q}}(dx)dt$ is the compensator of $\mu_X(dx, dt)$.
- The Lévy-Khintchine formula is intimately connected to the structure of the process X:

$$X_t = \sigma W_t + Y_t$$

Pricing contingent claims on Lévy driven assets

- **Stock price process:**

$$e^{-rt}S_t = S_0 e^{ct+X_t}, \quad X_t \ \text{ Lévy process}, \quad e^{-ct} = E_{\mathbb{Q}}[e^{X_t}]$$

- **European (call) Option price:**

$$f = f(t, S_t) = E_{\mathbb{Q}}[e^{-r(T-t)}h(S_T)|\mathcal{F}_t]$$

PIDE for Pricing European Vanillas

- **European (call) Option price:**

$$f = f(t, S_t) = E_{\mathbb{Q}}[e^{-r(T-t)} h(S_T) | \mathcal{F}_t]$$

has to satisfy the following **Partial Integro-Differential Equation (PIDE)**

$$\frac{\partial f}{\partial t}(t, S_{t\_}) + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial S^2}(t, S_{t\_}) S_{t\_}^2 + r S_{t\_} \frac{\partial f}{\partial S}(t, S_{t\_}) - r f(t, S_{t\_})$$

$$+ \int_{\mathbb{R}} [f(t, S_{t\_} e^x) - f(t, S_{t\_}) - \frac{\partial f}{\partial S}(t, S_{t\_}) S_{t\_}(e^x - 1)] \nu_{\mathbb{Q}}(dx) = 0$$

$$f(T, S_T) = h(S_T) := (S_T - K)_+$$

The PIDE for Pricing European Options

- assume that the Lévy measure $\nu_{\mathbb{Q}}(dx)$ has the Lévy density $k_{\mathbb{Q}}(x)$: $\nu_{\mathbb{Q}}(dx) = k_{\mathbb{Q}}(x)dx$
- change the variables $x = \log(S)$ (logarithmic price) and $\tau = T - t$ (time to maturity)

We obtain for $u(\tau, x) = f(S, t)$

$$\frac{\partial u}{\partial \tau} \underbrace{- \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial u}{\partial x} - ru}_{A_{\mathrm{Black-Scholes}}} + A_{\mathrm{jump}}[u] = 0, \qquad x \in \mathbb{R}$$

$$u|_{\tau=0} = g := (e^x - K)^+$$

$$A_{\mathrm{jump}}[\varphi](x) := \int_{\mathbb{R}} \{\varphi(x+y) - \varphi(x) - y\frac{d\varphi}{dx}(x)\chi_{\{|y|\le 1\}}(y)\}k(y)\,dy + c_{\exp}\frac{d\varphi}{dx}(x)$$

The CGMY model
P. Carr, H. Geman, D.B. Madan, M. Yor 2000

- CGMY process generalizes the VG (variance gamma) process by adding a parameter permitting finite or infinite activity and finite or infinite variation.

- CGMY Lévy density is given by

$$k_{CGMY}(x) = \begin{cases} C\dfrac{e^{-G|x|}}{|x|^{1+Y}} & \text{if } x < 0 \\ C\dfrac{e^{-M|x|}}{|x|^{1+Y}} & \text{if } x > 0, \end{cases}$$

$C > 0$, $G, M \geq 0$ and $Y < 2$. $Y = 0 =$ VG process.

Pricing methodologies

$$\frac{\partial u}{\partial \tau} + \underbrace{A_{\text{Black-Scholes}}[u] + A_{\text{jump}}[u]}_{\mathcal{A}[u]} = 0, \qquad x \in \mathbb{R}$$

$$u|_{\tau=0} = g := (e^x - K)^+$$

- ✍ **well-posedness in spaces of possibly exponentially growing solutions**; give a suitable variational formulation
- ✍ **truncation of the PIDE to a bounded domain** (an essential step for numerical simulation as well as for modeling certain types of contracts)
- ✍ due to the jump part of the Lévy process, this localization cannot be effected by simple restriction to the bounded domain plus suitable boundary conditions! but must also take into account information from beyond the computational domain

Variational framework; r = 0

- If $r \neq 0$ the transformation $u(\tau, x) = e^{-r\tau}\tilde{u}(\tau, x + r\tau)$ leads to a parabolic problem for $\tilde{u}$ with $r = 0$.
- Let k_{CGMY} denote the Lévy density of the CGMY-model ($Y \in \mathbb{R}$, $Y < 2$)

$$\frac{\partial u}{\partial \tau}(\tau, x) + \underbrace{A_{\text{Black-Scholes}}[u(\tau,\cdot)](x) + A_{\text{jump}}[u(\tau,\cdot)](x)}_{\mathcal{A}[u(\tau,\cdot)](x)} = 0$$

$$u(0, x) = g(x) := (e^x - K)^+$$

$$A_{\text{jump}}[\varphi](x) := \int_{\mathbb{R}} \left\{ \varphi(x + y) - \varphi(x) - y\frac{d\varphi}{dx}(x)\chi_{\{|y| \le 1\}}(y) \right\} k_{CGMY}(y)\, dy + c_{\text{exp}}\frac{d\varphi}{dx}(x)$$

Variational framework; r = 0

- **Weighted Sobolev spaces.** Let $\eta \in L^1_{\text{loc}}(\mathbb{R})$, $\eta' \in L^\infty(\mathbb{R})$.

$$H^1_\eta(\mathbb{R}) := \{\varphi \in L^1_{\text{loc}}(\mathbb{R}) \ : \ e^\eta \varphi, e^\eta \varphi' \in L^2(\mathbb{R})\}.$$

- We observe that $g \in H^1_{-\zeta}(\mathbb{R})$ for all ζ of the form

$$\zeta(x) = \begin{cases} \mu_1 |x| & \text{if } x < 0 \\ \mu_2 |x| & \text{if } x > 0 \end{cases}$$

for all $\mu_1 > 0$ and $\mu_2 > 1$.

Variational framework

Theorem. Let

$$a^{\eta}(\varphi,\psi) := \int_{\mathbb{R}} \mathcal{A}[\varphi](x)\psi(x)e^{2\eta(x)}dx.$$

- ☞ Assume that $\int_{\mathbb{R}} e^{\eta(y)}|y|\chi_{\{|y|\geq 1\}}(y)k_{CGMY}(y)\, dy < +\infty$. Then, $\exists\ \alpha_\eta, \beta_\eta, C_\eta > 0$ s.t.

$$|a^{-\eta}(\varphi,\psi)| \leq C_\eta \|\varphi\|_{H^1_{-\eta}(\mathbb{R})}\|\psi\|_{H^1_{-\eta}(\mathbb{R})} \quad \forall\, \varphi,\psi \in H^1_{-\eta}(\mathbb{R})$$

$$a^{-\eta}(\varphi,\varphi) \geq \alpha_\eta \|\varphi\|^2_{H^1_{-\eta}(\mathbb{R})} - \beta_\eta \|\varphi\|^2_{L^2_{-\eta}(\mathbb{R})} \quad \forall\, \varphi \in H^1_{-\eta}(\mathbb{R}).$$

- ☞ Assume that $\int_{\mathbb{R}} e^{\eta(-y)}|y|\chi_{\{|y|\geq 1\}}(y)k_{CGMY}(y)dy < +\infty$. Then, $\exists\ \alpha'_\eta, \beta'_\eta, C'_\eta > 0$ s.t.

$$|a^{\eta}(\varphi,\psi)| \leq C'_\eta \|\varphi\|_{H^1_{\eta}(\mathbb{R})}\|\psi\|_{H^1_{\eta}(\mathbb{R})} \quad \forall\, \varphi,\psi \in H^1_{\eta}(\mathbb{R})$$

$$a^{\eta}(\varphi,\varphi) \geq \alpha'_\eta \|\varphi\|^2_{H^1_{\eta}(\mathbb{R})} - \beta'_\eta \|\varphi\|^2_{L^2_{\eta}(\mathbb{R})} \quad \forall\, \varphi \in H^1_{\eta}(\mathbb{R}).$$

Variational framework

➔. . . ➔

There exists a unique $u \in L^2((0,T); H^1_{-\zeta}(\mathbb{R})) \cap H^1((0,T); (H^1_{-\zeta}(\mathbb{R}))^*)$ s.t.

$$\frac{d}{d\tau}(u(\tau,\cdot), v)_{L^2_{-\zeta}(\mathbb{R})} + a^{-\zeta}(u(\tau,\cdot), v) = 0 \quad \forall\, v \in H^1_{-\zeta}(\mathbb{R})$$

$$u(0,\cdot) = g \in H^1_{-\zeta}(\mathbb{R}).$$

Excess to pay-off $U = u - g$

Theorem.

Let $r = 0$.

Let $H^1_\eta(\mathbb{R}) := \{v \mid e^\eta v, e^\eta \frac{dv}{dx} \in L^2(\mathbb{R})\}$.

Let $\mathcal{A} = A_{\text{Black-Scholes}} + A_{\text{jump}}$.

$$\rightarrow \quad \mathcal{A}[g] \in (H^1_\eta(\mathbb{R}))^*$$

for all η such that $\int_{\mathbb{R}} e^{\eta(-y)} |y| \chi_{\{|y| \geq 1\}}(y) k_{CGMY}(y) dy < +\infty$ (in particular for $\eta = 0$).

$\rightarrow$ $U = u - g : (0, T) \mapsto H^1_\eta(\mathbb{R})$ solves

$$\frac{\partial U}{\partial \tau}(\tau, x) + \mathcal{A}[U(\tau, \cdot)](x) = -\mathcal{A}[g](x) \in (H^1_\eta(\mathbb{R}))^*$$

$$U(0, x) = 0$$

Localization to a bounded domain

Solve in $(0,T)\times(-R,R)$:

$$\frac{\partial U_R}{\partial \tau} + \mathcal{A}_R[U_R] = -\mathcal{A}[g]|_{(-R,R)} \quad \text{in} \quad (0,T)\times(-R,R)$$
$$U_R(\tau,\cdot)|_{\partial\Omega_R} = 0 \quad \text{on} \quad \partial\Omega_R \quad \forall\, 0<\tau<T$$
$$U_R|_{\tau=0} = 0 \quad \text{in} \quad (-R,R),$$

with $\mathcal{A}_R$ being the restriction of $\mathcal{A}$ to $H_0^1(-R,R)$.

Theorem. For $T>0$ fixed, there exist constants $b>0$ and $C>0$ independent of R, and $R_0 = R_0(T) > 0$ sufficiently large such that for all $R > R_0$ it holds

$$\|(U-U_R)(\tau,\cdot)\|_{L^2(\mathbb{R})} \le Ce^{-bR}, \quad 0\le\tau\le T.$$

Localization to a bounded domain

- $U(\tau,\cdot) \in H^1_\eta(\mathbb{R})$ ➔ $U \to 0$ at $\pm\infty$ exponentially fast ➔ fix $R > 0$ sufficiently large
- Find v ($\approx u - g$) : $(0,T) \mapsto H^1_0(-R,R)$ s.t.

$$\frac{\partial v}{\partial \tau}(\tau,x) + \mathcal{A}_R[v(\tau,\cdot)](x) = -\mathcal{A}[g] \quad \text{in} \quad (0,T)\times(-R,R)$$
$$v(0,x) = 0$$

$$\frac{d}{d\tau}(v,w) + a(v,w) = (\underbrace{F}_{-\mathcal{A}[g]}, w) \quad \forall\, w \in H^1_0(-R,R).$$
$$v(0,x) = 0$$
$$a(\varphi,\psi) := \langle \mathcal{A}[\varphi], \psi\rangle_{(H^1_0(-R,R))^* \times H^1_0(-R,R)}$$

Discretization

Discretization in $x = \ln(S) \in (-R, R)$.

Let $V_N \subset H_0^1(-R, R)$, $V_N = \text{Span}\,\{\Phi_i\}_{i=1}^N$

The price *semi-discrete* problem reads: Find $v_N : (0, T) \mapsto V_N$ such that

$$\frac{d}{d\tau}(v_N, w_N) + a(v_N, w_N) = (F, w_N) \quad \forall\, w_N \in V_N.$$

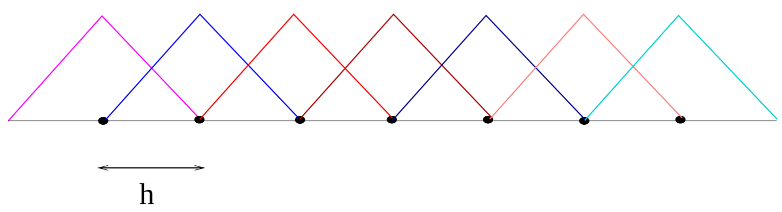
The semi-discrete problem is an initial value problem for $N = \dim(V_N)$ ODEs

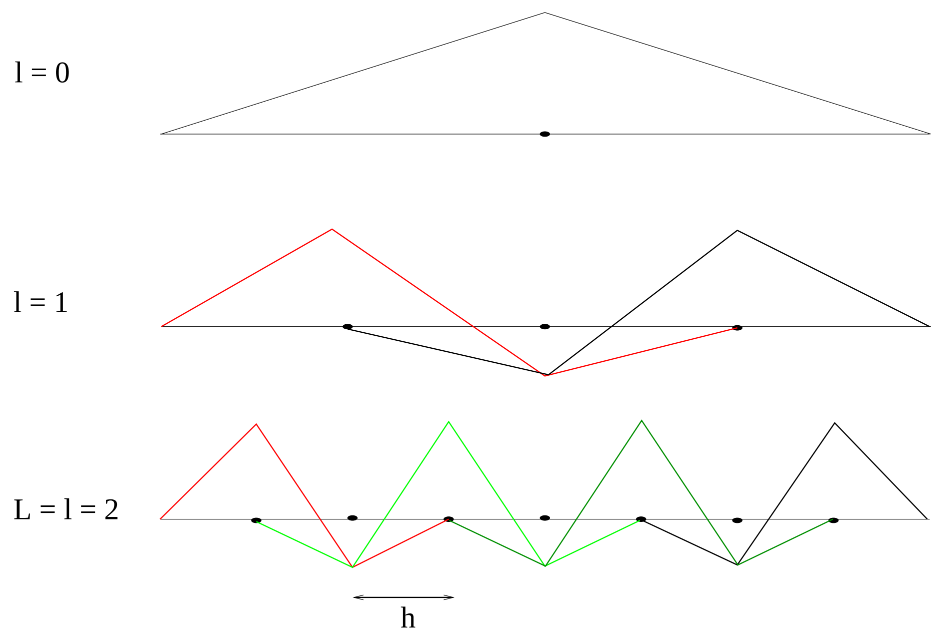
$$\mathbf{M}\frac{d}{d\tau}\underline{v}_N + \mathbf{A}\underline{v}_N = \underline{F}, \quad \underline{v}_N(0) = 0.$$

$$M_{ij} = (\Phi_j, \Phi_i)_{L^2(-R,R)},\ A_{ij} = a(\Phi_j, \Phi_i) = (\mathcal{A}[\Phi_j], \Phi_i),\ F_i = (F, \Phi_i)$$

Finite Element Spaces V_N

$V_N = V_h$ space of continuous, piecewise linears with respect to a mesh of width h on $(-R, R)$.





Time Discretization

Time discretization using θ scheme.

$k := T/M$ **time-step**, $t^m = mk$, $m = 0, \ldots, M$.

For $m = 0, \ldots, M$ find $v_N^{m+1} \in V_N (\approx v_N(t^{m+1}))$ s.t.

$$\left(\frac{v_N^{m+1} - v_N^m}{k}, w_N\right) + a(\underbrace{v_N^{m+\theta}}_{\theta v_N^{m+1} + (1-\theta) v_N^m}, w_N) = (F, w_N) \; \forall w_N \in V_N$$

$$(\mathbf{M} + k\theta\mathbf{A})\underline{v}_N^{m+1} = (\mathbf{M} - k(1-\theta)\mathbf{A})\underline{v}_N^m + k\underline{F}, \quad m = 0, \ldots, M-1.$$

➔ a linear system has to be solved for each time-step.

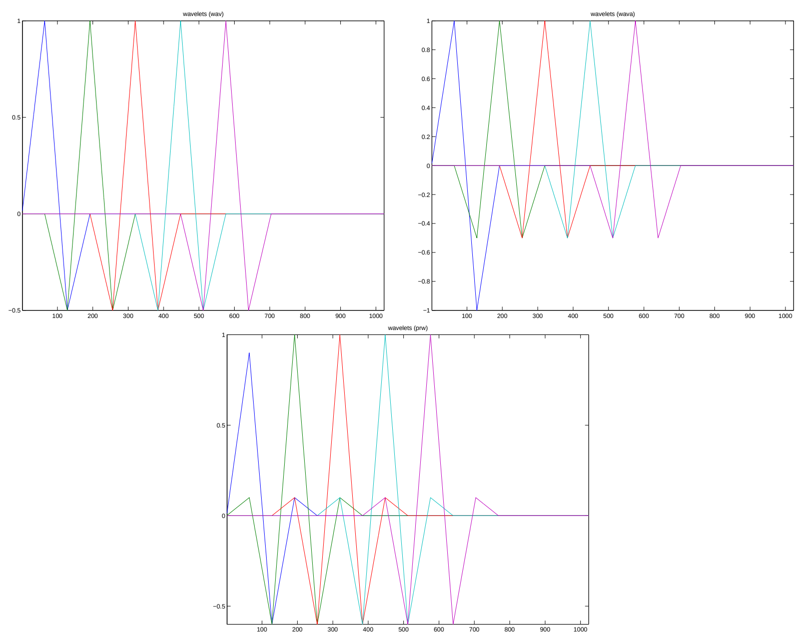
If the operator $\mathcal{A}$ is non-local standard FEM lead to **dense** matrices $\mathbf{A}$ and hence at least $O(N^2)$ complexity per time-step.

Wavelets

$V_N = V_h$ spaces of continuous p.w. linear functions on meshes $\{\mathcal{T}_h\}_h$ of Ω

$$V^0 \subset V^1 \subset \cdots \subset V^L = V_h, \quad N = N_L = \dim V_h = C2^L$$

hierarchical basis $\quad V^L = \text{Span}\,\{\psi_j^l \mid 0 \leq l \leq L, 1 \leq j \leq M^l\}$



Wavelet norm equivalences

Let $N = 2^L$ and $\{\psi_j^l\}_{\substack{l=0,\dots,L \\ j=0,\dots,M^l}}$ denote the wavelet basis.

$\forall\, v \in V_N :$ $\quad v(x) = \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_{j,l}\psi_j^l(x)$ with $v_{j,l} = (v, \tilde{\psi}_j^l)$

$\forall\, v \in V = H_0^1(-R,R) :$ $\quad v(x) = \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} v_{j,l}\psi_j^l(x)$ with $v_{j,l} = (v, \tilde{\psi}_j^l)$

norm equivalence

$$\forall v \in \tilde{H}^{Y/2}(-R,R) : \quad c_1 \|v\|^2_{\tilde{H}^{Y/2}} \leq \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} |v_l^j|^2 2^{lY} \leq c_2 \|v\|^2_{\tilde{H}^{Y/2}}, \quad 0 \leq Y \leq 2$$

$$\tilde{H}^{Y/2}(-R,R) := \{v|_{(-R,R)} \mid v \in H^{Y/2}(\mathbb{R}),\ v|_{\mathbb{R}\setminus(-R,R)} = 0\} \ \text{ for } \ 0 \leq Y \leq 2$$

Wavelet compression

Let $Y \in [0,2]$ and $\Omega \subset \mathbb{R}$

$\mathcal{A} \in \Psi^Y(\Omega)$ (pseudo-differential operator of order Y)

$\mathcal{A} : V \to V^*$, with $V := \tilde{H}^{Y/2}(\Omega)$

Schwartz kernel theorem ➔

$$a(u,v) := (\mathcal{A}[u], v)_{V^* \times V} = \langle k, v(x) \otimes u(y) \rangle \quad \text{with} \;\; k \in \mathcal{D}'(\Omega \times \Omega)$$

$k(x,y) \in C^\infty(\Omega \times (\mathbb{R} \backslash \{y = 0\}))$ satisfies **Calderon-Zygmund type estimates**

$$|\partial_y^\alpha k(x,y)| \leq C(\alpha)|y|^{-1+Y+|\alpha|}$$

uniform with respect to x.

Matrix compression

Stiffness matrix $\mathbf{A}$: $A_{(j,l),(j',l')} := a(\psi_j^l, \psi_{j'}^{l'})$

Compressed stiffness matrix $\tilde{\mathbf{A}}$:

$$\tilde{A}_{(j,l),(j',l')} := \begin{cases} A_{(j,l),(j',l')} & \text{if dist}\,(\text{supp}\,(\psi_j^l), \text{supp}\,(\psi_{j'}^{l'})) < \delta_{l,l'} \\ 0 & \text{otherwise} \end{cases}$$

with

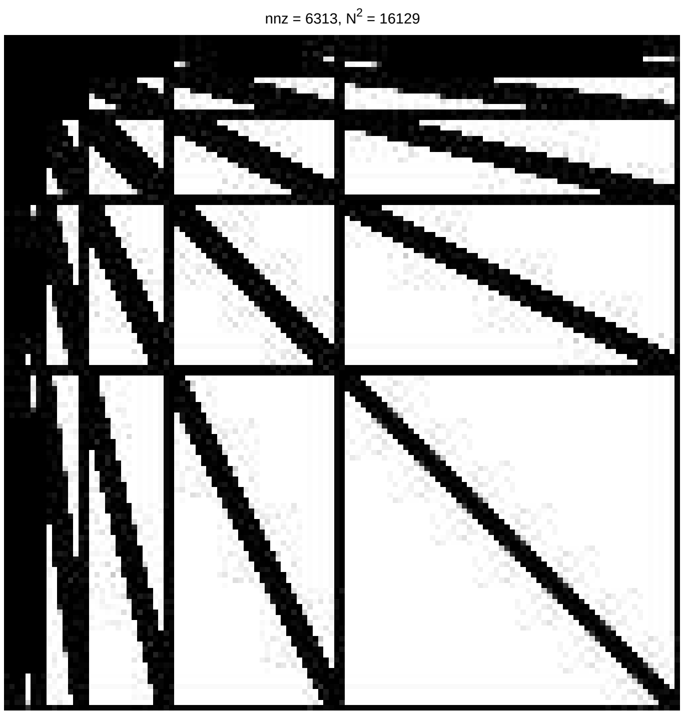
$$\delta_{l,l'} := c \max\{2^{-L+\hat{\alpha}(2L-l-l')}, 2^{-l}, 2^{-l'}\}$$

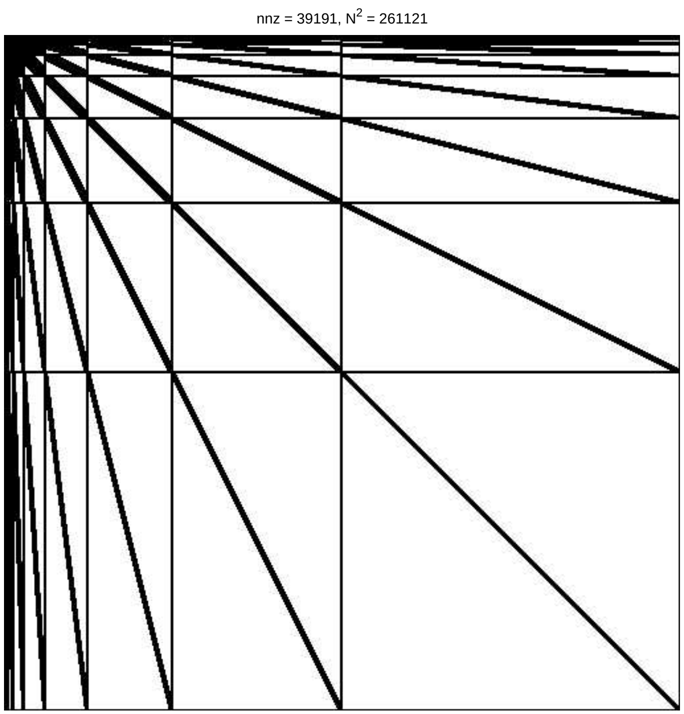
$\hat{\alpha} < 1$ ➔ $\tilde{\mathbf{A}}$ has $O(N \log(N))$ non-zero entries

Wavelet compression

Y. Meyer (1990), G.Beylkin, R. Coifman, V.Rokhlin (1992), T. von Petersdorff, C. Schwab (1996), R. Schneider (1998)

➔ reduce $O(N^2)$ complexity to $O(N \ln(N))$ by a *wavelet*-based matrix compression.





Consistency

Denote by $\tilde{a}(\cdot,\cdot)$ the **perturbed** bilinear form.

Proposition.

Assume $c > 0$ sufficiently large and $1 \geq \hat{\alpha} \geq \dfrac{2p+2}{2p+2+Y}$.

Then $\forall\, u \in H^{p+1}(\Omega) \cap \tilde{H}^{Y/2}(\Omega)$ and $\forall\, v \in V_N$ it holds

$$\boxed{|a(P_L u, v) - \tilde{a}(P_L u, v)| \leq C h^{p+1-Y/2} |\log(h)|^{\nu} \|u\|_{H^{p+1}(\Omega)} \|v\|_{\tilde{H}^{Y/2}(\Omega)}}$$

Convergence

For $v_N \in V_N$ and $f \in V_N^*$ define

$$\|v_N\|_{\tilde{a}} := (\tilde{a}(v_N, v_N))^{1/2}, \qquad \|f\|_{\tilde{*}} := \sup_{v_N \in V_N} \frac{(f, v_N)}{\|v_N\|_{\tilde{a}}}, \qquad \lambda_{\tilde{A}} := \sup_{v_N \in V_N} \frac{\|v_N\|^2}{\|v_N\|_{\tilde{*}}^2}.$$

Proposition. (Stability) For $0 \leq \theta < \frac{1}{2}$ assume the time-step restriction

$$\sigma := k(1 - 2\theta)\lambda_{\tilde{A}} < 2$$

Then the sequence $\{\tilde{v}_N^m\}_{m=0}^M$ satisfies the **stability** estimate

$$\|\tilde{v}_N^M\|^2 + C_1 k \sum_{m=0}^{M-1} \|\tilde{v}_N^{m+\theta}\|_{\tilde{a}}^2 \leq C_2 T \|f\|_{\tilde{*}}^2.$$

Proposition. (Convergence)

$$\left\|v^M - \tilde{v}_N^M\right\|^2 + k \sum_{m=0}^{M-1} \|v^{m+\theta} - \tilde{v}_N^{m+\theta}\|_a^2 \leq C\{h^{2p} |\log h|^{2\nu} + k^{2\mu}\},$$

where $C > 0$ depends on R, $\mu = 1$ if $\theta \neq \frac{1}{2}$ and $\mu = 2$ else.

Complexity

☞ **wavelet compression** $O(N^2)$ entries ➔$O(N\ln(N))$ entries

☞ **stability** for $\theta \in [1/2, 1]$ the θ-scheme is unconditionally stable, for $\theta \in [0, 1/2)$ the θ-scheme is stable if the CFL condition holds

$$k \leq C\frac{1}{1-2\theta}\begin{cases} h^2 & \text{if } \sigma > 0 \\ h^Y & \text{if } \sigma = 0. \end{cases}$$

☞ 'cond$(\mathbf{K} + k\theta\mathbf{A}) = O(h^{-2})$' $(\sigma > 0)$

☞ **wavelet preconditioning**

Let $\mathbf{B} := \mathbf{K} + k\theta\mathbf{A}$ and let $\widehat{B} := \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-1}$ with

$$D_{(j,l),(j,l)} = \begin{cases} \operatorname{diag}\left(1 + k\theta 2^{l}\right) & \text{if } \sigma > 0 \\ \operatorname{diag}\left(1 + k\theta 2^{lY/2}\right) & \text{if } \sigma = 0 \end{cases}$$

Then $\lambda_{\min}(\widehat{B} + \widehat{B}^{\top})/2 \geq C_1$ and $\|\widehat{B}\|_2 \leq C_2$ with C_1, C_2 independent of k, h

Complexity

- ☞ **linear complexity**, incomplete GMRES at each time step, $n_{GMRES} = O(\ln N)$ iterations
- ☞ total work for M time steps is $O(MN(\ln(N))^2)$
- ☞ $\Delta t = O(\Delta x)$ ➔$O(N^2(\ln(N))^2)$
- ☞ Can we get $O(N\log(N))$ complexity?
- ☞ $E(t) : u(0) \mapsto u(t)$ is **analytic semi-group** ➔ high order time discretization

Numerical Results

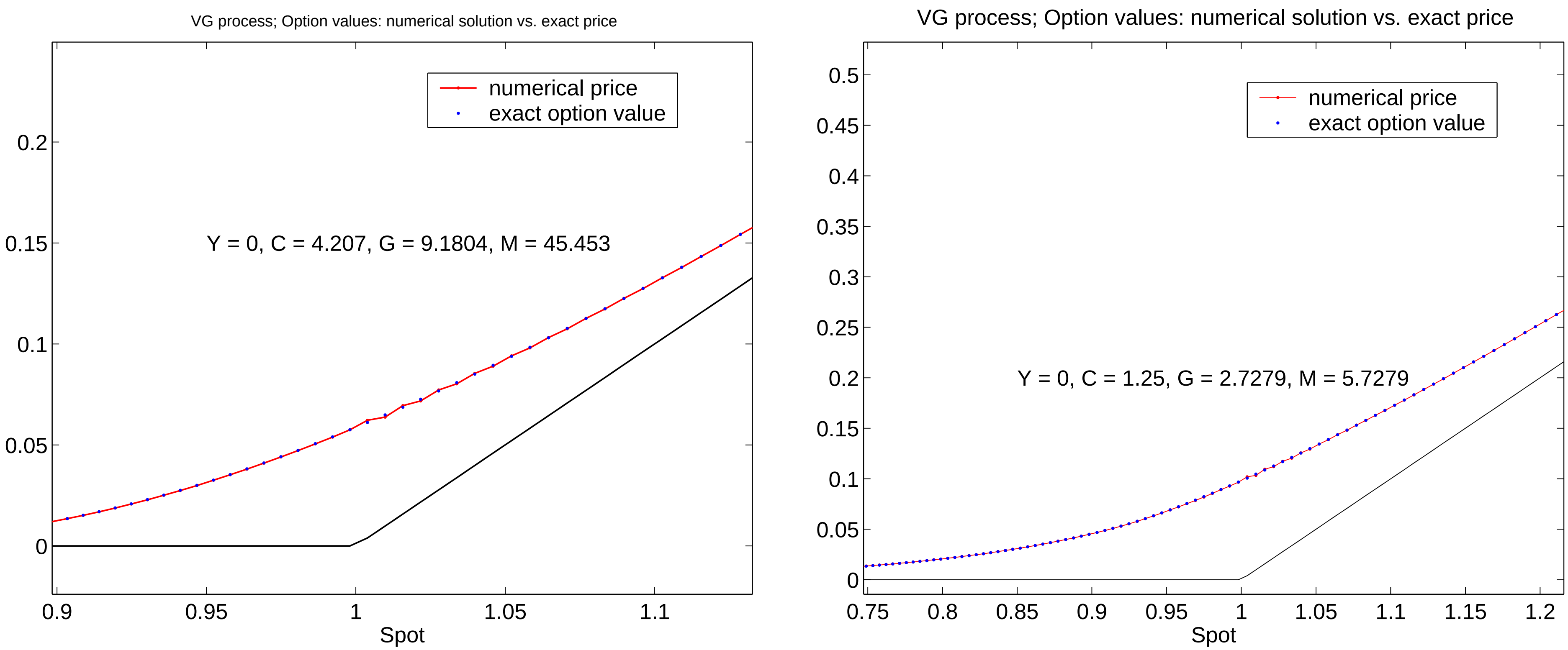


Figure 1: VG as price process

Numerical Results

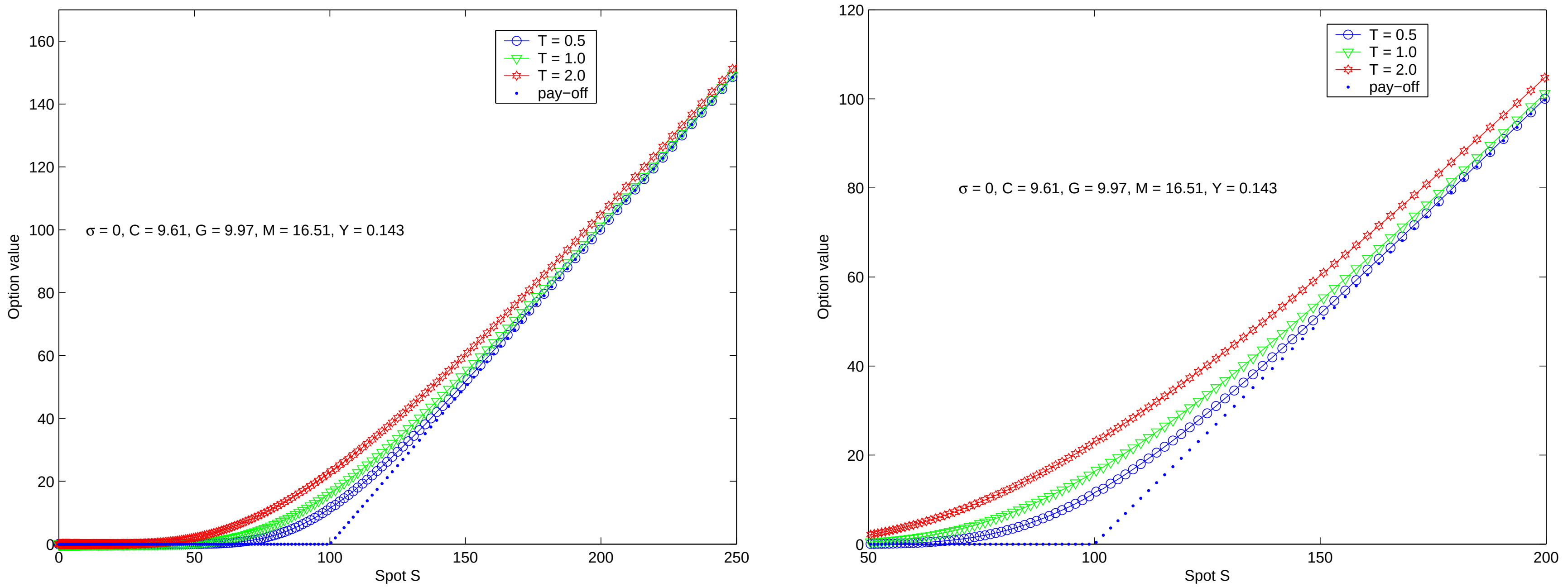


Figure 2: Option prices versus the stock price S for the case of an European call contract on pure jump Lévy driven assets ($\sigma = 0$) at different maturities (left and right (zoom)); CGMY parameters are: $Y = 0.1430$, $C = 9.61$, $G = 9.97$ and $M = 16.51$.

Numerical Results

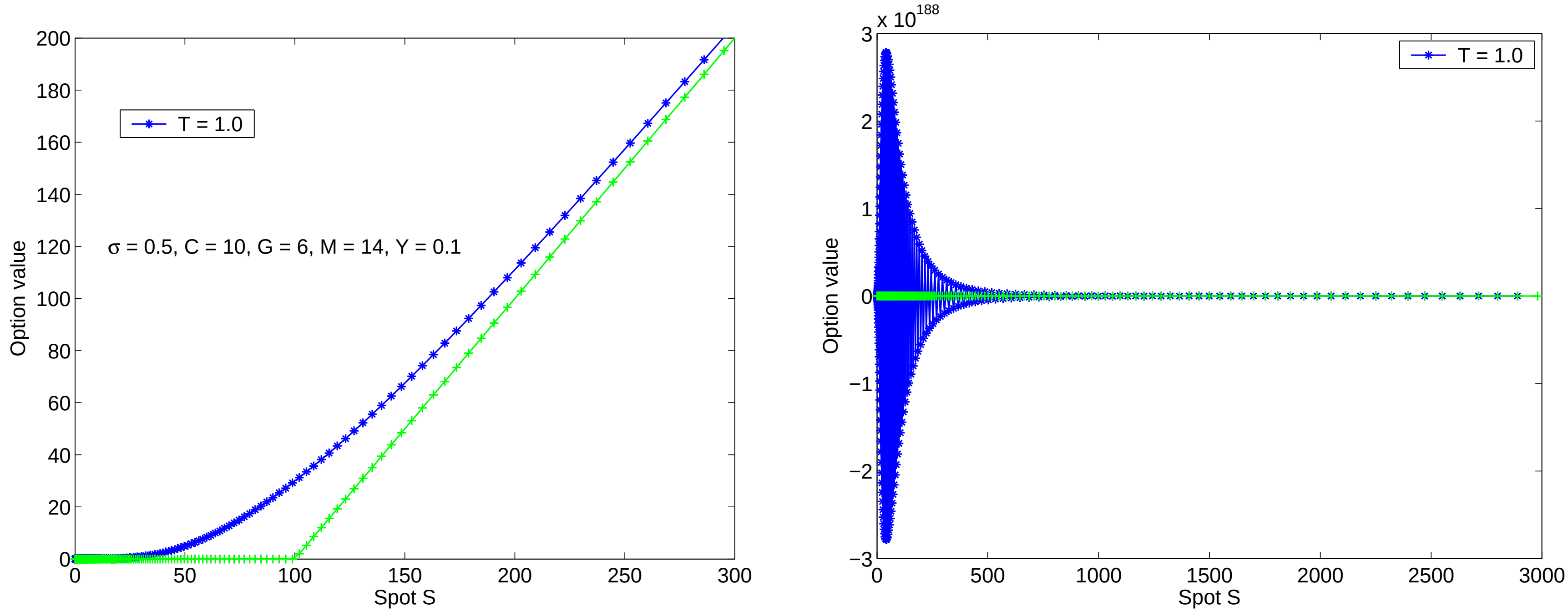


Figure 3: Explicit Euler scheme ($\theta = 0.0$), $h = 0.0312$ ($L = 8$, $R = 8$), $T = 1.0$; CGMY parameters: $C = 10.0$, $G = 6.0$, $M = 14.0$, $Y = 0.1$, $\sigma = 0.5$; stable $k = h^2$ (left), unstable: $k = 2h^2$

Numerical Results

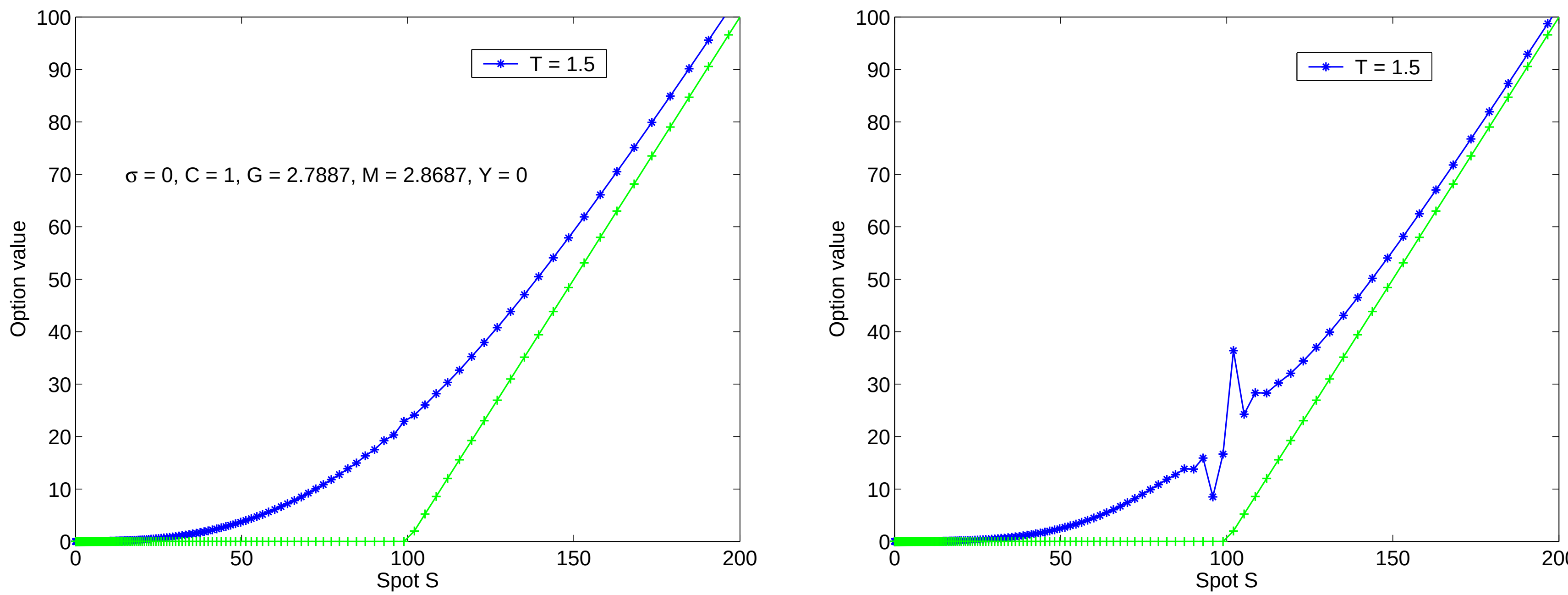


Figure 4: Explicit Euler scheme, pure jump VG process: $\theta = 0.0$, $L = 8$; α is the coefficient in the convection term $\alpha\frac{\partial u}{\partial x}$. $|\alpha|\frac{k}{h} \leq 1$ stable; $|\alpha|\frac{k}{h} > 1$ unstable VG parameters: $\sigma_{VG} = 0.5$, $\nu_{VG} = 1.0$, $\theta_{VG} = -0.01$: CGMY parameters: $Y = 0$, $G = 2.78$ and $M = 2.86$.

Numerical Results

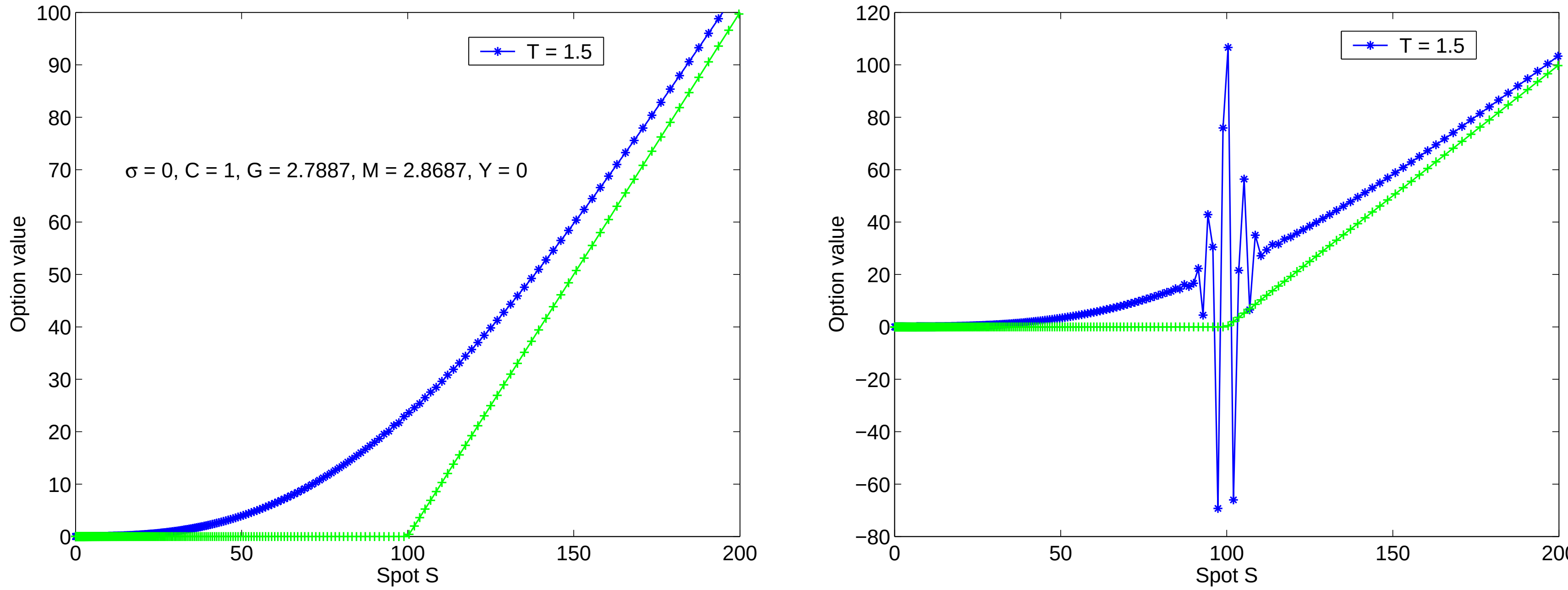


Figure 5: Explicit Euler scheme, pure jump VG process: $\theta = 0.0$, $L = 9$; $|\alpha|\frac{k}{h} \leq 1$ (left) stable, $|\alpha|\frac{k}{h} > 1$ (right) unstable; VG parameters: $\sigma_{VG} = 0.5$, $\nu_{VG} = 1.0$, $\theta_{VG} = -0.01$: CGMY parameters: $Y = 0$, $G = 2.78$ and $M = 2.86$.