Model equations for dispersive fluid flow

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One-dimensional Wave Models

Nonlinear and dispersive waves are modeled by Korteweg and de Vries (KdV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \qquad (1)$$

Peregrine & Benjamin, Bona & Mahoney (PBBM) equation (obtained from KdV by using $u_t \approx -u_x$)

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (2)$$

Camassa and Holm (CH) equation (has the soliton property like KdV; PBBM does not!)

$$u_t + u_x + uu_x - \alpha u_{xxt} = \frac{2}{3}\alpha u_x u_{xx} + \frac{1}{3}\alpha u u_{xxx} \qquad (3)$$

The CH equation (3) has a suggestive form

$$
\begin{aligned}
v_t + u_x + u v_x + 2 v u_x &= 0 \\
u &= \left(1 - \alpha \frac{\partial^2}{\partial x^2}\right)^{-1} v
\end{aligned}
\tag{4}
$$

Means u solves $\left(1 - \alpha \frac{\partial^2}{\partial x^2}\right) u = v$ with $u \to 0$ at ∞.

CH equation (3/4) is a nonlinear perturbation of PBBM and so also of KdV and can be viewed as a nonlinear advection equation for a variable v which has a simple relation to the advection velocity u.

(4) forms the template for a three-dimensional fluid flow model including dispersion.

3-D dispersive fluid models

The 3-D analog of the Camassa-Holm equation is

$$\mathbf{v}_t - \nu\Delta\mathbf{u} + (\mathbf{curl}\,\mathbf{v}) \times \mathbf{u} + \nabla p = 0 \tag{5}$$

where $(1 - \alpha\Delta)\,\mathbf{u} = \mathbf{v}$, $\alpha > 0$ and $\nabla \cdot \mathbf{u} = 0$.

$\alpha = 0$ implies $\mathbf{u} = \mathbf{v}$; (5) becomes Navier-Stokes

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{curl}\,\mathbf{u}) \times \mathbf{u} + \nabla p = 0 \tag{6}$$

where pressure $\tilde{p} = p - \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$.

Nonlinear term $(\mathbf{curl}\,\mathbf{u}) \times \mathbf{u}$ differs from usual one $\mathbf{u} \cdot \nabla\mathbf{u}$ via this transformation involving a re-definition of the "pressure" variable.

α model history

The α-model equation (5) first appeared as a model of Rivlin and Ericksen: a continuum of material with velocity $\mathbf{u}$ described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \nabla \cdot \mathbf{T}, \tag{7}$$

where $\mathbf{T} = \mathbf{T}(\mathbf{u})$ is the stress and we define

$$\frac{\mathrm{d}}{\mathrm{d}t}w := w_t + \mathbf{u} \cdot \nabla w \tag{8}$$

for any w (either scalar, vector or tensor valued).

Rivlin-Ericksen grade n model

Assume that the stress tensor $\mathbf{T}$ has the form (for the grade n model)

$$\mathbf{T} = -\tilde{p}\mathbf{I} + \mathcal{S}^n(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \tag{9}$$

where $\tilde{p} =$ pressure and $\mathbf{a}_j$ are Rivlin-Ericksen tensors defined recursively by (recall the notation (8))

$$\mathbf{a}_1 = \mathbf{L} + \mathbf{L}^T \quad , \quad \mathbf{L} = \nabla\, \mathbf{u}\,,$$

$$\mathbf{a}_j = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_{j-1} + \mathbf{a}_{j-1}\mathbf{L} + \mathbf{L}^T\mathbf{a}_{j-1}\,,$$

$\mathcal{S}^n = \sum_{i=1}^n \mathbf{S}_i$ and each $\mathbf{S}_i$ is a polynomial in the $\mathbf{a}_j$:

$$\mathbf{S}_1 = \eta\mathbf{a}_1\ ,\ \mathbf{S}_2 = \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2,\ \text{etc.}$$

Grade 2 Model

$$\mathbf{a}_1 = \nabla\,\mathbf{u} + \nabla\,\mathbf{u}^T\,, \quad \mathbf{a}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_1 + \mathbf{a}_1\mathbf{L} + \mathbf{L}^T\mathbf{a}_1\,,$$

$$\mathbf{T} = \mathbf{T}^2 = -\tilde{p}\mathbf{I} + \eta\mathbf{a}_1 + \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2\,, \qquad (10)$$

where the parameters η, α_i are material constants. Physical arguments show that

$$\eta \geq 0 \quad , \quad \alpha_1 \geq 0 \quad \text{and } \alpha_1 + \alpha_2 = 0\,.$$

Setting $\alpha = \alpha_1$ (and $\alpha_2 = -\alpha$) and substituting (10) into (7) yields the dispersive fluid equations (5).

Geometry of maps

When $\nu = 0$ in (5), we get a grade-two variant of the Euler equations. Consider the class $\mathcal{V}^s$ of vector fields $\mathbf{u}$ satisfying (a) $\mathbf{u} \in H^s(\Omega)$ (b) $\nabla \cdot \mathbf{u} = 0$ and (c) $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Given $\mathbf{u}(\cdot, t) \in \mathcal{V}^s$, consider the flow $\mathbf{f_u} = \mathbf{f_u}(\mathbf{x}, t)$ generated by $\mathbf{u}$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f_u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{f_u}(\mathbf{x}, t), t) \qquad (11)$$

where $\mathbf{f_u}(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$. For each t, $\mathbf{f_u}(\cdot, t) \in \mathcal{D}_\Omega =$ the space of volume preserving diffeomorphisms of Ω. Thus the map $t \to \mathbf{f_u}(\cdot, t)$ is a curve in $\mathcal{D}_\Omega$.

Geometry of $\mathcal{D}_\Omega$

$\mathcal{D}_\Omega$ has a natural group structure: composition. The tangent space $T_{\mathcal{D}_\Omega}(\mathcal{I})$ to $\mathcal{D}_\Omega$ at $\mathcal{I}$ can be identified with the space $\mathcal{V}^s$ of divergence-free vector fields. Putting an inner-product on $\mathcal{V}^s$ puts a metric on the tangent space to $\mathcal{D}_\Omega$ at $\mathcal{I}$. For example,

$$< \tau, \sigma >_{L^2} = \int_\Omega \tau(\mathbf{x}) \cdot \sigma(\mathbf{x}) \, d\mathbf{x} \tag{12}$$

$$< \tau, \sigma >_{H^1_\alpha} = \int_\Omega \tau(\mathbf{x}) \cdot \sigma(\mathbf{x}) + \alpha \nabla \tau(\mathbf{x}) : \nabla \sigma(\mathbf{x}) \, d\mathbf{x}. \tag{13}$$

Using the group structure allows us to translate this metric to the entire tangent bundle invariantly.

Geodesics come from solutions

Remarkably $\mathbf{u}$ solves the Euler equations if and only if the curve $t \to \mathbf{f_u}(\cdot, t)$ is a geodesic in $\mathcal{D}_\Omega$ with metric (12) given by the L^2 inner-product on $\mathcal{V}^s$.

Even more remarkably, $\mathbf{u}$ solves α-model (5) if and only if the curve $t \to \mathbf{f_u}(\cdot, t)$ is a geodesic in $\mathcal{D}_\Omega$ with metric (13) given by the H^1_α inner-product on $\mathcal{V}^s$.

This structural property of the α-model (5) exhibits a key property of a good model and makes it clear that it does not appear by chance.

Turbulence models

Statistical properties of ensembles of solutions of the Navier-Stokes equations satisfy averaged equations. If $\mathbf{u}$ denotes an ensemble average of solutions, it may satisfy a Rivlin-Erkicksen model, as was observed in the 1950's by Rivlin.

A modified Grade-two model has been used by Chen, Foias, Holm, Olson and Titi (1998) to provide an accurate model of turbulence experiments done in a channel.

Grade two and the blob

Oliver and Shkoller (2000) observed that the vortex-blob method of Chorin

$$\begin{aligned} \partial_t + \mathbf{u}\cdot\nabla q &= 0 \\ \mathbf{u} &= K^\alpha * q \end{aligned} \tag{14}$$

more closely represents the Grade-2/α equation than it does the Euler equation.

Here K^α is the integral kernal inverse of $(1-\alpha\Delta)$curl.

The (point) vortex method corresponds to $\alpha = 0$.

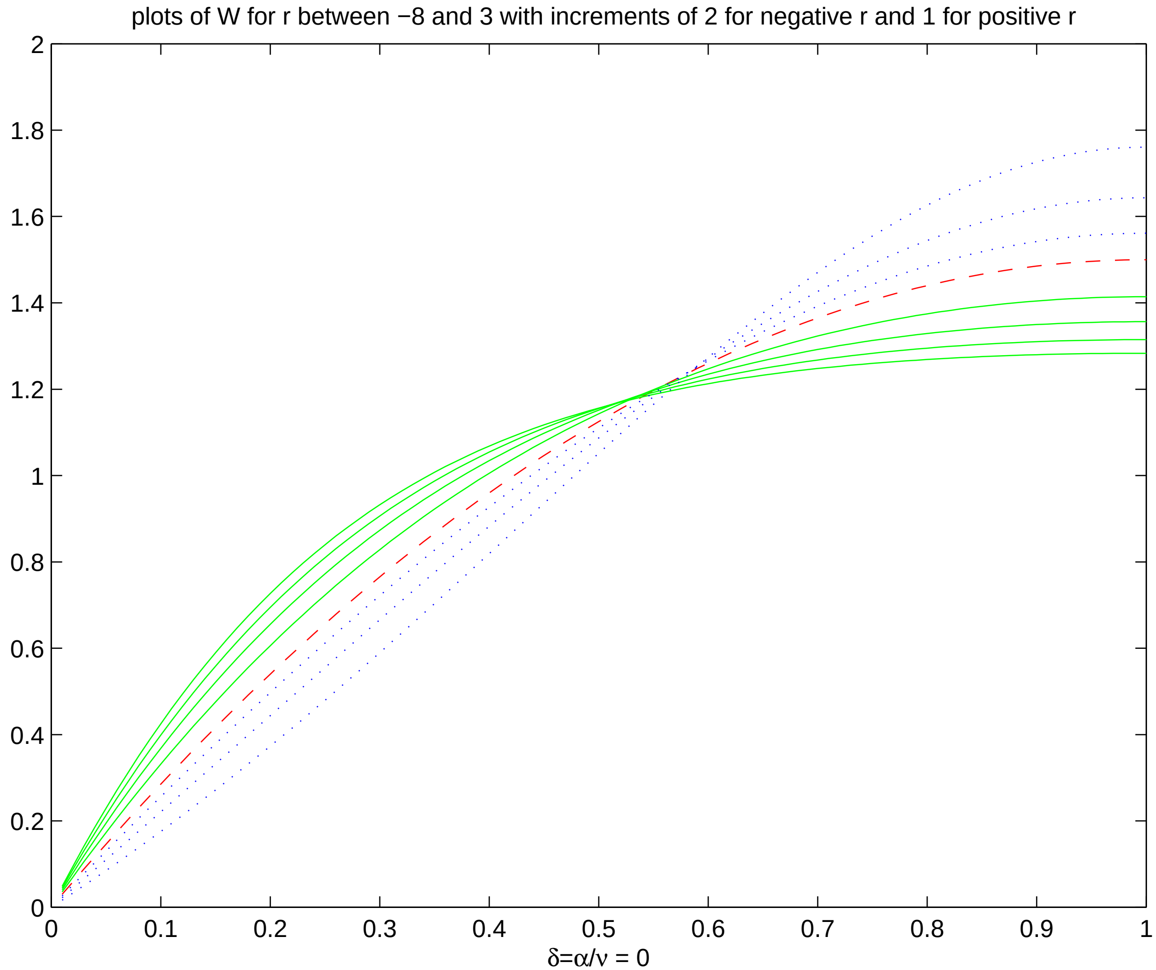
Lubrication model solutions

How does the α term affect the flow?

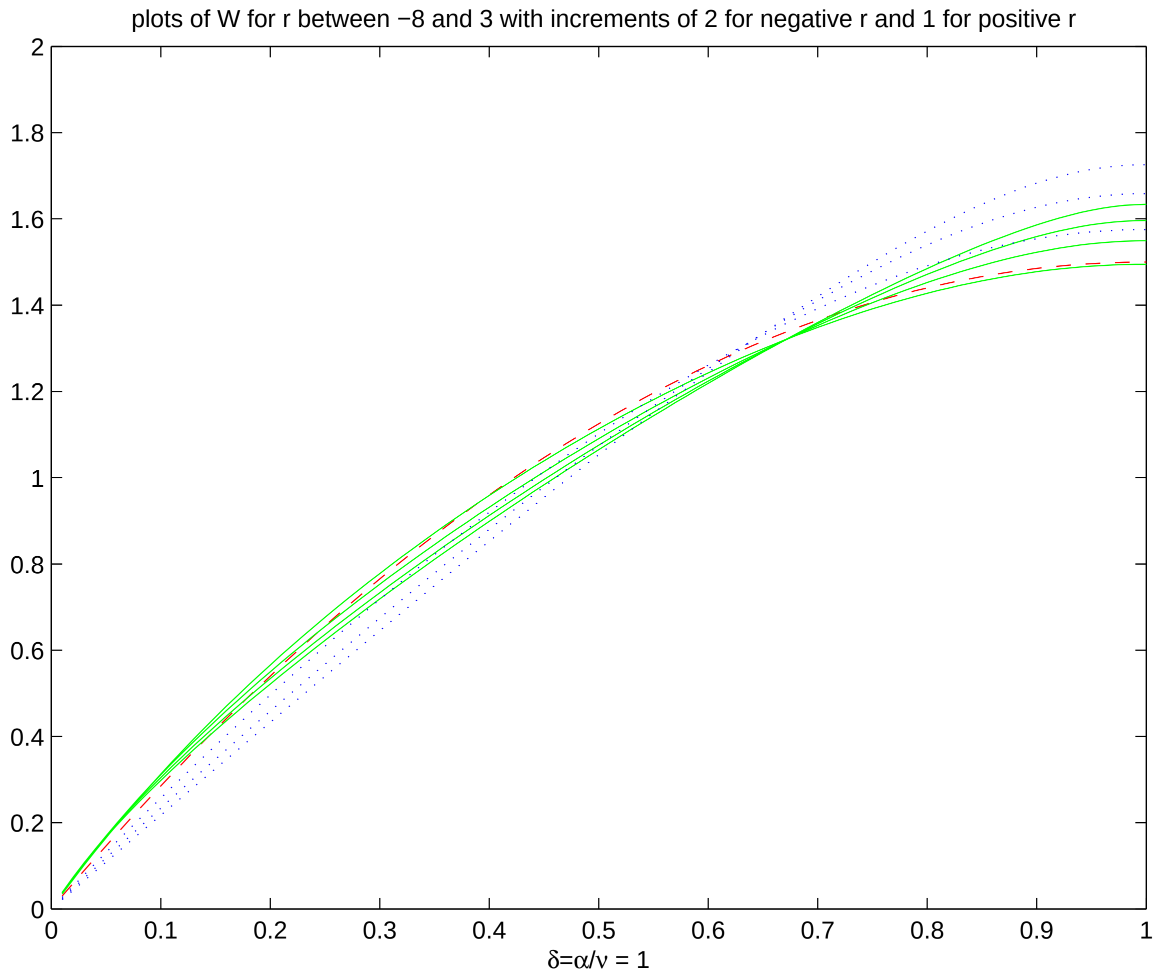
Lubrication theory provides flow patterns in a long channel with slowly varying width.

The shape depends on the parameter $r := t'R$ where R is the Reynolds number and t is the width of the channel.

When $\alpha > 0$ the effect of R is diminished.



Newtonian flow in a wavy chanel.



$\alpha \approx \nu = 1/R \implies$ parabolic flow re-established.

Stability of the grade-two model

Taking the **curl** of (5) and introducing the variable

$$\mathbf{z} = \mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,(\mathbf{u} - \alpha\Delta\mathbf{u}) \tag{15}$$

This leads to a transport equation for $\mathbf{z}$:

$$\alpha\mathbf{z}_t + \nu\mathbf{z} + \alpha\mathbf{u}\cdot\nabla\mathbf{z} - \alpha\mathbf{z}\cdot\nabla\mathbf{u} = \nu\,\mathbf{curl}\,\mathbf{u} \tag{16}$$

The steady versions of (5) and (16) in 2-D read

$$\begin{aligned} -\nu\Delta\mathbf{u} + z(u_2 - u_1) + \nabla p &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \nu z + \alpha\mathbf{u}\cdot\nabla z &= \nu\,\mathrm{curl}\,\mathbf{u} \end{aligned} \tag{17}$$

where we now $\mathbf{u}$ denotes a 2-vector valued function and $\mathrm{curl}\,\mathbf{u} := u_{1,2} - u_{2,1}$.

Dependence of $\mathbf{u}$ on z

The first two equations in (17) are well posed with $\mathbf{u} \in H^1$ as long as $z \in L^r$ for $r > 1$: given such a z we can find a unique $\mathbf{u} \in H^1$ (and $p \in L^2$) to the modified Stokes equation

$$\begin{aligned} -\nu \Delta \mathbf{u} + z(u_2 - u_1) + \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{18}$$

with appropriate bounds for $\mathbf{u}$ and p in terms of $\|z\|_{L^r}$.

However, the third (transport) equation in (17) is more problematic.

Transport equation

Let us write the general form of the transport equation in (17), after dividing by ν, as

$$z + \mathcal{W}\mathbf{u} \cdot \nabla z = f. \tag{19}$$

If all we know is that $\mathbf{u} \in H^1$, then $f \in L^2$ is the best we can hope for in (17). But then we could not hope for more than $z \in L^2$ either, as (19) provides no smoothing. And for $z \in L^2$ (and $\mathbf{u} \in H^1$), the term $\mathbf{u} \cdot \nabla z$ is a concern.

Certainly, $\mathbf{u} \cdot \nabla z$ will not in general be in L^2 for unrelated z and $\mathbf{u}$.

Transport equation solution

Miraculously, it is possible to show that (provided $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$), the problem (17), i.e.,

$$z + \mathcal{W}\mathbf{u} \cdot \nabla z = f$$

has a unique solution $z \in L^2$ for any $f \in L^2$.

In fact, you can say more, in that z lies in the space

$$X_{\mathbf{u}} := \left\{ w \in L^2 \ : \ \mathbf{u} \cdot \nabla w \in L^2 \right\} \tag{20}$$

and $\|z\|_{L^2} \leq \|f\|_{L^2}$.

Stability of numerical discretizations

A standard complication is the divergence zero condition in (17).

The major new ingredient in the system (17) is the transport equation (19):

$$\nu z + \alpha \mathbf{u} \cdot \nabla z = \nu \, \mathbf{curl} \, \mathbf{u} \tag{21}$$

Suppose that $\mathbf{u}_h = \mathbf{u}_h(z)$ and $p_h = p_h(z)$ denote finite element functions defined in appropriate spaces for solving the modified Stokes equation (18) for a given $z \in L^2$. Let W_h denote a subspace of H^1 for simplicity in which we will seek approximations z_h to the solutions of (21).

Numerical discretization of transport equation

We define a "master" variational problem for approximating (21) $(\nu \mathbf{z} + \alpha \mathbf{u} \cdot \nabla \mathbf{z} = \nu\, \mathbf{curl}\, \mathbf{u})$ as:

Find $z_h \in W_h$ such that

$$\begin{aligned} &\nu\,(z_h, w) + \alpha\,(\mathbf{u}_h \cdot \nabla z_h, w) \\ &\qquad + \tfrac{1}{2}\alpha\,((\nabla \cdot \mathbf{u}_h) z_h, w) = \nu\,(\mathbf{curl}\,\mathbf{u}_h, w) \end{aligned} \tag{22}$$

for all $w \in W_h$. Here, the forms (v, w) denote the $L^2(\Omega)$ inner-product.

Note that we have allowed for the possibility that $\nabla \cdot \mathbf{u}_h$ is not exactly zero.

Convergence

Coupled system for $\mathbf{u}_h$, p_h and z_h has solutions which converge to the system (17) if approximation scheme for Stokes-like equation (18) satisfies either

- $\nabla \cdot \mathbf{u}_h$ identically zero (true for high-degree polynomials on triangles), or
- $\nabla \cdot \mathbf{u}_h$ is orthogonal to products $z_h w_h$ for any $z_h, w_h \in W_h$ (e.g. $z_h w_h \in \Pi_h \, \forall z_h, w_h \in W_h$, holds if $\Pi =$ quadratics and $W_h =$ linears).

Improved Convergence for z

If we modify the transport approximation with a standard upwinding technique (streamline diffusion), we again prove stability of the corresponding numerical approximation and prove quasi-optimal rates of convergence for $\mathbf{u} - \mathbf{u}_h$, $p - p_h$ and suffer only the usual half-order degradation of approximation for $z - z_h$ common for streamline diffusion schemes.

Unfortunately, our estimates involve constants proportional to $1/\alpha$ and would not be uniform as $\alpha \to 0$.

Convergence for general Π_h, W_h

With no assumption about the approximation scheme for (18) with regard to $\nabla \cdot \mathbf{u}_h$, if

- Ω is a convex polygon

- and $\mathbf{u}$ and z satisfy some extra smoothness,

we again get convergence rates for the discrete approximation. However, these results require us to make a strong restriction on the domain Ω and so do not seem to be ideal.

Sign of α?

Rheologists tell us that often $\alpha < 0$.

Grade-two model is unstable for $\alpha < 0$!

Slemrod (1999) has suggested that truncation of the Rivlin-Ericksen expansion is at fault and a rational approximation should be used.

Dunn (1982) observed that if α is a function of $\|\nabla \mathbf{u} + \nabla \mathbf{u}^t\|^2$, stability is recovered for $\alpha < 0$.

One objective is to demonstrate the possibility of shear thinning without resorting to η dependence on strain-rate.

Energy estimates

If we take the dot-product of $\mathbf{u}$ and integrate over the flow domain Ω, we obtain an energy balance equation of the form

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega \|\mathbf{u}\|^2 + \tfrac{1}{2}\beta(\|\nabla\mathbf{u}+\nabla\mathbf{u}^t\|^2)\,dx\right)\\ +\eta\int_\Omega \|\nabla\mathbf{u}+\nabla\mathbf{u}^t\|^2\,dx = 0\end{aligned} \tag{23}$$

where $\beta' = \alpha$. Stability requires $\beta \geq 0$, or

$$\int_0^r \alpha(s)\,ds \geq -\beta(0) \quad \forall\, r > 0\,. \tag{24}$$

Stability of the rest state follows from $\beta(0) = 0$, but is this necessary?

Example: $\beta(r) = r/(1+r^2)$ has $\alpha < 0$ for $r > 1$.

Rheological properties

The viscometric functions commonly measured include the first normal stress difference N_1 which Dunn computes to be

$$N_1 = -2\gamma^2\alpha(2\gamma^2) \tag{25}$$

where γ is the shear rate. This can clearly be either positive or negative depending on the sign of α.

N_1 is typically positive, esp. for small γ, but N_1/κ^2 is not constant. In fact, it would appear that $N_1(\kappa) \approx \gamma^q$ for $q < 2$ leading to a singularity in α.

Stability of the rest state

Let $\{\phi_i\}$ be a complete set of eigenfunctions for the Stokes equations, which we write in variational form as

$$\langle \phi_i, v \rangle = \lambda_i(\phi_i, v) \; \forall v \in V \tag{26}$$

where V is the set of divergence zero functions in $H^1(\Omega)$ which vanish on the boundary. Assume that $(\phi_i, \phi_j) = \delta_{ij}$. Then if α is constant, we can write the grade-two equations as

$$(\mathbf{u}_t, v) + \alpha \langle \mathbf{u}_t, v \rangle + \nu \langle \mathbf{u}, v \rangle = 0\,. \tag{27}$$

We can expand the solution as

$$\mathbf{u} = \sum_i c_i(t) \phi_i\,. \tag{28}$$

Using (28) in (27), we obtain ordinary differential equations for the coefficients c_i of the form

$$c_i'(t)(1+\alpha\lambda_i)+\nu\lambda_i c_i(t)=0\,, \qquad (29)$$

which can be re-written as

$$c_i'(t)+\frac{\nu}{\lambda_i^{-1}+\alpha}c_i(t)=0 \qquad (30)$$

where $c_i(0)=(\mathbf{u}_0,\phi_i)$. Clearly if $\alpha<0$ then $\lambda_i^{-1}+\alpha<0$ for i sufficiently large, and we get exponential increase of these modes. On the other hand, if

$$\lambda_i^{-1}+\alpha>0 \qquad (31)$$

then the i-th mode is stable.

However, if α depends on $\nabla\mathbf{u} + \nabla\mathbf{u}^t$ then the picture is more complicated. We have

$$\begin{aligned} \lambda_i &= \lambda_i(\phi_i, \phi_i) \\ &= \langle \phi_i, \phi_i \rangle \\ &= \int_\Omega \|\nabla\phi_i + \nabla\phi_i^t\|^2 \, dx \, . \end{aligned} \tag{32}$$

That is, $\lambda_i/|\Omega|$ is the mean of $\|\nabla\phi_i + \nabla\phi_i^t\|^2$ over Ω, where $|\Omega|$ denotes the volume of Ω. Thus we can think that

$$\alpha(\|\nabla\phi_i + \nabla\phi_i^t\|^2) \approx \alpha(\lambda_i/|\Omega|) \, . \tag{33}$$

Combining this with the stability condition (31), we find that we have a stable mode if

$$\lambda_i^{-1} + \alpha(\lambda_i/|\Omega|) > 0 \, . \tag{34}$$

Since we are interested in the case $\alpha < 0$, let us set $\tilde{\alpha} = -\alpha$. Then (34) becomes

$$\tilde{\alpha}(\lambda_i/|\Omega|) < \lambda_i^{-1}\,, \tag{35}$$

which would be satisfied if we knew that

$$-\alpha(r) = \tilde{\alpha}(r) < \frac{1}{|\Omega| r}\,. \tag{36}$$

Notice that

$$\alpha(r) := \frac{br}{1 + ar^2}. \tag{37}$$

satisfies this condition for suitable a and b. Also, if α is monotone for large r, then the integrability of $-\alpha(r)$ implies that $-\alpha(r) \leq \delta/r$ for $r > r_\delta$ for any $\delta > 0$.

Oldroyd-B model

The upper-convected Maxwellian derivative is defined by

$$\frac{\nabla}{\nabla t}\mathbf{f} := \frac{\mathsf{D}}{\mathsf{D}t}\mathbf{f} - (\nabla\mathbf{u})\mathbf{f} - \mathbf{f}\nabla\mathbf{u}^T\,, \tag{38}$$

for any tensor $\mathbf{f}$. The Oldroyd-B fluid has the following constitutive equation for the extra stess tensor T:

$$\lambda\frac{\nabla}{\nabla t}\mathbf{T} + \mathbf{T} = \eta\mathbf{a}_1\,, \tag{39}$$

in its simplest form, where $\mathbf{a}_1 := \nabla\mathbf{u} + \nabla\mathbf{u}^t$.

Suppose that $\alpha_1 = -\lambda\eta$ and $\alpha_2 = 2\lambda\eta$. Then the grade-two model becomes

$$\begin{aligned} \mathbf{T} =& \eta\mathbf{a}_1 - \lambda\eta\frac{\Delta}{\Delta t}\mathbf{a}_1 + 2\lambda\eta\mathbf{a}_1^2 \\ =& \eta\mathbf{a}_1 - \lambda\eta\frac{\nabla}{\nabla t}\mathbf{a}_1 \end{aligned} \tag{40}$$

This is to be contrasted with the Oldroyd-B model (39), which we can write as

$$\begin{aligned} \mathbf{T} =& \eta\mathbf{a}_1 - \lambda\frac{\nabla}{\nabla t}\mathbf{T} \\ \approx& \eta\mathbf{a}_1 - \lambda\eta\frac{\nabla}{\nabla t}\mathbf{a}_1 \, , \end{aligned} \tag{41}$$

provided λ is very small. Thus we expect that the Oldroyd-B model will be very similar to the grade-2 model with the choice $\alpha_1 = -\lambda\eta$ and $\alpha_2 = 2\lambda\eta$.

Note that this would imply that $\alpha_1 < 0$ if $\lambda > 0$, and $\alpha_1 + \alpha_2 = \lambda\eta > 0$. This would imply that the model is not shear thinning.

We can understand the relationship of the two models by letting the symbol z denote the operator

$$z := \lambda \frac{\nabla}{\nabla t}. \tag{42}$$

Then the two models can be written as

$$\begin{aligned} (1+z)\mathbf{T} =& \eta \mathbf{a}_1 \\ \mathbf{T} =& \eta(1-z)\mathbf{a}_1 \end{aligned} \tag{43}$$

Thus the two models are related simply by the approximation

$$(1+z)^{-1} \approx 1 - z\,. \tag{44}$$

Conclusions

Reviewed model for dispersive flow in 2 & 3 D.

Lubrication models illustrate laminar flows.

Stability was described in the 2-D case. Numerical schemes for approximating the equations and their corresponding stability and convergence properties have been proved.

The sign of α provides open questions.

Oldroyd-B and Grade-2 may be similar.